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LEONARDO OF PISA AND HIS LIBER QUADRATORUM.

By R. B. McCLENON, Grinnell College.

The thirteenth century is a period of great fascination for the historian, whether his chief interest is in political, social, or intellectual movements. During this century great and far-reaching changes were taking place in all lines of human activity. It was the century in which culminated the long struggle between the Papacy and the Empire; it brought the beginnings of civil liberty in England; it saw the building of the great Gothic cathedrals, and the establishment and rapid growth of universities in Paris, Bologna, Naples, Oxford, and many other centers. The crusades had awakened the European peoples out of their lethargy of previous centuries, and had brought them face to face with the more advanced intellectual development of the East. Countless travelers passed back and forth between Italy and Egypt, Asia Minor, Syria, and Bagdad; and not a few adventurous and enterprising spirits dared to penetrate as far as India and China. The name of Marco Polo will occur to everyone, and he is only the most famous among many who in those stirring days truly discovered new worlds.

Among the many valuable gifts which the Orient transmitted to the Occident at this time, undoubtedly the most precious was its scientific knowledge, and in particular the Arabian and Hindu mathematics. The transfer of knowledge and ideas from East to West is one of the most interesting phenomena of this interesting period, and accordingly it is worth while to consider the work of one of the pioneers in this movement.

Leonardo of Pisa, known also as Fibonacci,[1] in the last years of the twelfth century made a tour of the East, saw the great markets of Egypt and Asia Minor,

[1] This is probably a contraction for "Filiorum Bonacci," or possibly for "Filius Bonacci"; that is, "of the family of Bonacci" or "Bonacci's son." See Boncompagni, *Della Vita e delle Opere di Leonardo Pisano, matematico del secolo decimoterzo*, Rome, 1852, pp. 8–12.

went as far as Syria, and returned through Constantinople and Greece.[1] Unlike most travelers, Leonardo was not content with giving a mere glance at the strange and new sights that met him, but he studied carefully the customs of the people, and especially sought instruction in the arithmetic system that was being found so advantageous by the Oriental merchants. He recognized its superiority over the clumsy Roman numeral system which was used in the West, and accordingly decided to study the Hindu-Arabic system thoroughly and to write a book which should explain to the Italians its use and applications. Thus the result of Leonardo's travels was the monumental *Liber Abaci* (1202), the greatest arithmetic of the middle ages, and the first one to show by examples from every field the great superiority of the Hindu-Arabic numeral system over the Roman system exemplified by Boethius.[2] It is true that Leonardo's *Liber Abaci* was not the first book written in Italy in which the Hindu-Arabic numerals were used and explained,[3] but no work had been previously produced which in either the extent or the value of its contents could for a moment be compared with this. Even today it would be thoroughly worth while for any teacher of mathematics to become familiar with many portions of this great work. It is valuable reading both on account of the mathematical insight and originality of the author, which constantly awaken our admiration, and also on account of the concrete problems, which often give much interesting and significant information about commercial customs and economic conditions in the early thirteenth century.

Besides the *Liber Abaci*, Leonardo of Pisa wrote an extensive work on geometry, which he called *Practica Geometriæ*. This contains a wide variety of interesting theorems, and while it shows no such originality as to enable us to rank Leonardo among the great geometers of history, it is excellently written, and the rigor and elegance of the proofs are deserving of high praise. A good idea of a small portion of the *Practica Geometriæ* can be obtained from Archibald's very successful restoration of Euclid's *Divisions of Figures*.[4]

The other works of Leonardo of Pisa that are known are *Flos*, a *Letter to Magister Theodorus*, and the *Liber Quadratorum*. These three works are so original and instructive, and show so well the remarkable genius of this brilliant mathematician of the thirteenth century, that it is highly desirable that they be made available in English translation. It is my intention to publish such a translation when conditions are more favorable, but in the meantime a short account of the *Liber Quadratorum* will bring to those whose attention has not yet been called to it some idea of the interesting and valuable character of the book.

The *Liber Quadratorum* is dedicated to the Emperor Frederick II, who

[1] *Scritti di Leonardo Pisano*, 2 vols., Rome, 1857–61. Vol. I, p. 1.

[2] Boethius, ed. Friedlein, Leipzig, 1867. The arithmetic occupies pages 1–173. This was the arithmetic that was very generally taught throughout Europe before the thirteenth century, and its use continued to be widespread long after better works were in the field.

[3] Smith and Karpinski, *The Hindu-Arabic Numerals*, Boston and London, 1911. Chapter VII gives an account of the first European writings on these numerals.

[4] Archibald, *Euclid's Book on Divisions of Figures; with a Restoration based on Woepcke's Text and on the Practica Geometriæ of Leonardo Pisano*, Cambridge, England, 1915.

throughout his whole career showed a lively and intelligent interest in art and science, and who had taken favorable notice of Leonardo's *Liber Abaci.* In the dedication, dated in 1225, Leonardo relates that he had been presented to the Emperor at court in Pisa, and that Magister Johannes of Palermo had there proposed a problem[1] as a test of Leonardo's mathematical power. The problem was, to find a square number which when either increased or diminished by 5 should still give a square number as result. Leonardo gave a correct answer, $11\frac{97}{144}$. For $11\frac{97}{144} = (3\frac{5}{12})^2$, $6\frac{97}{144} = (2\frac{7}{12})^2$, and $16\frac{97}{144} = (4\frac{1}{12})^2$. Through considering this problem and others allied to it, Leonardo was led to write the *Liber Quadratorum.*[2] It should be said that this problem had been considered by Arab writers with whose works Leonardo was unquestionably familiar; but his methods are original, and our admiration for them is not diminished by careful study of what had been done by his Arabian predecessors.[3]

In the *Liber Quadratorum,* Leonardo has given us a well-arranged, brilliantly-written collection of theorems from indeterminate analysis involving equations of the second degree. Many of the theorems themselves are original, and in the case of many others the proofs are so. The usual method of proof employed is to reason upon general numbers, which Leonardo represents by line segments. He has, it is scarcely necessary to say, no algebraic symbolism, so that each result of a new operation (unless it be a simple addition or subtraction) has to be represented by a new line. But for one who had studied the "geometric algebra" of the Greeks, as Leonardo had, in the form in which the Arabs used it,[4] this method offered some of the advantages of our symbolism; and at any rate it is marvelous with what ease Leonardo keeps in his mind the relation between two lines and with what skill he chooses the right road to bring him to the goal he is seeking.

To give some idea of the contents of this remarkable work, there follows a list of the most important results it contains. The numbering of the propositions is not found in the original.

PROPOSITION I. THEOREM. Every square number[5] can be formed as a sum of successive odd numbers beginning with unity. That is,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

PROPOSITION II. PROBLEM. To find two square numbers whose sum is a square number. "I take any odd square I please, . . . and find the other from

[1] In the introduction to *Flos* we are told that two other problems were propounded at the same time. *Scritti,* II, p. 227.

[2] *Scritti,* II, p. 253.

[3] See, for example, Woepcke, *Recherches sur plusieurs ouvrages de Leonard de Pise, et sur les rapports qui existent entre ces ouvrages et les travaux mathématiques des Arabes,* Rome, 1859.

[4] Heath, T. L., *The Thirteen Books of Euclid's Elements,* Cambridge, 1908. Vol. I, pp. 372–374, 383–385, 386–388; Zeuthen, H. G., *Geschichte der Mathematik im Altertum und Mittelalter,* Copenhagen, 1896, pp. 44–53; Karpinski, L. C., *Robert of Chester's Latin translation of the Algebra of Al-Khowarizmi,* New York, 1915, pp. 77–89.

[5] Throughout this article, unless otherwise stated, the word "number" is to be understood as meaning "positive integer."

the sum of all the odd numbers from unity up to that odd square itself."[1] Thus, if $2n + 1$ is a square ($= x^2$) then

$$1 + 3 + 5 + \cdots + (2n - 1) + x^2 = n^2 + (2n + 1)$$
$$= \text{a sum of two squares} = (n + 1)^2$$

This is equivalent to Pythagoras's rule for obtaining rational right triangles, as stated by Proclus,[2] viz.,

$$\left(\frac{x^2 - 1}{2}\right)^2 + x^2 = \left(\frac{x^2 + 1}{2}\right)^2.$$

For, inasmuch as $2n + 1 = x^2$, we have $n = \dfrac{x^2 - 1}{2}$ and $n + 1 = \dfrac{x^2 + 1}{2}$.

PROPOSITION III. THEOREM. $\left(\dfrac{n^2}{4} - 1\right)^2 + n^2 = \left(\dfrac{n^2}{4} + 1\right)^2.$

This enables us to obtain rational right triangles in which the hypotenuse exceeds one of the legs by 2. It is attributed by Proclus to Plato[2]. Leonardo also gives the rule in case the hypotenuse is to exceed one leg by 3, and indicates what the result would be if the hypotenuse exceeds one leg by any number whatever.

PROPOSITION IV. THEOREM. "Any square exceeds the square which immediately precedes it by the amount of the sum of their roots." That is, $n^2 - (n - 1)^2 = n + (n - 1)$. It follows from this that when the sum of two consecutive numbers is a square number, then the square of the greater will equal the sum of two squares. For, if $n + (n - 1) = u^2$, then $n^2 - (n - 1)^2 = u^2$ or $n^2 = u^2 + (n - 1)^2$.

PROPOSITION V. PROBLEM. Given $a^2 + b^2 = c^2$, to find two integral or fractional numbers x, y, such that $x^2 + y^2 = c^2$. Solution: Find two other numbers m and n such that[3] $m^2 + n^2 = q^2$. If $q^2 \neq c^2$, multiply the preceding equation by c^2/q^2, obtaining

$$\left(\frac{c}{q} \cdot m\right)^2 + \left(\frac{c}{q} \cdot n\right)^2 = c^2$$

so that $x = c/q \cdot m$, $y = c/q \cdot n$ is a solution.

PROPOSITION VI. THEOREM. "If four numbers not in proportion are given, the first being less than the second, and the third less than the fourth, and if the sum of the squares of the first and second is multiplied by the sum of the squares of the third and fourth, there will result a number which will be equal in two ways to the sum of two square numbers." That is,

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2 = (ad + bc)^2 + (ac - bd)^2.$$

This very important theorem should be called Leonardo's Theorem, for it is

[1] The use of quotation marks indicates a literal translation of Leonardo's words; in other cases the exposition follows his thought without adhering closely to his form of expression.

[2] Proclus, ed. Friedlein, Leipzig, 1873, p. 428.

[3] This is possible by Proposition II or Proposition III.

not found definitely stated, to say nothing of being proved, in any earlier work. Leonardo considers also the case where a, b, c, and d are in proportion, and shows that then $(a^2 + b^2)\cdot(c^2 + d^2)$ is equal to a square and the sum of two squares. This gives him still another way of finding rational right triangles.[1]

PROPOSITION VII. THEOREM. $(x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$.[2] Leonardo proves this very simply as a corollary of Proposition VI.

PROPOSITION VIII. PROBLEM. "To find two numbers the sum of whose squares is a number, not a square, formed from the addition of two given squares." That is, to find x and y such that $x^2 + y^2 = a^2 + b^2$. Choose any two numbers c and d, such that $c^2 + d^2$ is a square, and write $(a^2 + b^2)(c^2 + d^2)$ as a sum of two squares, let us say $p^2 + q^2$; this we can do by Proposition VI. Construct the right triangle whose legs are p and q; then the similar triangle whose hypotenuse is equal to $\sqrt{c^2 + d^2}$ will have as its legs the two required numbers x and y.

PROPOSITION IX. THEOREM.

$$6(1^2 + 2^2 + 3^2 + \cdots + n^2) = n(n + 1)(2n + 1).$$

The proof of this is strikingly original, and proceeds from the identity

$$n(n + 1)(2n + 1) = n(n - 1)(2n - 1) + 6n^2.$$

Hence

$$\begin{aligned} n(n - 1)(2n - 1) &= (n - 1)(n - 2)(2n - 3) + 6(n - 1)^2, \\ &\cdots\cdots\cdots \\ 2\cdot3\cdot(2 + 3) &= 1\cdot2\cdot(1 + 2) + 6\cdot2^2, \\ 1\cdot2\cdot(1 + 2) &= \qquad\qquad 6\cdot1^2. \end{aligned}$$

It follows by addition that

$$n(n + 1)(2n + 1) = 6(1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2).$$

PROPOSITION X. THEOREM.

$$12[1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2] = (2n - 1)(2n + 1)4n.$$

Leonardo gives a proof very similar to that of Proposition IX.

PROPOSITION XI. THEOREM.

$$12[2^2 + 4^2 + 6^2 + \cdots + (2n)^2] = 2n(2n + 2)(4n + 2),$$

and likewise

$$18[3^2 + 6^2 + 9^2 + \cdots + (3n)^2] = 3n(3n + 3)(6n + 3),$$

and

$$24[4^2 + 8^2 + 12^2 + \cdots + (4n)^2] = 4n(4n + 4)(8n + 4),$$

[1] For instance, letting $a = 6$, $b = 4$, $c = 3$, $d = 2$,

$$(36 + 16)(9 + 4) = 676 = (6\cdot3 + 4\cdot2)^2 = (6\cdot2 + 4\cdot3)^2 + (6\cdot3 - 4\cdot2)^2 = 26^2 = 24^2 + 10^2.$$

[2] This is Euclid's general solution of the problem of finding rational right triangles. Heath, *op. cit.*, III, p. 63. (Euclid's Elements, X, Lemma to Theorem 29.)

and in general

$$6a[a^2 + (2a)^2 + (3a)^2 + \cdots + (na)^2] = na(na + a)(2na + a).$$

Here Leonardo has almost discovered the general result

$$a^2 + (a + d)^2 + (a + 2d)^2 + \cdots + [a + (n - 1)d]^2 = \frac{6na^2 + 6n(n - 1)ad + n(n - 1)(2n - 1)d^2}{6}.$$

His method needed no change at all, in fact.

PROPOSITION XII. THEOREM. If $x + y$ is even, $xy(x + y)(x - y)$ is divisible by 24; and in any case $4xy(x + y)(x - y)$ is divisible by 24. A number of this form is called by Leonardo a *congruum,* and he proceeds to show that it furnishes the solution to a problem proposed by Johannes of Palermo.

PROPOSITION XIII. PROBLEM. "To find a number which, being added to, or subtracted from, a square number, leaves in either case a square number." Leonardo's solution of this, the problem which had stimulated him to write the *Liber Quadratorum,* is so very ingenious and original that it is a matter of regret that its length prevents its inclusion here. It is not too much to say that this is the finest piece of reasoning in number theory of which we have any record, before the time of Fermat. Leonardo obtains his solution by establishing the identities

$$(x^2 + y^2)^2 - 4xy(x^2 - y^2) = (y^2 + 2xy - x^2)^2$$

and

$$(x^2 + y^2)^2 + 4xy(x^2 - y^2) = (x^2 + 2xy - y^2)^2.$$

PROPOSITION XIV. PROBLEM. To find a number of the form $4xy(x+y)(x-y)$ which is divisible by 5, the quotient being a square. Take $x = 5$, and y equal to a square such that $x + y$ and $x - y$ are also squares. The least possible value for y is 4, in which case

$$4xy(x + y)(x - y) = 4 \cdot 5 \cdot 4 \cdot 9 \cdot 1 = 720.$$

PROPOSITION XV. PROBLEM. "To find a square number which, being increased or diminished by 5, gives a square number. Let a congruum be taken whose fifth part is a square, such as 720, whose fifth part is 144; divide by this the squares congruent to 720,[1] the first of which is 961, the second 1681, and the third 2401. The root of the first square is 31, of the second is 41, and of the third is 49. Thus there results for the first square $6\frac{97}{144}$, whose root is $2\frac{7}{12}$, which results from the division of 31 by the root of 144, that is, by 12; and for the second, that is, for the required square, there will result $11\frac{97}{144}$, whose root is $3\frac{5}{12}$, which results from the division of 41 by 12; and for the last square there will result $16\frac{97}{144}$, whose root is $4\frac{1}{12}$."

[1] That is, the three squares in arithmetic progression, whose common difference is the congruum 720. They are obtained by Proposition XIII, thus: Taking $x = 5$ and $y = 4$, $y^2 + 2xy - x^2 = 31$, the root of the first square; $x^2 + y^2 = 41$, the root of the second square; and $x^2 + 2xy - y^2 = 49$, the root of the third square.

PROPOSITION XVI. THEOREM. When $x > y$, $(x + y)/(x - y) \neq x/y$. It follows that $x(x - y)$ is not equal to $y(x + y)$, and "from this," Leonardo says, "it may be shown that no square number can be a congruum." For if $xy(x + y)(x - y)$ could be a square, either $x(x - y)$ must be equal to $y(x + y)$, which this proposition proves to be impossible, or else the four factors must severally be squares, which is also impossible. Leonardo to be sure overlooked the necessity of proving this last assertion, which remained unproved until the time of Fermat.[1]

PROPOSITION XVII. PROBLEM. To solve in rational numbers the pair of equations

$$x^2 + x = u^2,$$
$$x^2 - x = v^2.$$

The solution is obtained by means of any set of three squares in arithmetic progression, that is, by means of Proposition XIII. Let us take x_1^2, x_2^2, and x_3^2 for the three squares, and let the common difference, that is, the congruum, be d. Leonardo says that the solution of the problem is obtained by giving x the value x_2^2/d. For then

$$x^2 + x = \frac{x_2^4}{d^2} + \frac{x_2^2}{d} = \frac{x_2^2(x_2^2 + d)}{d^2} = \frac{x_2^2x_3^2}{d^2};$$

and

$$x^2 - x = \frac{x_2^4}{d^2} - \frac{x_2^2}{d} = \frac{x_2^2(x_2^2 - d)}{d^2} = \frac{x_2^2x_1^2}{d^2}.$$[2]

PROPOSITION XVIII. PROBLEM. To solve in rational numbers the pair of equations

$$x^2 + 2x = u^2,$$
$$x^2 - 2x = v^2.$$

The method is similar to that in Proposition XVII, the value of x being found to be $2x_2^2/d$. Leonardo adds, "You will understand how the result can be obtained in the same way if three or more times the root is to be added or subtracted."

PROPOSITION XIX. PROBLEM. To solve (in integers) the pair of equations

$$x^2 + y^2 = u^2,$$
$$x^2 + y^2 + z^2 = v^2.$$

Take for x and y any two numbers that are prime to each other and such that the

[1] Fermat, *Oeuvres*, Paris, 1891, vol. 1, p. 340; Heath, *Diophantus of Alexandria*, Cambridge, 1910, p. 293.

[2] The simplest numerical example would be $x_1^2 = 1$, $x_2^2 = 25$, $x_3^2 = 49$, and this is the illustration given by Leonardo. It leads to $x = \frac{25}{24}$, from which we have $x^2 + x = \frac{1225}{576} = (\frac{35}{24})^2$ and $x^2 - x = \frac{25}{576} = (\frac{5}{24})^2$.

sum of their squares is a square, let us say u^2. Adding all the odd numbers from unity to $u^2 - 2$[1], the result is $((u^2 - 1)/2)^2$.

Now

$$\left(\frac{u^2-1}{2}\right)^2 + u^2 = \left(\frac{u^2+1}{2}\right)^2.$$

Thus

$$z^2 = \left(\frac{u^2-1}{2}\right)^2,$$

and

$$v^2 = \left(\frac{u^2+1}{2}\right)$$

PROPOSITION XX. PROBLEM. To solve in rational numbers the set of equations

$$\begin{aligned} x + y + z + x^2 &= u^2, \\ x + y + z + x^2 + y^2 &= v^2, \\ x + y + z + x^2 + y^2 + z^2 &= w^2. \end{aligned}$$

By an extension of the method used in Proposition XIX Leonardo obtains the results $x = 3\frac{1}{5}$, $y = 9\frac{3}{5}$, $z = 28\frac{4}{5}$. He even goes farther and obtains the integral solutions $x = 35$, $y = 144$, $z = 360$. He continues, "And not only can three numbers be found in many ways by this method but also four can be found by means of four square numbers, two of which in order, or three, or all four added together make a square number. . . . I found these four numbers, the first of which is 1295, the second $4566\frac{6}{7}$, the third $11417\frac{1}{7}$, and the fourth 79920." In the midst of the explanation of how these values were obtained, the MS. of the *Liber Quadratorum* breaks off abruptly. It is probable, however, that the original work included little more than what the one known Ms. gives. At all events, considering both the originality and power of his methods, and the importance of his results, we are abundantly justified in ranking Leonardo of Pisa as the greatest genius in the field of number theory who appeared between the time of Diophantus and that of Fermat.

[1] Here u^2 is odd, because it is the sum of the squares of two numbers x and y which are prime to each other. It is not possible that both x and y are odd, since $(2m + 1)^2 + (2n + 1)^2 = 4m^2 + 4m + 4n^2 + 4n + 2$, and this is divisible by 2 but not by 4, and hence can not be a square. Thus, of the numbers x and y, one must be even and the other odd, hence $x^2 + y^2$ is odd.

ON THE FORM OF THE POWER SERIES FOR AN ALGEBRAIC FUNCTION.

By E. J. WILCZYNSKI, University of Chicago.

It is a very familiar fact that the expansion of a rational function into a power series is characterized by the property that its coefficients are connected by a so-called scale of relation. In other words: if $w = f(z)$ is a rational function of z, and if w can be expanded into a power series of the form

$$w = f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_k z^k + \cdots,$$

there will exist a positive integer n, such that for all values of $k \geqq n$, a relation of the form

$$a_k = A_1 a_{k-1} + A_2 a_{k-2} + \cdots + A_p a_{k-p}, \qquad (k = n, n+1, n+2, \cdots),$$

will hold between $p+1$ consecutive coefficients of the expansion, the quantities $A_1, A_2, \cdots, A_p$ being constants independent of k. Conversely any expansion of this form defines w as a rational function of z.

Thus, the existence of a scale of relation is a necessary and sufficient condition for a power-series which defines a rational function. The corresponding condition for the power series expansion of a general algebraic function seems to have escaped notice, although it may be obtained by the following simple and elementary process.

Let us consider the algebraic function w, of z, which is defined by the algebraic equation

$$F(z, w) = \varphi_0(z) w^n + \varphi_1(z) w^{n-1} + \cdots + \varphi_{n-1}(z) w + \varphi_n(z) = 0, \tag{1}$$

where the coefficients $\varphi_k(z)$ are polynomials of degree no higher than m, so that we may write

$$\varphi_k(z) = A_0^{(k)} + A_1^{(k)} z + \cdots + A_m^{(k)} z^m, \tag{2}$$

where the quantities $A_i^{(k)}$ are constants.

Let us suppose that one of the roots of (1) may be expanded as a power series in the neighborhood of $z = 0$, and let

$$w = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{i=0}^{\infty} a_i z^i \tag{3}$$

be this power series. From (3) we may obtain the expansions of $w^2, w^3, \cdots, w^n$ by the familiar process of multiplication of power-series. Let

$$w^k = \sum_{i=0}^{\infty} a_i^{(k)} z^i, \qquad (k = 0, 1, 2, \cdots, n), \tag{4}$$

be the expansions obtained in this way.

If (3) actually represents a root of (1), substitution of (4) in (1) must give rise

to an identity; that is, the total coefficient of each power of z must be equal to zero. Let us make this substitution, and equate to zero the coefficient of z^k. For all values of $k \leqq m$, we find

$$
\begin{aligned}
(5)\quad & (A_0{}^{(0)}a_k{}^{(n)} + \cdots + A_m{}^{(0)}a_{k-m}{}^{(n)}) + (A_0{}^{(1)}a_k{}^{(n-1)} + \cdots + A_m{}^{(1)}a_{k-m}{}^{(n-1)}) \\
& + \cdots + (A_0{}^{(i)}a_k{}^{(n-i)} + \cdots + A_m{}^{(i)}a_{k-m}{}^{(n-i)}) \\
& + \cdots + (A_0{}^{(n-1)}a_k + \cdots + A_m{}^{(n-1)}a_{k-m}) \\
& + A_k{}^{(n)} = 0, \qquad (k \leqq m),
\end{aligned}
$$

and for all higher values of k,

$$
\begin{aligned}
(6)\quad & (A_0{}^{(0)}a_k{}^{(n)} + \cdots + A_m{}^{(0)}a_{k-m}{}^{(n)}) + (A_0{}^{(1)}a_k{}^{(n-1)} + \cdots + A_m{}^{(1)}a_{k-m}{}^{(n-1)}) \\
& + \cdots + (A_0{}^{(i)}a_k{}^{(n-i)} + \cdots + A_m{}^{(i)}a_{k-m}{}^{(n-i)}) \\
& + \cdots + (A_0{}^{(n-1)}a_k + \cdots + A_m{}^{(n-1)}a_{k-m}) = 0, \\
& \qquad (k = m + 1, m + 2, m + 3, \cdots).
\end{aligned}
$$

If $n = 1$, equation (1) defines w as a rational function of z, and the relations (6) reduce to the familiar scale of relation. In the general case, relation (6) replaces the scale of relation. As in the scale of relation, the coefficients $A_i{}^{(j)}$ which occur in (6) are independent of k. In fact they are merely the constant coefficients which appear in the defining algebraic equation (1). But the relation (6) is *not* in general, a *linear* relation between the coefficients of the power series. It *is* linear in the $n(m + 1)$ quantities $a_k, \cdots a_{k-m}, a_k{}^{(2)}, \cdots, a_{k-m}{}^{(2)}, \cdots, a_k{}^{(n)}, \cdots, a_{k-m}{}^{(n)}$, but it is of degree n with respect to the quantities $a_1, \cdots, a_k$. We shall therefore speak of a relation of form (6) as *a scale of relation of degree n*. Such a scale is of the same form as the familiar simplest case except that it is concerned not only with the coefficients of the expansion of w itself, but also with the coefficients of the expansion of $w^2, w^3, \cdots, w^n$. We may summarize our result as follows, including an obvious generalization which consists in replacing z by $z - \alpha$.

If w is an algebraic function of z, defined by an equation of degree n in w and of degree m in z, and if w can be expanded in a convergent series proceeding according to positive integral powers of $z - \alpha$, then the coefficients of this expansion satisfy a scale of relation of degree n, which will contain at most $(m + 1)n$ terms.

The converse is immediate. If the coefficients a_k of a power series satisfy a given scale of relation of degree n, the coefficients, $A_i{}^{(j)}$, $(j = 0, 1, \cdots, n - 1;$ $i = 0, 1, \cdots, m)$, of the scale will determine the coefficients $\varphi_0(z), \varphi_1(z), \cdots, \varphi_{n-1}(z)$ of an algebraic equation of form (1) between w and z The constant coefficients of $\varphi_n(z)$ may then be determined by means of (5). These latter coefficients depend not only upon the given scale of relation but also upon the values of the first $m + 1$ coefficients of the given power series. The algebraic equation between w and z, obtained in this way, will have as one of its roots the function with the given expansion.

Thus, *any convergent power-series, whose coefficients satisfy a scale of relation of degree n, defines an algebraic function which is a root of an equation of degree n in w, and whose coefficients are integral rational functions of z.*

Of course we are concerned only with the form of the power series expansion for w whenever such an expansion *exists*. We have not discussed the exceptional cases when such an expansion is impossible. But these exceptional cases are well known. Thus if α is a zero of $\varphi_0(z)$ the expansion of w in powers of $z - \alpha$ may contain negative powers. If α is a root of the equation obtained by eliminating w from $F(z, w) = 0$ and $\partial F/\partial w = 0$, there may be fractional powers of $z - \alpha$ in such an expansion. If $z = \alpha$ is a value of z different from these critical values, an expansion in powers of $z - \alpha$ involving only positive integral powers is known to exist, and the coefficients of the expansion will obey the law which we have indicated. As a matter of fact such expansions frequently exist even in the neighborhood of the critical values. Moreover it is easy to make the modifications necessary to describe the form of the expansion when fractional or negative powers occur in it. Finally it may be stated that the critical values of z might also be rediscovered by investigating directly those cases in which the series obtained from (5) and (6) fail to converge.

The equation (1) may be reducible, that is, it may be possible to express its left member as a product of two integral rational functions of lower degree. In that case we shall still have a scale of relation of form (6) but with smaller values for m or n or both.

The scale of relation for the coefficients of the power-series expansion of an algebraic function will actually be a scale of degree n (and not lower), if the equation (1) which defines this algebraic function is an irreducible equation of degree n in w.

We have found a necessary and sufficient condition in order that the function w defined by the given power-series may be algebraic. We may express this condition in another way.

Let us write the matrix of the coefficients of $A_0^{(0)} \cdots A_m^{(n-1)}$, in equations (6), that is, the matrix of mn columns and infinitely many rows:

$$
\begin{array}{l}
a_{m+1}, \cdots, a_1, a_{m+1}^{(2)}, \cdots, a_1^{(2)}, \cdots, a_{m+1}^{(n)}, \cdots, a_1^{(n)}, \\
a_{m+2}, \cdots, a_2, a_{m+2}^{(2)}, \cdots, a_2^{(2)}, \cdots, a_{m+2}^{(n)}, \cdots, a_2^{(n)}, \\
a_{m+3}, \cdots, a_3, a_{m+3}^{(2)}, \cdots, a_3^{(2)}, \cdots, a_{m+3}^{(n)}, \cdots, a_3^{(n)}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m+k}, \cdots, a_k, a_{m+k}^{(2)}, \cdots, a_k^{(2)}, \cdots, a_{m+k}^{(n)}, \cdots, a_k^{(n)}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{array}
\tag{7}
$$

Clearly, the existence of relations of the form (6) is equivalent to the vanishing of all of the determinants of order mn of this matrix. Therefore we obtain the following result:

If the function defined by the power series (3) is an algebraic function there must exist two finite positive integers, m and n, such that all of the determinants of order mn of the matrix (7) will vanish. Conversely if, for given values of m and n, all of the determinants of order mn of the matrix (7) do vanish, then w will be a root of an algebraic equation of degree m in z and of degree n in w.

It is clear that the systematic investigation of all such determinants will disclose the lowest values of m and n which will serve for a given power series, and that we may then actually determine the irreducible algebraic equation which the given power-series will actually satisfy. If no finite values of m and n can be found which satisfy the conditions of the theorem, the given power-series defines a transcendental function.

The remarkable theorem of EISENSTEIN,[1] which furnishes a necessary condition on the coefficients a_k in order that w may be algebraic, in the special case when the coefficients a_k are all rational numbers, may be regarded as a consequence of the general theorems stated in this paper.

CONCERNING A METHOD FOR FINDING A PARTICULAR INTEGRAL.[2]

By ARTHUR B. COBLE, University of Illinois.

The problem considered here is the determination of a particular integral U of the linear differential equation with constant coefficients,

$$(1) \qquad f(D)y \equiv (k_0D^n + k_1D^{n-1} + \cdots + k_n)y = X\left(D = \frac{d}{dx}\right),$$

in the special case when the right-hand member X is the sum of a number of terms $g_1(x), \cdots, g_j(x)$ such that these terms and all of the terms which arise from them by differentiation can be expressed linearly with constant coefficients by means of a finite number

$$(2) \qquad g_1(x),\ g_2(x),\ \cdots,\ g_l(x)$$

of the terms. Such terms for example are x^ρ, $e^{\alpha x}$, $\sin^\rho \beta x$, $\cos^\rho \gamma x$, $\sinh^\rho \delta x$, $\cosh^\rho \epsilon x$ (ρ any positive integer), and any products of these.

As is customary we call $f(m) = 0$ the *auxiliary equation;* and the complete solution of the differential equation $f(D)y = 0$, the *complementary function* of (1). Each r-fold root, $m = \alpha$, of $f(m) = 0$ contributes a part

$$(3) \qquad G(\alpha, r) = (\lambda_1x^{r-1} + \lambda_2x^{r-2} + \cdots + \lambda_r)e^{\alpha x}$$

to the complementary function.

The rule for the determination of the particular integral U of (1) can be stated as follows:

I. *Substitute for y in* (1) *the trial integral* $U = c_1g_1(x) + c_2g_2(x) + \cdots + c_lg_l(x)$ *and determine the coefficients $c_1, \cdots, c_l$ by equating coefficients of $g_1, \cdots, g_l$. If*

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[1] Stated by him without proof. For a proof consult Heine, "Der Eisenstein'sche Satz über Reihenentwickelungen aller algebraischer Funktionen," *Crelle's Journal*, Vol. 45 (1853), pp. 285–382.

[2] Read before the Md.-Va.-D. C. Section of the Mathematical Association, at Annapolis, December 15, 1917.

however X contains a term $g(x)$ or a term $x^p g(x)$ where $g(x)$ already occurs in the complementary function corresponding to an r-fold root of the auxiliary equation, then in making up the trial integral U, the term $x^{p+r}g(x)$ and its derivatives are to be used.[1]

This method for determining U has several advantages. It requires only differentiations and the solution of a system of linear equations. It furnishes an integral expressed in terms of functions similar to those that occur in X. It applies to nearly all types of equations that actually occur in practice. It has the further pedagogic advantage that in any particular example the method justifies itself by furnishing the required integral. The following proof of the range and validity of the method is perhaps of interest only to the teacher or more mature student.

We shall first find an explicit form for the right-hand member X of (1), and second prove that the set described in I of linear equations in $c_1, \cdots, c_l$ is consistent.

By hypothesis the linearly independent functions $g_1(x), \cdots, g_l(x)$ derived from the terms in X and their derivatives are such that

$$\begin{aligned} D_x g_1(x) &= g_1' = a_{11}g_1 + a_{12}g_2 + \cdots + a_{1l}g_l, \\ D_x g_2(x) &= g_2' = a_{21}g_1 + a_{22}g_2 + \cdots + a_{2l}g_l, \\ &\cdots\cdots\cdots\cdots \\ D_x g_l(x) &= g_l' = a_{l1}g_1 + a_{l2}g_2 + \cdots + a_{ll}g_l.\text{[2]} \end{aligned} \tag{4}$$

If we set

$$g_i = b_1h_1(x) + b_2h_2(x) + \cdots + b_lh_l(x),$$

then

$$g_i' = b_1h_1' + b_2h_2' + \cdots + b_lh_l';$$

whence *the l functions g and the l derivatives g' are cogredient under linear transformation with constant coefficients.* If then the characteristic equation of the linear transformation (4) has roots $\beta_1, \cdots, \beta_k$ of respective multiplicities $s_1, \cdots, s_k$ $(s_1 + \cdots + s_k = l)$ we can find l linearly independent linear functions $h_1(x), \cdots, h_l(x)$ of $g_1(x), \cdots, g_l(x)$ such that the system of equations (4) takes the form[3]

$$\begin{aligned} h_1' &= \beta h_1 + h_2 , \\ h_2' &= \qquad \beta h_2 + h_3 , \\ &\cdots\cdots\cdots\cdots \\ h_{s-1}' &= \qquad\qquad \beta h_{s-1} + h_s , \\ h_s' &= \qquad\qquad\qquad \beta h_s, \end{aligned} \tag{5}$$

where (5) is a sample of that set of equations of the system which corresponds to a root β of multiplicity s of the characteristic equation.

[1] Cohen's *Differential Equations*, p. 112. So far as the writer is aware the method is first set forth there under the name of "the method of undetermined coefficients."

[2] For a similar system of differential equations cf. Riemann-Weber, *Partielle Differentialgleichungen*, p. 138.

[3] Bôcher's *Higher Algebra*, Chap. XXI.

The solutions of (5) satisfy the system

$$
(6)\quad \begin{aligned}
&h_s' - \beta h_s &&= 0,\\
&h_{s-1}'' - 2\beta h_{s-1}' + \beta^2 h_{s-1} &&= 0,\\
&h_{s-2}''' - 3\beta h_{s-2}'' + 3\beta^2 h_{s-2}' - \beta_3 h_{s-2} &&= 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&h_1^{(s)} - \binom{s}{1}\beta h_1^{(s-1)} + \binom{s}{2}\beta^2 h_1^{(s-2)} \cdots + (-1)^s\binom{s}{s}\beta^s h_1 &&= 0.
\end{aligned}
$$

This is evidently true of the first equation; we have only to verify that it is true of the last when the earlier ones are satisfied. By virtue of the numerical relation

$$\binom{s-1}{a-1} + \binom{s-1}{a} = \binom{s}{a}$$

the last equation can be written

$$[h_1^{(s)} - \beta h_1^{(s-1)}] - \binom{s-1}{1}\beta[h_1^{(s-1)} - \beta h_1^{(s-2)}] + \binom{s-1}{2}\beta^2[h_1^{(s-2)} - \beta h_1^{(s-3)}]$$
$$\cdots + (-1)^{s-1}\binom{s-1}{s-1}\beta^{s-1}[h_1' - \beta h_1] = 0.$$

Since $h_1' - \beta h_1 = h_2$ this is satisfied due to the preceding equation in (6). The complete solution of the last equation of (6) is

$$h_1 = e^{\beta x}H(x, s), \qquad H(x, s) = a_1 x^{s-1} + a_2 x^{s-2} + \cdots + a_s$$

and the values of $h_2, \cdots, h_s$ derived in turn from (6) are

$$h_2 = e^{\beta x}D_x H(x, s), \qquad h_3 = e^{\beta x}D_x^2 H(x, s), \cdots, h_s = e^{\beta x}D_x^{s-1}H(x, s).$$

Thus $h_1, \cdots, h_s$, and therefore the original g's, can be expressed linearly in terms of $x^{s-1}e^{\beta x}, x^{s-2}e^{\beta x}, \cdots, e^{\beta x}$. Hence

II. *The right-hand member X of* (1) *can be expressed as*

$$X = G_1(\beta_1, s_1) + G_2(\beta_2, s_2) + \cdots + G_k(\beta_k, s_k)$$

where the β's are the distinct roots with respective multiplicities s of the characteristic equation of the matrix $|a_{ij}|$ of the system of equations (4) *and the functions G are of the form* (3).

We can obtain a particular integral U, for each part G, of X and by adding the U, get the required integral. If the $X = G(\beta, s)$ we shall take the general case and suppose that β, the s-fold root of the characteristic equation, is also an r-fold root of the auxiliary equation $f(m) = 0$. Then

III. *The method* (4) *furnishes one and only one particular integral of the form*

$$e^{\beta x}[c_1 x^{r+s-1} + c_2 x^{r+s-2} + \cdots + c_s x^r].$$

For if in $f(D)y$ we set

$$y = e^{\beta x}(b_1 x^{t-1} + b_2 x^{t-2} + \cdots + b_{t-1}x + b_t)$$

then the coefficients of $e^{\beta x}x^{t-1}$, $e^{\beta x}x^{t-2}$, $\cdots$, $e^{\beta x}$ in the result are respectively

$$
\begin{aligned}
&f(\beta)\cdot b_1,\\
&f(\beta)\cdot b_2+\binom{t-1}{1}f'(\beta)\cdot b_1,\\
(7)\quad &f(\beta)\cdot b_3+\binom{t-2}{1}f'(\beta)\cdot b_2+\binom{t-1}{2}f''(\beta)\cdot b_1,\\
&f(\beta)\cdot b_4+\binom{t-3}{1}f'(\beta)\cdot b_3+\binom{t-2}{2}f''(\beta)\cdot b_2+\binom{t-1}{3}f'''(\beta)\cdot b_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

the law of formation being sufficiently obvious. Under the conditions above, for $t = r + s$, we have to set $b_1, \cdots, b_s = c_1, \cdots, c_s$ when $f(\beta) = f'(\beta) = \cdots = f^{(r-1)}(\beta) = 0$ while $f^{(r)}(\beta) \neq 0$ and equate these coefficients in order to those of $G(\beta, s)$ of which the first r are zero and the last s in order are $\lambda_1, \cdots, \lambda_s$ as in (3). Then the first r equations are identically satisfied and the last s equations suffice to determine $c_1, \cdots, c_s$ in terms of $\lambda_1, \cdots, \lambda_s$. In fact since $f^{(r)}(\beta) \neq 0$ the $(r+1)$-st equation determines c_1 in terms of λ_1; for the same reason the next equation determines c_2 in terms of λ_2 and c_1, etc. Indeed to within a numerical factor the determinant of the last s equations in $c_1, \cdots, c_s$ is $[f^{(r)}(\beta)]^s \neq 0$.

If X is actually expressed in the form given in II the system (7) is immediately available, but as a rule other forms of X are preferable.

RECENT PUBLICATIONS.

WHO WAS THE FIRST INVENTOR OF THE CALCULUS?

The Geometrical Lectures of Isaac Barrow. Translated by J. M. CHILD, Chicago and London, The Open Court Publishing Co., 1916. xiv + 218 pages.

An English translation of so important a work as Isaac Barrow's *Lectiones geometricæ* will be greatly welcomed. Few American mathematicians have had access to a translation into English by E. Stone, published in 1735; according to a statement made by W. Whewell in the preface to his Latin edition of *The Mathematical Works of Isaac Barrow,* Cambridge, 1860, Stone's translation "was so badly executed that it cannot be of use to any one." Few readers will object to Child's omission of certain parts of Barrow's geometrical lectures, parts which seem to be of little or no interest at the present time. Child's historical introduction and critical notes greatly assist in the deeper comprehension of Barrow's genial work. In fact, Child has aimed to do much more than simply to supply a translation. He has made a searching study of Barrow and has arrived at

startling conclusions on the historical question relating to the first invention of the calculus. He places his conclusions in italics in the first sentence of his preface, as follows:

"ISAAC BARROW *was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibniz was also in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's book that he purchased in 1673.*"

These claims are far-reaching. Either they are true and should be accepted or they are false and should be rejected.

Before entering upon an examination of the evidence brought forth by Child, it may be of interest to review a similar claim set up for another man as inventor of the calculus who, like Barrow, was active before the time of Newton and Leibniz. Such a claim has been made in favor of Pierre de Fermat (1601–1665) who died when Newton was just beginning his great career and when Leibniz had not yet brought out his first mathematical publication. Fermat was declared to be the first inventor of the calculus by Lagrange, Laplace, and apparently also by P. Tannery, than whom no more distinguished mathematical triumvirate can easily be found. Lagrange expressed himself as follows:[1]

"One may regard Fermat as the first inventor of the new calculus. In his method *De maximis et minimis* he equates the quantity of which one seeks the maximum or the minimum to the expression of the same quantity in which the unknown is increased by the indeterminate quantity. In this equation he causes the radicals and fractions, if any such there be, to disappear and after having crossed out the terms common to the two numbers, he divides all others by the indeterminate quantity which occurs in them as a factor; then he takes this quantity zero and he has an equation which serves to determine the unknown sought. . . . It is easy to see at first glance that the rule of the differential calculus which consists in equating to zero the differential of the expression of which one seeks a maximum or a minimum, obtained by letting the unknown of that expression vary, gives the same result, because it is the same fundamentally and the terms one neglects as infinitely small in the differential calculus are those which are suppressed as zeroes in the procedure of Fermat. His method of tangents depends on the same principle. In the equation involving the abscissa and ordinate which he calls the specific property of the curve, he augments or diminishes the abscissa by an indeterminate quantity and he regards the new ordinate as belonging both to the curve and to the tangent; this furnishes him with an equation which he treats as that for a case of a maximum or a minimum. . . . Here again one sees the analogy of the method of Fermat with that of the differential calculus; for, the indeterminate quantity by which one augments the abscissa x corresponds to its differential dx, and the quantity ye/t, which is the corresponding augmentation[2] of y, corresponds to the differential dy. It is also remarkable that in the paper which contains the discovery of the differential calculus, printed in the Leipsic Acts of the month of October, 1684, under the title, *Nova methodus pro maximis et minimis* etc., Leibnitz calls dy a line which is to the arbitrary increment dx as the ordinate y is to the subtangent; this brings his analysis and that of Fermat nearer together. One sees therefore that the latter has opened the quarry by an idea that is very original, but somewhat obscure, which consists in introducing in the equation an indeterminate which should be zero by the nature of the question, but which is not made to vanish until after the entire equation has been divided by that same quantity. This idea has become the germ of new calculi which have caused geometry and mechanics to make such progress, but one may say that it has brought also the obscurity of the principles of these calculi. And now that one has a quite clear idea of these principles, one sees

[1] J. Lagrange, "Lecons sur le calcul des fonctions," leçon dix-huitième, *Œuvres de Lagrange*, publiées par J. A. Serret, Tome X, p. 294.

[2] Fermat lets e be the increment of x, and t the subtangent for the point x, y on the curve.

that the indeterminate quantity which Fermat added to the unknown simply serves to form the *derived function* which must be zero in the case of a maximum or minimum, and which serves in general to determine the position of tangents of curves. But the geometers contemporary with Fermat did not seize the spirit of this new kind of calculus; they did not regard it but a special artifice, applicable simply to certain cases and subject to many difficulties, . . . moreover, this invention which appeared a little before the *Géométrie* of Descartes remained sterile during nearly forty years. . . . Finally Barrow contrived to substitute for the quantities which were supposed to be zero according to Fermat quantities that were real but infinitely small, and he published in 1674 his *method of tangents*, which is nothing but a construction of the method of Fermat by means of the infinitely small triangle, formed by the increments of the abscissa e, the ordinate ey/t, and of the infinitely small arc of the curve regarded as a polygon. This contributed to the creation of the system of infinitesimals and of the differential calculus."

Such is Lagrange's interpretation of the work of Fermat and of the place it should occupy in the history of the calculus. Even more positive is the dictum of Pierre Simon Laplace who, in his *Essai philosophique sur le calcul des probabilities*, speaks of Fermat in the following terms:[1]

"This great geometrician [Fermat] expresses by the character E the increment of the abscissa; and considering only the first power of this increment, he determines exactly as we do by differential calculus the subtangents of the curves, their points of inflection, the *maxima* and *minima* of their ordinates, and in general those of rational functions. We see likewise by his beautiful solution of the problem of the refraction of light inserted in the *Collection of the Letters of Descartes* that he knows how to extend his methods to irrational functions in freeing them from irrationalities by the elevation of the roots to powers. Fermat should be regarded, then, as the true discoverer of Differential Calculus. Newton has since rendered this calculus more analytical in his *Method of Fluxions*, and simplified and generalized the processes by his beautiful theorem of the binomial. Finally, about the same time Leibnitz has enriched differential calculus by a notation which, by indicating the passage from the finite to the infinitely small, adds to the advantage of expressing the general results of calculus, that of giving the first approximate values of the differences and of the sums of the quantities; this notation is adapted of itself to the calculus of partial differentials."

P. Tannery, the noted historian of mathematics, expressed himself more recently as follows:[2]

"Fermat is also honored with the invention of the differential calculus on account of his method of maxima and minima and of tangents, which, of the prior processes, is in reality the nearest to the algorithm of Leibniz; one could with equal justice, attribute to him the invention of the integral calculus; his treatise *De æquationum localium transmutatione*, etc., gives indeed the method of integration by parts as well as rules of integration, except the general powers of variables, their sines and powers thereof. However, it must be remarked that one does not find in his writings a single word on the main point, the relation between the two branches of the infinitesimal calculus."

We proceed to quote two opinions, one English, the other French, relating to the attitude taken by Lagrange and Laplace. In the *Edinburgh Review* for September, 1814, p. 324, we read (in an anonymous review now known to have been from the pen of John Playfair):

"To a passage of the latter [Laplace], however, we cannot but advert, and with much less satisfaction than we have generally felt in pointing out any of the remarks of this celebrated writer to the attention of our readers. '*Il paraît que Fermat, le veritable inventeur du calcul differentiel,*

[1] *A Philosophical Essay on Probabilities*, by Pierre Simon, Marquis de Laplace. Transl. by F. W. Truscott and F. L. Emory, New York, 1902, Part I, Chapter V, p. 46. The French original appeared in 1812.

[2] P. Tannery, Article "Fermat" in *La Grande Encyclopédie* (Berthelot).

a consideré ce calcui comme une dérivation de celui des differences finies,' etc. Against the affirmation that FERMAT is the real inventor of the Differential Calculus, we must enter a strong and solemn protestation. The age in which that discovery was made, has been unanimous in ascribing the honour of it either to NEWTON or LEIBNITZ; or, as seems to us much the fairest and most probable opinion, to both; that is, to each independently of the other, the priority in respect of time being somewhat on the side of the English mathematician. The writers of the history of the mathematical sciences have given their suffrages to the same effect;—MONTUCLA, for instance, who has treated the subject with great impartiality, and BOSSUT, with no prejudices certainly in favour of the English philosopher. In the great controversy, to which this invention gave rise, all the claims were likely to be well considered; and the ultimate and fair decision, in which all sides seem to have acquiesced, is that which has just been mentioned. It ought to be on good grounds, that a decision, passed by such competent judges, and that has now been in force for a hundred years, should all at once be reversed.—Fermat . . . had certainly approached very near to the differential or fluxionary calculus, as his friend ROBERVAL had also done. He considered the infinitely small quantities introduced in his method of drawing tangents, and of resolving *maxima* and *minima*, as derived from finite differences; and, as Laplace remarks, he has extended his method to a case, when the variable quantity is irrational. He was, therefore, very near to the method of fluxions; with the principle of it, he was perfectly acquainted;—and so at the same time were both ROBERVAL and WALLIS, though men much inferior to FERMAT. The truth is, that the discovery of the new calculus was so gradually approximated, that more than one had come quite near it, and were perfectly acquainted with its principles, before any of the writings of NEWTON or LEIBNITZ were known. That which must give, in such a case, the right of being considered as the true inventor, is the extension of the principle to its full range; connecting with it a new calculus, and new analytical operations; the invention of a new algorithm with corresponding symbols. These last form the public acts, by which the invention becomes known to the world at large the judge by which the matter must be finally decided. Great, therefore, as is the merit of FERMAT which no body can be more willing than ourselves to acknowledge; and near as he was to the greatest invention of modern times, we cannot admit that his property in it is to be put on a footing with that of NEWTON or of LEIBNITZ;—we should fear, that in doing so, we were violating one of the most sacred and august monuments that posterity ever raised in honour of the dead."

Poisson says:[1]

"As a magnitude approaches its maximum or its minimum, it varies less and less and its differential vanishes as it reaches one or the other of these extreme values. Starting from this principle, Fermat had the happy idea, for the determination of the maximum or minimum of a quantity, to assign to the variable upon which it depends, an infinitely small increment and to equate to zero the corresponding increment of that quantity previously reduced to the same order of magnitude as that of the variable. It is in this manner that he determined the path of light in passing from one medium into another upon the supposition conforming with the theory he had adopted, that the time of passage be a minimum. Lagrange considered him, for that reason, as the first inventor of the differential calculus, but this calculus consists of a set of rules suitable for finding immediately the differences of all functions, rather than of the use one makes of the infinitely small variations in the resolution of this or that species of problems; and from that point of view, the creation of the differential calculus does not go back beyond Leibnitz, the inventor of the algorithm and of the notation which have generally prevailed since the origin of the calculus and to which infinitesimal analysis is chiefly indebted for its progress. It should be observed, moreover, that the binomial formula which supplied Newton and Leibnitz the means of expressing very simply the differential of any power whatever, integral or fractional, positive or negative—that this formula was unknown to Fermat, that he could not differentiate the radicals which presented themselves in his problem, and that he replaced this operation by geometrical constructions and special devices, the avoidance of which is the special object of the differential calculus."

We have now quoted the views of Lagrange, Laplace, Poisson, Paul Tannery and an Edinburgh reviewer, on the invention of the calculus. We have quoted

[1] Poisson "Mémoire sur le calcul des Variations," *Mémoires de l'académie royale des sciences de l'institut de France*, Tome XII, Paris, 1833, p. 223.

the conclusion reached by Child in favor of Barrow. The practical question arises, which of these conflicting opinions is correct. It is easy to see that the answer to be given to this question hinges upon the conception we have as to what constitutes an invention of the calculus. Is it the creation of a method like that of Fermat? Is it the invention of a set of rules and a notation as demanded by Poisson?

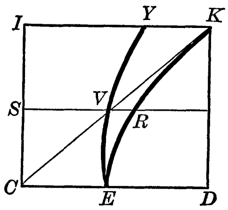
Mr. Child endeavors to answer this question in his preface: "By the 'Infinitesimal Calculus,' I intend 'a complete set of standard forms for both the differential and integral sections of the subject, together with rules for their combination, such as for a product, a quotient, or a power of a function; and also a recognition and demonstration of the fact that differentiation and integration are inverse operations.'"

Every one will admit that a set of rules for differentiation and integration is an essential part of the calculus. But does this include all the essentials to be met by a successful candidate for the honor of invention of the calculus? Could the calculus have fulfilled its mission had it not possessed a suitable notation both for differentiation and integration? We do not mean that the very symbols introduced by Leibniz or by Newton are essential parts of a differential and integral calculus. But is not some sort of a suitable notation to designate the first differential or derivative as well as higher differentials or derivatives and to designate integration a necessary part of the calculus—a *sine qua non,* without which the calculus could not render its service in the resolution of complicated problems? We hold that by common and tacit agreement of all text book writers since the time of Newton and Leibniz such a notation is looked upon as an essential of the calculus. We cannot recall a single author of a text on this subject since the time of Newton and Leibniz who has not recognized the need of a suitable notation and who does not use one. On the strength of this unanimity of tacit testimony we must insist upon a suitable notation as a necessary requirement to be met by any one for whom the invention of the calculus is claimed. In his geometrical lectures Barrow uses few symbols. He does not even use the ordinary symbols in trigonometry. He uses the letters a and e for the designation of increments of variables, but this notation is altogether inadequate.

Passing to another topic, we observe that most texts have invoked the aid of geometry in the development of the calculus. Geometrical figures help in the grasp of abstract relations. Nevertheless, the calculus has been largely analytical. Algebraic, logarithmic and trigonometric symbols have been used habitually. We could not have rules for differentiation and integration, as ordinarily understood, unless the mathematical relations were expressed in analytical form. Now Barrow does not develop an *analytical* calculus. He establishes geometrical theorems which Child translates into analysis. In that way Child obtains a group of formulas for differentiation and integration, such as have been gathered also from other pre-Newtonian writers.[1] These formulas he sets down to Bar-

[1] See H. G. Zeuthen, *Geschichte der Mathematik im XVI. und XVII. Jahrhundert,* deutsche Ausg. von R. Meyer, Leipzig, 1903, "Integrationen vor der Integralrechnung," pp. 248–300.

row's credit. For example Barrow proves the theorem (Lecture XII, Appendix I, 9):



"Let ERK be an equilateral hyperbola (that is, one having equal axes), and let the axes be CED, CI; also let KI, KD be ordinates to these; let EVY be a curve such that, when any point R is taken at random on the hyperbola, and a straight line RVS is drawn parallel to DC, then SR CE, SV are in continued proportion; join CK; then the space $CEYI$ will be double the hyperbolic sector KCE."

The adjoining figure is our own; Child gives none. The determination of the area $CEYI$ by calculus involves the formula

$$\int_0^y \frac{dy}{\sqrt{a^2+y^2}} = \log \frac{y+\sqrt{a^2+y^2}}{a}.$$

Hence Child claims that Barrow possessed this formula in geometrical garb. Accordingly, the geometrical process of integrating

$$\int_0^y \frac{dy}{\sqrt{a^2+y^2}}$$

would be to construct the hyperbola $x^2 - y^2 = a^2$ and from it the curve EVY, yielding $CEYI$ as the area representing the definite integral. By the same argument it may be claimed that when Dinostratus of old used the quadratrix in the quadrature of the circle, he worked out the part of the integral calculus contained in the formula $\int_0^a \sqrt{a^2-x^2}dx = \frac{1}{4}\pi a^2$. Dinostratus and Barrow were clever men, but it seems to us that they did not create what by common agreement of mathematicians has been designated by the term differential and integral calculus. Two processes yielding equivalent results are not necessarily the same. It appears to us that what can be said of Barrow is that he worked out a set of geometric theorems suggesting to us constructions by which we can find lines, areas and volumes whose magnitudes are ordinarily found by the analytical processes of the calculus. But to say that Barrow invented a differential and integral calculus is to do violence to the habit of mathematical thought and expression of over two centuries. The invention rightly belongs to Newton and Leibniz.

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QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

The discussion given below is closely related to Professor Moritz's article "On the Construction of Certain Curves Given in Polar Coördinates" published

in the May, 1917, issue of this MONTHLY, pp. 213–220. Professor Moritz discussed "cyclic-harmonic" curves resulting from the motion of a point which has simple harmonic motion along a line, while at the same time this line rotates with a constant angular velocity about one of its fixed points. Professor Rigge gives below a very complete discussion of a method for tracing a cardioid as the resultant of a simple harmonic motion along a line and a uniform angular motion about a point not in that line.

CONCERNING A NEW METHOD OF TRACING CARDIOIDS.

By WILLIAM F. RIGGE, S.J., Creighton University, Omaha, Nebr.

A cardioid may under certain conditions be traced by a point when its motion is the resultant of a rectilinear simple harmonic movement and a uniform angular one of the same period. The harmonic motion may be obtained from any plane mechanical contrivance such as a revolving crank and a sliding slotted bar, while the angular one is best furnished by a disk rotating under the pen.

Fig. 1 will illustrate the definition just given as well as the conditions to be mentioned presently. The point A is the center of the disk which rotates in a clockwise direction with uniform angular speed. If the disk did not rotate, the tracing pen would move over the line EG parallel to the Y axis with simple harmonic motion, so that its distance from R, the middle point, would at any moment be the sine of the phase, the amplitude RE or RG being taken as the unit of our scale in this investigation. But when the disk does rotate, the combination of the rectilinear motion of the pen with the rotary one of the disk causes the pen B to trace the cardioid $BCKQZFB$ (R is only by accident on the curve), provided the following conditions are observed.

Conditions to be Observed.—First, the pen may be set down on the disk as at B at any initial phase α of its rectilinear harmonic motion. In Fig. 1 this initial phase α is taken as 52°, so that $RB = \sin 52°$, RE, as said, being unity.

Second, the point B, at which the pen is set down on the disk, must be on the unit circle whose center O is on the Y axis and whose distance from A, the center of the disk, is the sine of the phase $OA = \sin\alpha = \sin 52° = BR$. The angle AOB we will call β, the *starting* angle, and the circle just mentioned the *starting* circle. In Fig. 1 β is equal to 77°.

Study of the Conditions.—It is to be noted first that α and β are independent variables and may have any values whatever.

Secondly, the initial phase α fixes the center of the starting circle O on the axis of Y, so that O is in the same direction from A that B is from R, and $OA = BR$. It also fixes the center D of what we will call the *cusp*-circle, because it is the locus of the cusp of the cardioid that can be generated with the given initial phase α. The radius of this small circle is one half that of the starting circle, or one half of our chosen unit, its circumference passes through A and O, and it is internally tangent to the large circle at S on the axis of X, so that the angle $ASO = \alpha$. This puts S to the left of A when $\cos\alpha$ is positive, and to the right

when negative. As α grows from $0°$ to $360°$ the center O of the starting circle executes a simple harmonic motion along the axis of Y, about its central position at A, while the center D of the cusp circle moves with uniform angular speed in a circle of its own size about the center of the disk A.

Thirdly, the assumed point B at which the pen is set down on the starting circle, or the assumed value of AOB or β, determines the position on its circle of

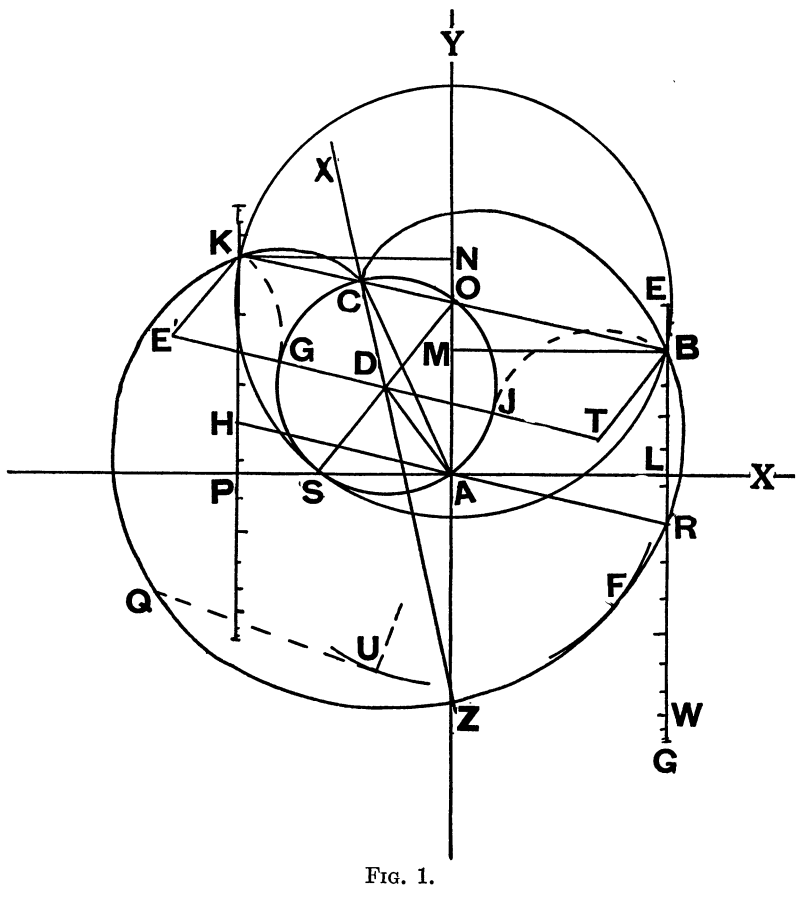

Fig. 1.

the cusp C of the cardioid that will be generated, since this cusp C must lie on the diameter through O and B. The same cardioid is drawn when β is increased or diminished by $180°$, and the pen set down at K instead of at B. Measured from their common point of tangency S on their respective circles in a positive or anti-clockwise direction the distances of the cusp C and the starting point B or K are equal, that is, arc $SAJOC$ = arc SB, and the first increased by a whole circumference is equal to the arc SBK, so that as β grows from $0°$ to $360°$ the cusp C swings round its circle twice with uniform speed. The axis CZ of the cardioid will of course pass through D.

From what has just been said about α and β, it follows that two or more pens may be set down at various points on the starting circle, fastened together, and made to move with the same simple harmonic mechanism: they will all trace equal cardioids which have their cusps on the same small cusp circle; and a pair of diametrically opposite pens will trace the same cardioid.

Other Initial Phases and Starting Angles.—Various figures are here given to illustrate the changes that come upon the position of the cardioid as α and β are given different values, the cardioid however being always of the same size as long as the amplitude of the simple harmonic motion remains the same. We will at this juncture call attention only to Figs. 1, 2, 3. Fig. 1, as said before,

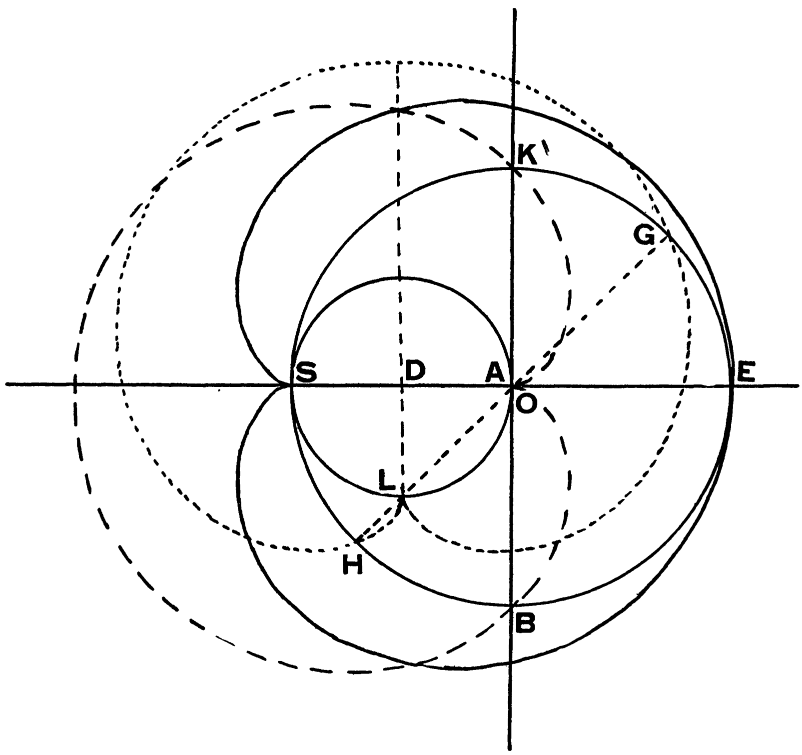

FIG. 2.

gives the values of $\alpha = 52°$ and $\beta = 77°$ or $257°$. In Fig. 2, we have $\alpha = 0°$, when the center O of the starting circle is at A, the center of the disk; the center D of the cusp circle is on the axis of X to the left of A, and the point of tangency of both circles is at S. When $\beta = 0°$ or $180°$, the pen is set down at B or K, and the cusp of the cardioid is at A. When $\beta = 90°$ or $270°$, the pen is started at E or S, and the cusp is at S. The cusp is always at the starting point when this is at the point of tangency or 180° away from it. When $\beta = 135°$ or $315°$, the pen is set down at G or H, and the cusp is at L. It will be seen that the cusp is always on the diameter through O and the starting point.

When $0^\circ < \alpha < 90^\circ$ we have a drawing like Fig. 1.

When $\alpha = 90^\circ$ as in Fig. 3, the center O of the starting circle is at the distance unity from A, and the center D of the cusp circle is halfway between them, their point of tangency being at A. When $\beta = 0^\circ$ or 180°, the pen is set down at A or K. When $\beta = 35^\circ$ or 215°, the point is B or F and the cusp is at C.

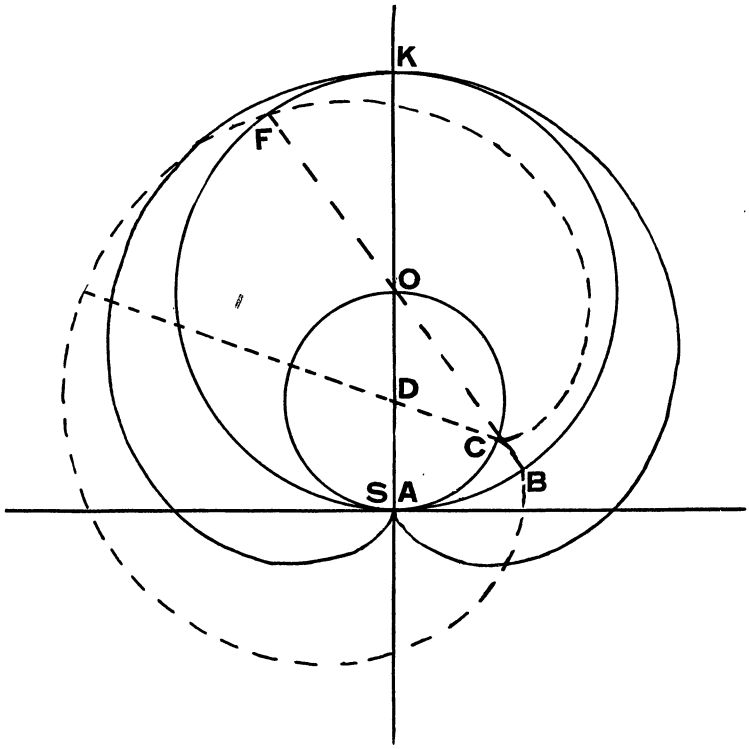

FIG. 3.

The easiest way to draw a cardioid is to take $\alpha = 90^\circ$ and $\beta = 0^\circ$, that is, to place the pen exactly at the center of the disk A when its phase is 90°. This is the way in which the writer had drawn a number of cardioids, when it occurred to him to displace the pen from A to some other point such as B, to see what would happen. Fortunately B was on the starting circle, and to his great surprise a perfect cardioid resulted. But the position of B had not been marked, so that in endeavoring to rediscover it later, hours of experimentation were necessary before its position was determined to be on the starting circle. This was changing β. Then the idea came to change the initial phase α. After much more experiment and analysis, the final law was found which is presented in this article.

When $90^\circ < \alpha < 180^\circ$, we have a diagram like Fig. 1 when this is turned half-way round the Y axis and looked at from the other side of the paper, the points D and S then being to the right of OA.

When $\alpha = 180°$, we have Fig. 2 when the paper is turned half-way round the point A with the same side of the page facing us. For the values of α from 180° to 360°, the points O and D are below the X axis, and the figures are obvious enough after a little study.

Object of this Article.—The chief object of this article is to prove that under the given conditions the curve so drawn is a cardioid.

The method to be pursued in the proof will be to suppose the cardioid generated as an epicycloid, and then to show that the pen traces the same curve. Taking the cusp circle, then, as the fixed one, the cusp C must evidently lie upon it and the axis of the cardioid must pass through its center. Placing the equal generating circle with its center at T in Fig. 1, when its tracing point is at B, we have the arcs COJ and BJ equal, and the lines CB and DT parallel. As the quadrilateral $DOBT$ has its opposite sides equal, they are also parallel, and hence OB is likewise parallel to DT, so that the lines CB and OB are coincident, and therefore the cusp C lies on the diameter KOB.

The Phase of the Pen at the Cusp.—For the sake of greater clearness we will suppose the disk to remain at rest and the pen to have its rotary motion in the opposite sense, that is, in an anti-clockwise direction. We will also assume for the present that the pen really traces the epicycloidal cardioid drawn by revolving the generating circle T with its pen at B about the fixed circle D. This assumption will be given a rigorous proof later, but it will serve well now to establish some of the conditions, such as the position of the cusp C and the phase of the pen when at the cusp.

It is evident that in passing through the cusp C the pen must be momentarily at rest on the paper and its rectilinear and rotary velocities must be equal and opposite. The diagram offers the suggestion that the pen is then at one of its two least distances from the center of revolution A, and that AC must be equal to $AL = MB = \sin\beta$. As there are two points on the cusp circle at this distance from A, we must take the one that is on the diameter KOB, as said before.

We must next find the two phases of the pen in its rectilinear motion when it is at L, at its minimum distance from A. As R is its mean position at the phase 0°, LR is equal to the sine of the phase when at L. Now as AO is equal and parallel to BR, OB and AR must also be equal to one another and parallel so that the triangles BOM and ARL are equal, and then $LR = OM = \cos\beta = \sin(90 \pm \beta)$. The phase of the pen when at the cusp is then either $90° - \beta = 90° - 77° = 13°$, or $90° + \beta = 90° + 77° = 167°$. In this instance we must take the latter value, because the direction of the linear motion of the pen must be downward at L in order to be to the right at C and annul its rotary motion which is counter-clockwise or to the left at C.

The rectilinear speed of the pen at L, or at C, is then $d[\sin(90° + \beta)] = \cos(90° + \beta)d\beta = -\sin\beta d\beta$. The rotary velocity at C is $CAd\theta = +\sin\beta d\theta = +\sin\beta d\beta$, because $d\theta$, the angular velocity of the pen, $d\beta$ or $d\alpha$, the variation of the rectilinear phase of the pen, are all equal and constant. Hence as the rectilinear velocity of the pen is $-\sin\beta d\beta$ and its rotary speed is $+\sin\beta d\beta$, it is at

rest at the cusp C and is then in the phase $90° + \beta$. The pen is in phase $90° - \beta = 13°$, in the present instance, when at F, where the rectilinear and rotary motions are also equal but in the same direction. There can be no other points besides C and F at which the two motions of the pen are exactly equal and either in directly opposite or in the same direction, because as the rotary motion must be at right angles to the radius through A, the two L points of the rectilinear motion are the only ones in which this is true.

As CA in the cusp circle is equal to $\sin\beta$, it follows that the angle $CDA = 2\beta$, and $DCA = DAC = 90° - \beta$. The angle $DSA = \alpha$, since $OA = \sin\alpha$ and SO = unity, and hence also $SA = \cos\alpha$, and $DAS = OCA = \alpha$. The position of the axis of the cardioid is therefore readily found. It is only by accident however that the cardioid passes through R, and that Z seems to lie on the Y axis.

The Second Starting Point K.—On account of the equality of the angles SDJ, SOB, DTB, CDJ, the (reflex) angle CDS is double the angle SOB, so that if the angular positions of the pen at B and of the cusp C are reckoned from the common point of tangency S of their circles, the angle of the cusp is twice that of the initial position of the pen and the arcs $SAJOC$ and SB are equal. If we add equals to these equals, the whole circumference of the D circle to the first and a semicircumference of the O circle to the second, it follows that the pen may be set down at K as well as at B in order to trace the same cardioid. The explanation given before for the point B then applies equally to the point K, if we replace LR by HP, BR by HK, and remember that as β is then increased by 180° or $KOA = 180° - \beta$, NO and HP are equal to $-\cos\beta = -\sin(90° \pm \beta) = \sin(270° \pm \beta) = 347°$ or 193° in the present case. The first of these $270° + \beta = 347°$ must evidently be here the phase of the K pen when at the cusp C, because the rectilinear motion at P must be upward, or to the right at C, to counteract the left rotary motion. As $90° + \beta$ and $270° + \beta$ differ 180°, the B and K pens are half a phase apart. And as their linear distance from one another is $KOB = 2 = (1 - \cos\theta) + (1 + \cos\theta)$, the line joining them passes through the cusp.

Preliminaries to the Proof that the Pen Traces the Cardioid.—We are now in a position to verify the assumption that the pen B traces the epicycloidal cardioid. First let us examine the rectilinear component of its motion. This would at any phase interval θ after passing through L, bring it, say, to W, so that its distance from L would be $LW = RW + RL = \sin[(90 + \beta) + \theta] - \cos\beta$. Here $90° + \beta$ is the phase of the pen at L, as we saw before, so that $(90° + \beta) + \theta$ is the phase after the interval θ. Taking $\theta = 135°$ as an example, this would make the phase $167° + 135° = 302°$, and put the pen at W, RW being equal to $\sin(90 + \beta + \theta)$. LW is in principle equal to the difference of RW and LR, that is, $RW - LR$ as would be evident if W were between L and R and both RW and LR positive. Here however, as RW is minus and LR plus, we have their numerical sum, which is minus. As $\sin((90° + \beta) + \theta) = \sin(90° + (\beta + \theta)) = \cos(\beta + \theta)$, the length of $LW = \cos(\beta + \theta) - \cos\beta$.

Secondly, the rotary component alone of the pen after its arrival at the cusp

in the phase $90° + \beta$, during the phase interval θ after that, swings the pen from C to U through the angle $\theta = CAU$, which, as said, is here taken as 135°, the center of rotation being A, and the radius $UA = CA = AL = \sin\beta$.

Thirdly, in compounding the two motions of the pen, if we allow the rotary motion first to carry the pen from the cusp C to the point U, and then the rectilinear component to move it from U to Q over the tangent $UQ = LW = \cos(\beta + \theta) - \cos\beta$, we must, if our contention is correct, find the point Q on our cardioid with a position angle $QCX' = \theta$. The actual proof however will be the reversal of this procedure, taking a point Q on the cardioid at the position angle $X'CQ = \theta$, and showing that the length of the tangent QU dropped from it to the A circle is equal to $\cos(\beta + \theta) - \cos\beta$.

The Proof.—Taking C as the pole or origin and CX' as the positive direction of the X axis, the equation of the cardioid is $\rho = 1 - \cos\theta$, $SO = OB$ [$= \frac{1}{2} CZ$] being unity.

The coördinates of Q are then

$h = \rho\cos\theta = \cos\theta - \cos^2\theta$;

$k = \rho\sin\theta = \sin\theta - \sin\theta\cos\theta$.

The radius of the A circle is $AU = AC = r = \sin\beta$.

The coördinates of its center are

$a = -AC\cos DCA = -\sin\beta\cos(90° - \beta) = -\sin^2\beta$;

$b = -AC\sin DCA = -\sin\beta\sin(90° - \beta) = -\sin\beta\cos\beta$.

The square of the tangent QU dropped from a point (h, k) to the circle $(x - a)^2 + (y - b)^2 - r^2 = 0$ being $(h - a)^2 + (k - b)^2 - r^2$, we here have the square of QU equal to $(\cos\theta - \cos^2\theta + \sin^2\beta)^2 + (\sin\theta - \sin\theta\cos\theta + \sin\beta\cos\beta)^2 - \sin^2\beta$.

Upon reducing, we find this value equal to the square of $\cos(\beta + \theta) - \cos\beta$. Therefore the pen B traces the cardioid $BCKQZFB$.

We may notice that the initial phase α does not appear in this analysis, because that merely fixes the positions of the centers of the starting and cusp circles O and D on the disk. The point B, however, with its starting angle β, being a point on the cardioid, determined the actual position of the cusp and the direction of the axis CZ. And lastly although definite numerical values of α, β, and θ, have been used in the figures for purposes of illustration, only general algebraic values have entered our equations.

Analysis and Synthesis of the Motions of the Pen.—The analysis just presented shows us how the cardioid is drawn mechanically. While the rotary motion swings the pen around the *tangent* circle (as we may call it) with uniform speed (see also Fig. 4) the rectilinear harmonic motion keeps it on the tangents to the successive points at the distance $\cos(\beta + \theta) - \cos\beta$. If we wished to plot these tangents as in Fig. 4, we would have to begin with a knowledge of the zero direction of the rotary motion. Its center is, of course, at A, and we know that the cusp C must be in the phase $90° + \beta$ (= 167°). Centering a protractor at A so that C reads $90° + \beta$, as in Fig. 4 in which the radii are drawn in dotted lines at intervals of 30°, we shall not only easily find the zero, but also see that L will indicate the initial phase α (= 52°), F will show $90° - \beta$ (= 13°), and D 180°.

As we wish to plot our tangents from the true 0° of the rotary motion, instead of from the cusp C from which we before reckoned the phase interval θ in our proof, we must subtract $90° + \beta$ from the θ there used. Changing back our $\cos(\beta + \theta)$ into $\sin(90° + \beta + \theta)$ from which it originated, and then subtracting $90° + \beta$ from θ, we have for the length of a tangent $\sin\theta - \cos\beta$, which, as $\cos\beta$ is constant, gives truly a rectilinear simple harmonic motion.

After the cardioid has actually been drawn, there is a very simple and expeditious way of finding at any time the phase of the pen at any point or conversely

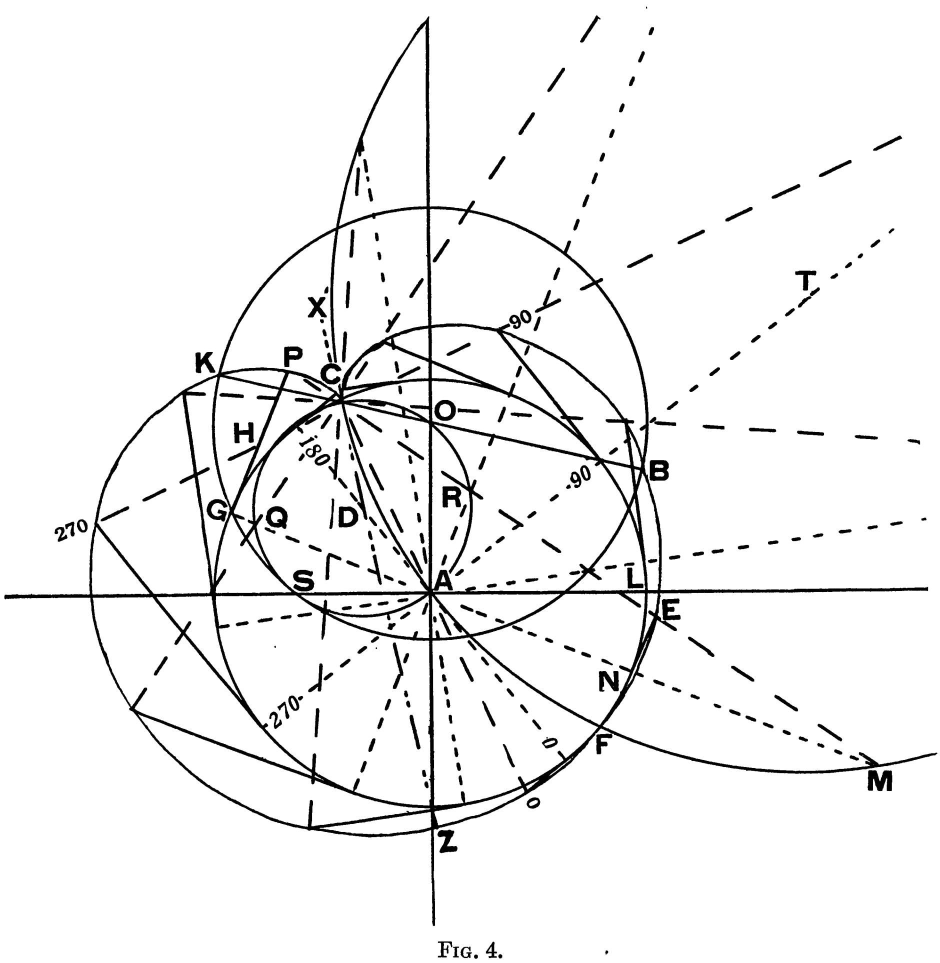

FIG. 4.

its position at a given phase. We need but center a protractor on the cusp C and take CA as the zero line. This has been done in Fig. 4, in which the radii vectores are drawn in dashed lines. The phases of the pen are then shown directly by the readings on the protractor. B will thus be found to show the initial phase α (= 52°), K will be $180° + \alpha$, X' $90° + \beta$ and Z $270° + \beta$. The points of the

cardioid at the phases 90° and 270° will be seen to have maximum, and those at $90 \pm \beta$ minimum distances from A.

While the mechanical method used in drawing the cardioid is really the tangent method just mentioned, the components of the movements, that is, the uniform angular speed of the pen about the center of the disk A, as well as its rectilinear simple harmonic motion, are bound to disappear individually from view on account of their combination. But if we refer now the motions of the pen to the cusp C, we find that the pen really moves about this cusp with uniform angular speed and also really moves in its direction with true simple harmonic motion, so that while the compound motion of the pen about the center of the disk is such as to make its components indistinguishable, these components, although united, are elegantly re-analyzed and exposed to view when we refer the motion to the cusp.

The Intersection Circle.—A study of Fig. 4 will reveal the peculiar property that the radii vectores passing through the cusp C and the radii of the tangent circle centered at A, that are in the same phase, intersect each other on what we may call the *intersection* circle which has its center at T. For example, the radius

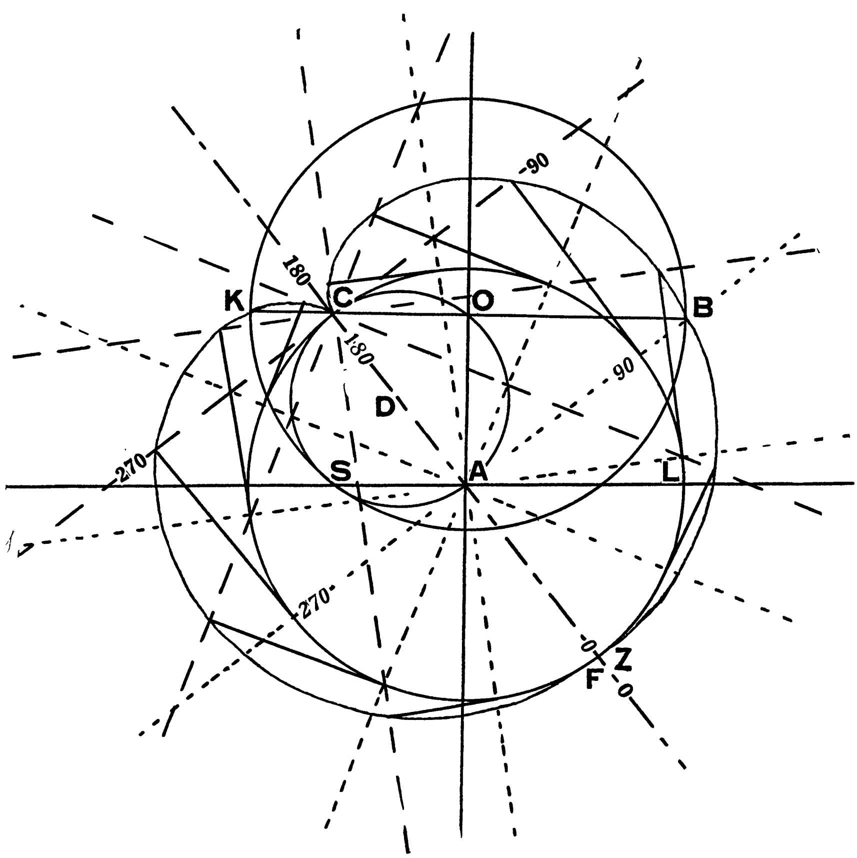

Fig. 5.

vector $PCREM$ of 30° (or 210°) and the radius $GQANM$ of the same phase, intersect in M which is on the intersection circle T. That this is really the case is due to the fact that the angle at M is constant and equal to $90° - \beta$. The 0° lines of the radii centered at C and A make this angle $CAD = 90° - \beta$, with one another, as we saw before in Fig. 1, so that all other radii in the same phase must also be inclined to one another at this angle, and hence the angle $PMG = 90° - \beta$. Therefore the intersection points of all the radii in equal phases lie on a circle, because they form triangles like MCA which has a constant base CA and a constant vertical angle M.

The length of the tangent NE (at 30°) is $+\sin\theta - \cos\beta$ and of GP (180° away) is $-\sin\theta - \cos\beta$ or $\sin\theta + \cos\beta$, if we règard only its absolute value. Their difference $PH = 2\cos\beta$, and their distance apart $GN = 2\sin\beta$. Let us imagine the line HE drawn parallel to GN which has been omitted in order not to crowd the figure too much. Then we have the proportion

$$\frac{PH}{HE} = \frac{PG}{GA + AM} \quad \text{or} \quad \frac{2\cos\beta}{2\sin\beta} = \frac{\sin\theta + \cos\beta}{\sin\beta + AM},$$

from which we find the chord $AM = \sin\theta\tan\beta$. This chord becomes a maximum when $\theta = 90°$, so that the diameter of the intersection circle is $\tan\beta$ and its radius $AT = \frac{1}{2}\tan\beta$. Its center T lies on the $90° - 270°$ radius through A, produced if necessary, and its circumference passes through C, A and F.

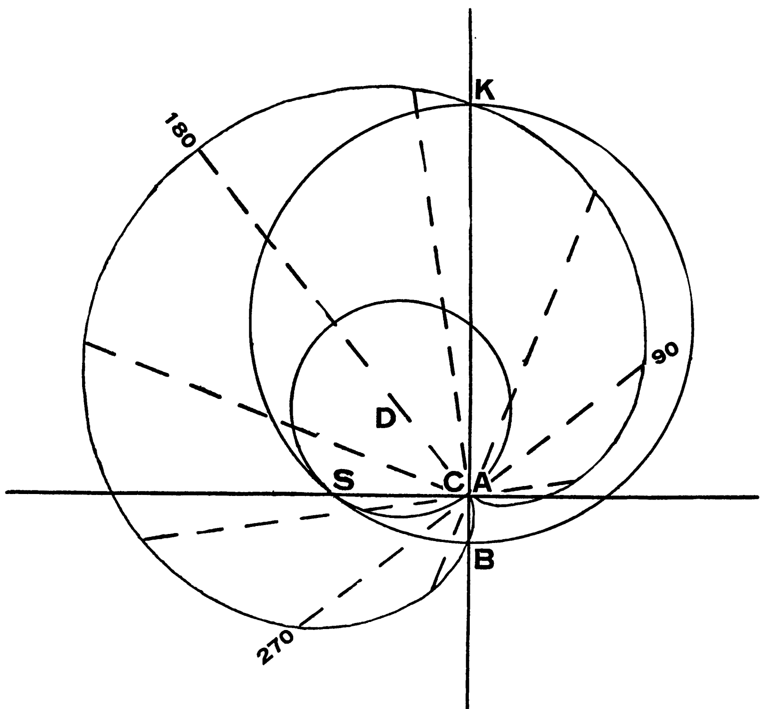

FIG. 6.

The points R and Q, at which a radius vector through C and a radius through A differing 90° in phase intersect, lie on the cusp circle, because the quadrilateral $CRAQ$ has two opposite angles at C and A equal to 90°, so that the other two are supplementary and are therefore inscribed in the circle passing through A and C.

In Fig. 5, α has its old value of 52°, but AOB or β has been made equal to 90°. The starting and cusp circles are in the same positions as in Fig. 1, because these depend upon α alone. The cusp C however, has shifted somewhat, so that it is now on the line DA since the angle $CAD = 90° - \beta = 0°$ in the present case. F is also on the line CDA, which is here the common 0° line of the radii passing through C and A. The radius AC or AL of the tangent circle is unity, and the lengths of the tangents $\sin\theta - \cos\beta$ drawn to it are now simply $\sin\theta$. The radius of the intersection circle $\frac{1}{2}\tan\beta$ is infinite, so that the radii through C and A at equal phases are parallel.

In Fig. 6, α is as usual 52°, but $\beta = 0°$. The starting and cusp circles and the 0° line DA are the same as before, but the cusp C is now at A. The radii of the tangent and intersection circles are zero. The lengths of the tangents $\sin\theta - \cos\beta$ are $\sin\theta - 1$ or $1 - \sin\theta$, and are ordinary radii vectores of the cardioid.

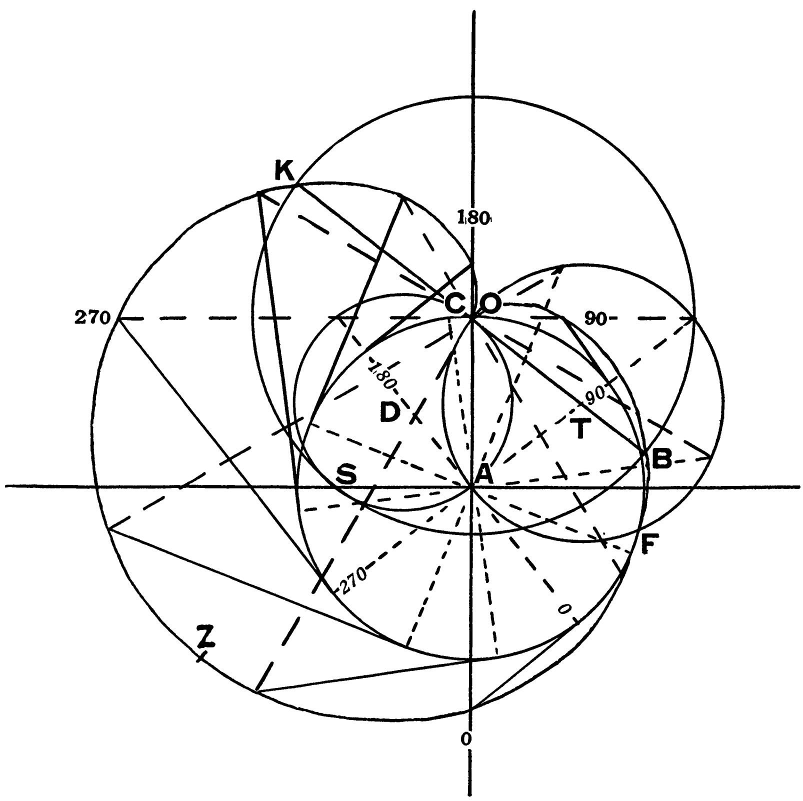

FIG. 7.

Finally in Fig. 7, $\alpha = \beta = 52°$. The starting and cusp circles and the 0° line of the tangent circle are the same as usual. The cusp C however is now at O, and the 0° line of the radii vectores through it is the negative direction of the Y axis. F has shifted somewhat and the intersection circle is much reduced.

COLLEGIATE MATHEMATICS FOR WAR SERVICE.

SEND COMMUNICATIONS FOR THIS DEPARTMENT TO H. BLUMBERG, University of Illinois.

STATISTICS IN RELATION TO THE WAR.

By ROXANA H. VIVIAN, Wellesley College.

Before 1914 there were many lines of work where the material dealt with was known to have the characteristics of statistical data, and to require treatment by one of the various statistical methods. Since 1914, not only are there many more fields where statistical methods are required because of their development in connection with war problems, but also, in the already established domains of statistics, the quantities to be collected and placed in systematic form for drawing conclusions have grown beyond the earlier proportions and require much more attention. Every new National Council or local committee needs immediate records of present conditions, and usually of the past as well. There is a growing interest in such statistical data and a desire for trained workers in large centers who can, and will, avoid expensive duplication in collecting, and who are able to present the authoritative abbreviated tables and charts that executive committees especially need.

This situation is withdrawing statistics from its position as a somewhat unpopular subject, and thoughtful workers and educators are making an effort to cancel the criticisms from which it has suffered because statistical problems were so often handled by inexperienced manipulators in ways that led to conclusions and plans that were disastrous in business and ludicrous in science. To schools and colleges this brings an opportunity for contributing certain preparation that is especially desired in the present emergency for the young men and young women who are being called to enter business, scientific, or government fields with the possibility of assuming some collection and supervision of statistical material, as well as the opportunity of giving to those who are going into different branches of military, naval or engineering service the underlying theory in those lines that the college class-room can so quickly furnish. For those who must present accurate tables of figures and draw far-reaching conclusions in the war problems of finance, research, national and international resources, the general background of college training is a substantial asset; and if to the general background there can be added even a brief elementary course in the mathematical foundations of statistics, a graduate is ready to enter more quickly into a responsible position and give reliable and necessary assistance.

This is not entirely a new point of view. For some years it has been evident that in many college subjects students need information in regard to statistical methods, as well as so-called statistics. Courses that deal with the general fundamental theory that is essential in any field have been introduced in various institutions. Except in a vocational institution where up-to-date files and computing machines are available, it is practically necessary to offer such work as a lecture course with special problems—that is, problems that require a minimum amount of time in collecting, tabulating, and making computations before drawing conclusions, and which yet afford opportunities for experimenting with all the known statistical methods and choosing the right one.

In attempting to present such work as a war emergency course, the situation differs very little from the problem of presenting the elements of statistical theory to a group of students whose college interests, or after-college plans, embrace all the natural sciences, psychology, education, economics, institutional accounting, commerical offices, municipal and governmental positions. The fundamental theory remains practically the same, the reading, required or suggested, can be slightly varied, and the problems can be chosen from the most essential fields of the present day, in the regions which are most open to the students. Such an arrangement is possible, although it may mean omission for the present of many of the interesting or vital sections of the subject, such as details of the history of government statistics in Belgium under Quêtelet, the development of insurance statistics beyond the possibility of business failure, the peculiar difficulties of early health statistics, and many more. The subject may still be presented after this adjustment as a combination course, which includes pure theory, supplemented by varied reading, with brief but ample work along especially suitable lines.

Books dealing with statistics can be found in many languages, but the standard collection for American students must be largely in English. During this year several new American books on statistics have been published and take their place among the older English and American volumes. Since they are all written from varying points of view, and since hardly one of the elementary books is sufficiently general, a reading shelf seems to be a prime necessity in beginning the study of statistics. In addition to this, careful references must be given to prevent a student from losing valuable time in securing the important first principles that are needed in all statistical work. Frequently a chapter will treat a simple collection by an advanced method, or a simple principle may be so veiled by logic, or by mathematics, as to be unrecognizable except by an expert in these lines. Part of the reading that can be done in such a collection becomes practice in extricating information, part furnishes the realization of how extremely varied and universal statistical problems and methods are, and the criticism of methods used by others leads often to the invention of new forms and short cuts by the student himself. Every one of these factors is an essential part of the mental resources of the future statistical clerk or statistician.

Daily, weekly, and monthly publications which students are reading today

with concern as to their accuracy and their predictions for the future, furnish problem material of deeper interest than many of the best and most convenient collections of preceding years. Questionnaires for registration show bewildering sources of difficulty in collecting material. Reports on daily receipts in Liberty Bond campaigns raise questions as to the complete reliability of published figures, and the necessity of checking results, and publishing the same with authorized signatures. Income tax classifications, national budget items, and appropriation lists with their large figures, are much more interesting to the average student and reveal problems which are more vital to him than the reports of corporations, organizations, and institutions, which were the usual forms of illustrating financial matters in statistics before the war. National vital statistics, and military and naval figures have put most of the scientific statistical data into the background; but since the methods remain the same the variation in material is relatively unimportant.

How such material shall be dealt with by students depends very much upon the recognized dividing line between academic and vocational courses. If adding machines are available, students can be taught to use them in a very short time, and more extensive problems can be undertaken without abnormal use of study hours for long clerical calculations. Unless such work includes obtaining information in regard to the mathematical principles of computation, and their application to machine computation, and unless the larger problems in collecting material and computing averages, percentages, deviations, and other figures are mentally worth the time they require, they lie outside the limits of a standard course for a college student. It is an interesting feature of the increasing demand for college courses in statistics that students will add, at an expense of their own time and money, an element of training which is vocational and not academic, namely, clerical experience with statistical collections once a week in some well-organized business office or municipal bureau where the exchange of collegiate theory for office experience seems to be a mutual and welcome courtesy.

The conclusion is rather a natural one that statistics with its steady general theory helping out the changing situation in these many fields is a worthy member of our mathematical family. The relation of the whole family to the war situation seems to be the customary one. It was jogging along as best it could and in the face of much unpopularity. Now the call comes to each member in this emergency; and statistics and all the other steeds are changing their harnesses and cutting across every field they can to reach the most strenuous and exacting positions and to carry the national problems to the speediest and best solution. Every institution and every instructor furnishing graduates who can assist in this particular drive is making a worthy contribution to the work of the war.

Note 1.—A few of the books advisable for a reading shelf for elementary statistical theory are the following:

Bailey & Cummings—Statistics.

Block—Traité Théorique et Pratique de Statistique.

Brinton—Graphic Methods.

Copeland—Business Statistics.
Elderton—Primer of Statistics.
King—Elements of Statistics.
Secrist—Introduction to Statistical Methods.
Thorndike—Mental and Social Measurements.
West—Introduction to Mathematical Statistics.
Yule—Introduction to the Theory of Statistics.

There are many that can be added along more general lines, and the *American Statistical Quarterly* and other magazines are furnishing an increasing number of articles on the statistical questions of the day.

Note 2.—Wellesley graduates who have had a one-hour per week course in statistical theory have secured positions in the Statistical Division of the Shipping Board and of the Labor Bureau. Their college work has covered, as completely as was possible in a short course, the collection and arrangement of statistical data, questionnaire forms, classification, graphic representation, computation of averages, percentages and deviations, methods of checking results, index figures, graphic comparison, and correlation.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

2737. Proposed by C. N. SCHMALL, New York City.

Employing Maclaurin's Theorem, or otherwise, expand the following three functions: (1) $\epsilon^{\tan^{-1}x}$ as far as x^6; (2) $\epsilon^{\sin x}$ as far as x^8; and (3) $\tan x$ as far as x^9.

2738. Proposed by W. D. CAIRNS, Oberlin College.

Prove that between any two points on a unit circle with its center at the origin there is a point whose coördinates are rational, *i. e.*, prove that the rational points on this circle are dense.

2739. Proposed by JOS. B. REYNOLDS, Lehigh University.

A particle is describing a smooth closed curve in a vertical plane. This curve is three feet high, $19\pi/4\sqrt{2}$ ft. long, and of such form that the pressure is always constant. Find the time of a complete revolution.

2740. Proposed by E. W. CHITTENDEN, University of Illinois.

Establish the identity (the determinant is of order n):

$$\begin{vmatrix} a-\lambda, & a, & a, & \cdots, & a \\ a, & a-\lambda, & a, & \cdots, & a \\ a, & a, & a-\lambda, & \cdots, & a \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a, & a, & a, & \cdots, & a-\lambda \end{vmatrix} = (-1)^{n-1}\lambda^{n-1}(na-\lambda).$$

2741. Proposed by HORACE L. OLSON, New Hampshire College.

Prove or disprove the following statement: If the three sides and the area of a triangle are integers, at least one of the three altitudes is an integer.

2742. Proposed by C. N. SCHMALL, New York City.

In Gregory's *Examples in the Differential and Integral Calculus*, 1841, Chap. VII, p. 124, ex. 22, I find the following celebrated problem: "To find a point within a triangle from which if lines be drawn to the angular points their sum may be the least possible." The author remarks that "the direct solution of this problem is long and complicated, etc."

Required a simple, brief solution.

2743. Proposed by H. S. UHLER, Yale University.

A certain compound pendulum (Borda) has the geometrical configuration resulting from the following assemblage of parts. The bob is a sphere supported directly by a rod of circular cross-section the axis of which coincides with an extended diameter of the sphere. (The rod does not project below the under surface of the sphere.) The upper end of the cylindrical rod is fitted perfectly in a hole in a rectangular block. The axis of the cylinder is parallel to the four vertical edges of the rectangular parallelepiped and it intersects the lower plane at the center of this rectangle. (The rod does not project above the upper surface of the block.) The knife-edges are formed by a right prism the lateral edges of which are horizontal and the cross section of which is an inverted isosceles triangle. The horizontal lateral face of the prism coincides with the upper plane of the rectangular block. The plane which contains the lowest lateral edge (axis of rotation) of the prism and bisects the associated diedral angle (median plane) also contains the (prolonged) axis of the rod. The triangular bases of the prism are parallel to, and equidistant from, the respective nearer vertical faces of the rectangular block. All parts are made of homogeneous invar and there are no cavities in the interior of the complete pendulum.

(*a*) Let $R \equiv$ radius of sphere,
$r \equiv$ radius of rod,
$h \equiv$ length of a vertical edge of the block (height),
$t \equiv$ length of a horizontal edge of the block which is parallel to the lateral edges of the prism (thickness),
$w \equiv$ length of a horizontal edge of the block which is perpendicular to the lateral edges of the prism (width),
$l \equiv$ perpendicular dropped from center of sphere upon the lowest lateral edge of the prism,
$a \equiv$ (vertical) altitude of isosceles section of prism,
$b \equiv$ (horizontal) base of isosceles section of prism,
$e \equiv$ total lateral edge of prism projecting beyond both sides of the block.

For extremely small oscillations of the frictionless pendulum the complete period P_0 may be represented by

$$P_0 = 2\pi\sqrt{\frac{N_1 + N_2 + N_3 + N_4 + N_5}{g(D_1 + D_2 + D_3 + D_4 + D_5)}}.$$

Prove that the symbols under the radical stand for the following expressions:

$$D_1 \equiv -\tfrac{1}{3}a^2be,$$

$$D_2 \equiv -\tfrac{1}{2}htw(2a - h),$$

$$D_3 \equiv \tfrac{1}{2}\pi r^2[(l - R)^2 - (a - h)^2],$$

$$D_4 \equiv \tfrac{4}{3}\pi lR^3,$$

$$D_5 \equiv \tfrac{1}{12}\pi k^2[4(l - R)(3R - 2k) + k(4R - 3k)],$$

$$N_1 \equiv \tfrac{1}{48}abe(12a^2 + b^2),$$

$$N_2 \equiv \tfrac{1}{12}htw(12a^2 - 12ah + 4h^2 + w^2),$$

$$N_3 \equiv \tfrac{1}{12}\pi r^2(a - h + l - R)[(a - h + l - R)^2 + 3(a - h - l + R)^2 + 3r^2],$$

$$N_4 \equiv \tfrac{4}{15}\pi R^3(5l^2 + 2R^2),$$

$$N_5 \equiv \tfrac{1}{60}\pi k^2[40kR^2 - 35k^2R + 4k^3 + 20(3R - 2k)(l - R)^2 + 10k(4R - 3k)(l - R)];$$

where $k \equiv R - \sqrt{R^2 - r^2}$.

(*b*) Let $x \equiv l - R$. If the pendulum beats seconds show that x must be a root of the cubic $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$, where

$$a_0 \equiv 80\pi^3r^2,$$

$$a_1 \equiv 40\pi[4\pi^2r^3 - 3r^2(g - 2\pi^2R) + 4\pi^2(R^2 - r^2)^{3/2}],$$

$$a_2 \equiv -20\pi\{4(g - 2\pi^2R)[2R^3 + 3r^2R + 2(R^2 - r^2)^{3/2}] + 3\pi^2r^4\},$$

$$\begin{aligned} a_3 \equiv &- [80\pi^3r^2(h-a)^3 - 120\pi gr^2(h-a)^2 + 60\pi^3r^4(h-a) - 5\pi^2abe(12a^2+b^2) \\ &- 20\pi^2htw(12a^2 - 12ah + 4h^2 + w^2) - 80ga^2be + 120ghtw(h-2a) - 224\pi^3R^5 \\ &+ 160\pi gR^4 - 320\pi^3r^2R^3 + 240\pi gr^2R^2 + 60\pi^3r^4R - 60\pi gr^4 - 16\pi(14\pi^2R^2 + \pi^2r^2 \\ &- 10gR)(R^2 - r^2)^{3/2}]. \end{aligned}$$

(*c*) Given $a = 1$, $b = 0 \cdot 8$, $l = 2$, $g = 980 \cdot 5$, $h = 2$, $P_0 = 2$, $r = 0 \cdot 49/4$, $R = 12 \cdot 01/4$, $t = 0 \cdot 64$, and $w = 1 \cdot 2$; calculate l correctly to four, or more, decimal places.

(*d*) Compute the percentage error affecting the result if g were calculated from the true value of the period (2 secs), as obtained experimentally, while neglecting all moments of forces and of inertia except those pertaining to the spherical bob.

2744. Proposed by E. B. ESCOTT, Chicago, Ill.

An insurance company computes its quarterly premiums by adding 6 per cent. to the annual premium and dividing by 4. If a policyholder pays quarterly, what rate of interest is he paying?

2745. Proposed by G. I. HOPKINS, Manchester, N. H.

A recent English publication contains the following method of constructing a regular polygon of 17 sides: Draw the radius CB perpendicular to the diameter AQ of the circle whose center is B. On BC lay off BD equal to one-fourth of BC. On BA, lay off BE and draw DE making angle BDE one-fourth of angle BDA. On BQ lay off BF and draw DF making angle FDE 45°. On AF as diameter, construct semi-circle FRA intersecting CB in H. With E as center and EH as radius construct semi-circle LHK intersecting CB in H. At the points L and K draw the ordinates NL and MK. Bisect the arc NM and let P be the point of bisection. Then the chord $NP (= MP)$ is a side of the regular polygon of 17 sides. Is the method of construction correct?

SOLUTIONS OF PROBLEMS.

433 (Calculus). Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Solve the differential equation, $\dfrac{d^{1/2}y}{dx^{1/2}} - \dfrac{y}{x} = 0$.

DISCUSSION BY EMIL L. POST, Columbia University.

In the April number of the MONTHLY appeared two solutions of the fractional differential equation

$$\frac{d^{1/2}y}{dx^{1/2}} = \frac{y}{x}. \tag{1}$$

The first of these, $y = Cx^{-1/2}e^{-1/x}$, was obtained by reducing the given equation to an ordinary differential equation of the first order; while the second, $y = A_0(1 - i\sqrt{\pi}x^{-1/2} - 2x^{-1} + i\sqrt{\pi}x^{-3/2} + \dots)$ was found by equating coefficients in an assumed expansion in series. Now each of the two methods used would seem to indicate that only one solution was possible, *i. e.*, the solution found by the corresponding method, yet the two solutions are clearly irreducible. A discussion of this difficulty might serve as a beginning of that more thorough discussion of the subject which the proposer of the problem desires.

Now first of all what does $d^{1/2}y/dx^{1/2}$ mean? More generally, what does $d^\mu y/dx^\mu$ mean where μ is any number? Before 1860, certainly, the method followed in answering that question was that due to Liouville. Starting from the known fact that $(d^\mu/dx^\mu)e^{ax} = a^\mu e^{ax}$, when μ was a positive integer, and also when μ was a negative integer, if by $d^{-p}\cdot y/dx^{-p}$ we mean the pth indefinite integral of y, he then assumed the relation to hold for all values of μ, and proceeded from that to obtain everything else. The most important formula that thus resulted was

$$\frac{d^\mu x^n}{dx^\mu} = (-1)^\mu \frac{\Gamma(-n+\mu)}{\Gamma(-n)} x^{n-\mu}. \tag{2}$$

It was this formula that the proposer of the problem used when he solved it by expansion in a series of powers of x.

But this method was too narrow, when compared with the general and rigorous methods of analysis then in use, to last. It was Riemann who first gave a definition of $d^\mu y/dx^\mu$ essentially like those used now. He gave it in the form of a definite integral. More recent writers express it as a contour integral. One of the most significant features of this newer development is the introduction of limits of "differentiation" thereby making fractional derivatives more like definite integrals than ordinary derivatives, while the ordinary derivative appears as a peculiar (though singularly important!) degeneration. The analogy of the general binomial expansion expressed as an infinite series, with its particular terminating form for a positive integral power is complete.

A useful form applicable only when the real part of the index of differentiation is negative is

$$\left\{\frac{d^{-\mu}}{dx^{-\mu}}\right\}_{x_0}^{X} f(x) = \frac{1}{\Gamma(\mu)}\int_{x_0}^{X} (X-x)^{\mu-1} f(x)\,dx. \tag{3}$$

It is to be noticed that when $x_0 = 0$, it becomes Riemann's form, while for $x_0 = -\infty$, it gives all the consistent results obtained by Liouville's method. As it stands it is more general than either.

The theorem that justifies the definition is that

$$\left\{\frac{d^{\mu_1}}{dx^{\mu_1}}\right\}_{x_0}^{X}\left\{\frac{d^{\mu_2}}{dy^{\mu_2}}\right\}_{x_0}^{X} f(y) = \left\{\frac{d^{\mu_1+\mu_2}}{dx^{\mu_1+\mu_2}}\right\}_{x_0}^{X} f(x), \tag{4}$$

where for positive integral indices we get the ordinary derivatives. The generalization of Leibnitz's theorem also follows, *i, e.*,

$$\left\{\frac{d^\mu}{dx^\mu}\right\}_{x_0}^{X} u\cdot v = u\left\{\frac{d^\mu}{dx^\mu}\right\}_{x_0}^{X} v + \mu\frac{du}{dx}\left\{\frac{d^{\mu-1}}{dx^{\mu-1}}\right\}_{x_0}^{X} v + \frac{m(\mu-1)}{2^3}\frac{d^2u}{dx^2}\left\{\frac{d^{\mu-2}}{dx^{\mu-2}}\right\}_{x_0}^{X} v + \cdots. \tag{5}$$

We are now in a position to reconcile the two solutions of equation (1). In reducing (1) to an equation of the first order, we used equation (5), and then equation (4) in the form

$$\frac{d^{1/2}}{dx^{1/2}}\frac{d^{1/2}y}{dx^{1/2}} = \frac{dy}{dx},$$

and

$$\frac{d^{-1/2}}{dx^{-1/2}}\frac{d^{1/2}y}{dx^{1/2}} = y.$$

Now although this procedure is valid when the lower limit, x_0, is finite, it is not valid when $x_0 = -\infty$, for we then have

$$\frac{d^{-1/2}}{dx^{-1/2}}\frac{d^{1/2}y}{dx^{1/2}} = y - B \tag{6}$$

where B is a constant depending on y.[1] The second solution can be now obtained in the same ways as the first.

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[1] The following explains the failure of the general theorem. From (3), when extended for all values, we find

$$\left\{\frac{d^\mu}{dx^\mu}\right\}_{x_0}^{X} B = B\frac{(X-x_0)^{-\mu}}{\Gamma(1-\mu)}.$$

If x_0 is not infinite on taking the μth derivative of this result, we get back C_1B. If x_0 is infinite, the above result is zero, the $-\mu$th derivative leaves it zero, and so the constant is lost, as (6) indicates.

Clearing (1) of fractions, operating through by $d^{1/2}/dx^{1/2}$ and using (5) we find

$$x\frac{d^{1/2}}{dx^{1/2}}\frac{d^{1/2}y}{dx^{1/2}}+\frac{1}{2}\frac{d^{-1/2}}{dx^{-1/2}}\frac{d^{1/2}y}{dx^{1/2}}=\frac{d^{1/2}y}{dx^{1/2}}$$

Using (4), (6), and (1) this reduces to

$$x\frac{dy}{dx}+\tfrac{1}{2}(y-B)=\frac{y}{x}$$

whose solution is

$$y=x^{-1/2}e^{-1/x}\left[\frac{B}{2}\int x^{-1/2}e^{1/x}dx+C\right]. \tag{7}$$

Since the order of the equation solved was raised, the best we can say is that if a solution of (1) exists for $x_0=-\infty$ it is contained in (7). As a check, we expand (7) in series obtaining

$$y=B+Cx^{-1/2}-2Bx^{-1}-Cx^{-3/2}+\cdots,$$

which, on comparison with the known solution for $x_0=-\infty$, gives $B=A_0$; $C=-i\sqrt{\pi}A_0$. Since C is no longer arbitrary, we must change the indefinite integral of equation (7) to a definite integral. We then find,

$$y=A_0\left[1-i\sqrt{\pi}x^{-1/2}e^{-1/x}+x^{-1/2}e^{-1/x}\int_{-\infty}^{x}t^{-3/2}e^{1/t}dt\right], \tag{8}$$

which is valid for all except positive real values of x.

The derivation of the solution

$$y=Cx^{-1/2}e^{-1/x}, \tag{9}$$

which was obtained in the same manner as (7) except that $B=0$, was not rigorous since it was not checked. If equation (1) be written

$$y=\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{y}{x}\right)$$

we can check directly by the use of (3). In fact

$$\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{y}{x}\right)=\frac{c}{\Gamma(\frac{1}{2})}\int_{x_0}^{X}(X-x)^{-1/2}x^{-3/2}e^{-1/x}dx=\frac{c}{\Gamma(\frac{1}{2})}X^{-1/2}e^{-1/X}\int_0^{(1/x_0-1/X)}t^{-1/2}e^{-t}dt$$

which is obtained by letting $x=X/(1+tX)$.

Since $\int_0^\infty t^{-1/2}e^{-t}dt=\Gamma(\frac{1}{2})$, if we let $x_0=0$ we find

$$\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{y}{x}\right)=CX^{-1/2}e^{-1/X}=y$$

which checks the solution. Clearly $\{d^{-1/2}/dx^{-1/2}\}_0{}^X(y/x)$ exists only when the real part of X is positive. The two solutions thus supplement each other.

Suppose now that $x_0 \neq 0$, and yet is finite. Is there no solution? Clearly we always have the trivial solution $y=0$. If however we restrict ourselves to real values of the variable and let $x_0<0$, the function $y=g(x)$ where $g(x)=0$ for $x_0\leq x\leq 0$, and $g(x)=cx^{-1/2}e^{-1/x}$ for $x>0$ clearly satisfies the auxiliary differential equation, and also the original one; since

$$\left\{\frac{d^{-1/2}}{dx^{-1/2}}\right\}_{x_0}^{x}\left(\frac{g(x)}{x}\right)=\left\{\frac{d^{-1/2}}{dx^{-1/2}}\right\}_{x_0}^{x}cx^{-3/2}e^{-1/x}=g(x);\ x>0,$$

$$\left\{\frac{d^{-1/2}}{dx^{-1/2}}\right\}_{x_0}^{x}\left(\frac{g(x)}{x}\right)=0=g(x);\ x\leq 0.$$

However, (8) and (9) are the only analytic solutions here found. It is noteworthy that they correspond to the definition of the "generalized derivative" given by Liouville and Riemann respectively.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

AMERICAN MATHEMATICIANS IN WAR SERVICE

The following list includes the known names of mathematicians of the United States in national service occasioned by the war. So far as possible, the name of the institution from which each entered service, or the home address, and the branch of the service (including Y. M. C. A. and other non-combatant branches) are named. It is not ordinarily feasible to attempt to give present addresses. Corrections and additions will be gladly received by the Secretary-Treasurer of the Mathematical Association of America, W. D. Cairns, Oberlin, Ohio. A few known names from Canada are included in the list.

O. S. ADAMS, Coast and Geod. Survey; 1st Lieut. under War Dept.
L. K. ADKINS, La Crosse St. Normal; 1st Lieut., Field Art.
J. W. ALEXANDER, Princeton Univ.; Ord. Dept. with Major Moulton.
C. S. ALLIN, Queen's Univ.; Can. Engineers.
N. H. ANNING, Chilliwack (B. C.) Pub. Schools; Sgt., Can. Rwy. Troops.
J. J. ARNAUD, Coll. of the City of New York. Aberdeen Provg. Grd.
C. S. ATCHISON, Washington & Jefferson Coll. Auditor and cost accountant, Emergency Shipping Bd.
R. W. BARNARD, Univ. of Michigan; Instr. in math., F.A.C.O.T.S., Camp Taylor
D. F. BARROW, Yale Univ.; National service.
RALPH BEATLEY, Horace Mann Sch.; 2nd Lieut., Coast Art.
D. R. BELCHER, Adelbert Coll.; Corp., N. A., in France.
A. A. BENNETT, Univ. of Texas; Capt., Ord. R. C.
W. J. BERRY, Brooklyn Polytech. Inst.; 2nd Lieut., 308th Inf., N. A.
HERMAN BETZ, Univ. of Michigan; Ord. Dept., N. A.
E. G. BILL, Dartmouth Coll.; Milit. Service Branch, Dept. of Justice, Ottawa.
R. L. BLANCHARD, Brown Univ.; 1st Lieut., Art., in France.
H. F. BLICHFELDT, Stanford Univ.; Aberdeen Provg. Grd.
G. A. BLISS, Univ. of Chicago; Aberdeen Provg. Grd.
PIERRE BOUTROUX, Princeton Univ.; French nat'l service.
J. B. BRANDEBERRY, Toledo Univ.; Naval Aux. Reserve.
J. N. BROADLICK, High Sch., Pittsburgh, Kan.; Battery, 130th F. A.
B. H. BROWN, Grad. Sch. Brown Univ.; Corp., 1 Cl., M.D.
THOMAS BUCK, Univ. of Calif.; 1st Lieut., Ord. Dept. with Major Moulton.
R. W. BURGESS, Brown Univ.; Bureau of Statistics, Washington, D. C.
H. T. BURGESS, Univ. of Wisconsin; Forest Products Laboratory, U. S. Dept. of Forestry.
C. C. CAMP, Ottawa Univ.; Sgt., Chem. Warfare Serv., A.E.F.
A. D. CAMPBELL, Northwestern Univ.; Lieut., N.A.
J. W. CAMPBELL, Wesley Coll. Winnipeg; Art., overseas.

E. F. CANADAY, Columbia, Mo.; Co. B, 342 M.G.B.N.
J. A. CAPARÓ, Univ. of Notre Dame; U. S. Air Service Sch. of Radio Mechanics.
H. L. CARD, Lombard Coll.; Sgt., Co. C, 1st Regt., Engrs., A.E.F.
F. L. CARMICHAEL, Grad. Sch., Princeton Univ.; 2nd Lieut., Aberdeen Provg. Grd.
MARY E. CASTER, Paterson, N. J.; Teaching blind soldiers, France.
C. W. COBB, Amherst Coll.; Instr., Govt. Aviation Service.
J. F. CONNER, Catholic Univ. of America; 1st Lieut., Bur. of Supplies and Accts., U. S. Navy.
J. L. COOLIDGE, Harvard Univ.; Major, Ord. Dept. heading scientific mission to Europe.
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VOLUME XXVI FEBRUARY, 1919 NUMBER 2

ON CERTAIN CONSTRUCTIONS OF DESCRIPTIVE GEOMETRY.

EXTRACTS FROM LETTERS ADDRESSED BY

GINO LORIA, University of Genoa, Italy,
to W. H. ROEVER, Washington University, St. Louis.

I.

It will doubtless not have escaped you that the month of July of this year marks the first centenary of the death of Gaspard Monge. The war has rendered impossible the execution of my project to publish at this time, as a monument to the memory of this immortal savant, my *History of Descriptive Geometry,* finished four years ago and intended to form the third volume of my *Vorlesungen über darstellende Geometrie.* Nevertheless my mind has recently been occupied —perhaps because of this centenary—with a group of questions, bearing on developments which have been influenced (or are likely to be influenced) by the ideas of this great mathematician, concerning that branch of geometry in which you are especially interested. Will you permit me therefore to share with you some of my thoughts on this subject?

Descriptive geometry had birth in the mind of Monge as an auxiliary to the engineer, destined (as he says in the beginning of his work) "to draw the French nation away from the dependence which it has had up to this time on foreign industry." He concerned himself then with a science, which certainly in an embryonic form, is found among all civilized peoples. But while architects and painters desire in general to represent objects as they appear, Monge has given a process for performing (by drawings in a plane) all sorts of geometric operations on figures of three dimensions. Hence one may say, that, while the method of double (orthographic) projection as used, for example, by Vitruvius in his celebrated work on architecture is *static* in character, with Monge it becomes

dynamic. Consequently, this method, which in the first stages of its existence served almost exclusively the artist, was elevated by Monge to the level of geometric research. And in effect the disciples of Monge will often unconsciously have recourse to this method of research.

In order to justify this last statement I am going to cite a noteworthy example. In the course of his important studies on the reflection of light, A. Quetelet established the following remarkable theorem: "Let us conceive two right cones having their vertices in different points and their axes parallel to each other; the intersection of these cones projected on a plane perpendicular to their axes will give aplanetic lines (ovals of Descartes)." From this proposition follow two methods of describing these curves; the one is the application of the method of construction which serves to find the projections of the curve of intersection of two cones represented by the method of Monge[1] while the other is an application of the general method of construction for finding the curve of intersection of two surfaces of revolution with parallel axes.[2] That portion of the first of these constructions which yields the horizontal projection can be enunciated as follows: "Given two circles in a plane; if around a fixed point taken on the line which joins their centers a transversal be made to turn cutting the two circles each in two points, the rays from the centers of the circles to their respective points of intersection with the transversal will meet in four points of which the geometric locus will be a complete aplanetic having its two foci situated at the centers of the two circles." Now this enunciation can be found in *Note* XXI of *l'Aperçu historique* of M. Chasles[3] where it is presented as a corollary of "the theorem of Ptolemy on the triangle cut by a transversal." But it is my opinion that the great French geometer reached this conclusion by combining the theorem of Quetelet, which I have quoted above and which he cited at the beginning of this note, with what Monge had taught him at l'Ecole polytechnique. If he has preferred to reach this conclusion by the theory of transversals it is for the reason that thereby he avoids geometry of space and thus uses methods characteristic to himself.

This example (to which I will return later), shows that the method of double projection can be very successfully used in questions relative to the construction of plane curves. However, it is not the only example which leads one to conclude that *the methods of descriptive geometry should be among the ordinary tools of those who cultivate the geometry of plane curves*. Such a recommendation may appear superfluous since descriptive geometry has become, among the majority of civilized peoples, an integral part of the program of studies for candidates of the doctor's degree in mathematics; but I am going to cite a fact which appears to lead to a different conclusion.

Among the most ancient curves known one finds "les spiriques de Perseus"; they are the sections cut from a torus (annular surface) by a plane parallel to the axis of the surface. In order to describe them "in plano" R. de Sluse in

[1] See my *Vorlesungen* already cited, Vol. II, p. 205.
[2] *Vorlesungen*, Vol. II, p. 262.
[3] II ed., Paris, 1875, p. 351.

1637 devised a somewhat complicated construction in which an hyperbola was employed. In 1905 F. Gomese Teixeira indicated a method which has the advantage of requiring only the use of straight lines and circles. However, the simplest and most natural method is the classic one of constructing the curve of intersection of a plane with a surface of revolution of vertical axis. When the method of double projection is used to represent the surfaces in question and the second plane of projection is taken to be parallel to the cutting plane,[1] then the vertical projection of the curve becomes identical with the objective curve, and the end sought is attained by a solution the details of which can be found in a letter which I addressed to M. Teixeira and which he published in Vol. IX (pp. 193–196) of the *Annaes scientificos da Academia polytechnica do Porto.* The thing is so simple and natural that I could never persuade myself to make it the subject of a publication as such. However, it appears that no one had thought of it before I did. If it did not concern itself with such a little thing, one might see in all this a confirmation of the observation of Jacobi that discoveries are born of the contact of theories which were at first separate.

Permit me to return to the construction of the ovals of Descartes discovered by Quetelet. These curves, as you know, possess three real foci which are situated on a straight line. But one can consider curves entirely analogous to these of which one focus alone is real while the other two are conjugate imaginaries. A. Cayley was the first to consider these curves in his "Note on the cartesian with two imaginary axial foci." [2] He called them Cartesian lines and established some of their properties, but he did not arrive at a simple construction nor did he determine their form. Now all of these things can be obtained by a consideration of geometry of three dimensions because there exists a simple relation between these curves and the ovals of Descartes, which have already been considered in a preceding paragraph.

A cartesian curve may be obtained as the orthographic projection of the curve of intersection of two cones of revolution which are conjugate imaginaries and of which the axes are parallel to each other, provided the plane of projection is normal to the axes. As descriptive geometry concerns itself exclusively with real elements, this remark does not lead at once to the desired result; but it is natural to ask whether it is not possible to substitute for the two imaginary cones two real surfaces of the pencil of surfaces which they determine. Now I have recently demonstrated in the note "Fasci di quàdrichi rotonde e Curve cartesiane," [3] of which I have the honor to send you a copy, that "through the intersection of two real or conjugate imaginary cones of revolution of which the axes are parallel, there passes a third cone of revolution of which the axis is parallel to the axes of the given cones and also a sphere, both of these surfaces being real; if the two given cones are real, the intersection is composed of two branches and its projection on the plane perpendicular to the axes is a pair of conjugate ovals of

[1] See my *Vorlesungen*, Vol. II, p. 258.

[2] *Proc. of the London Math. Society*, Vol. III, 1871–72 or also *The Collected Papers*, Vol. VII, pp. 241–243.

[3] *Rend. della R. Accademia dei Lincei*, sèance du 17 Mars, 1918.

Descartes; but if the given cones are conjugate imaginaries, the intersection has only one branch and then its projection is a cartesian line." It is thus demonstrated that the ovals of Descartes and the cartesian curves may be obtained in the same manner, namely, as the orthographic projection of the curve of intersection of a real sphere by a real cone of revolution, and that the first or second type of curve is obtained according as the cone actually penetrates or merely "bites" the sphere. Now since a sphere and a cone of revolution may be regarded as two surfaces of revolution with parallel axes, the orthographic projection of their intersection may be obtained by the aid of a classic construction of descriptive geometry to which I have already had occasion to refer. In Fig. 1 I have carried through this construction, supposing, as is permissible, that the vertical plane of projection is parallel to the plane determined by the axis of the cone and the center of the sphere. In the figure the horizontal projection only of the curve of intersection is constructed, and thus I have arrived at the *first construction which I know of cartesian lines.*

I will finish with the remark that that which precedes enables us to complete another theorem of Quetelet. In effect, this geometer demonstrated that if one has a sphere and a right cone and makes a stereographic projection of the curve of penetration of these surfaces, the eye being placed at an extremity of the diameter of the sphere which is parallel to the axis of the cone and the plane of projection taken perpendicular to this axis, then the projection will be a pair of conjugate ovals of Descartes. Now, if the cone instead of penetrating the sphere, merely "bites" it, the projection will be a cartesian line.

II.

The application of considerations of geometry of space to geometry of the plane is one of the most brilliant ideas of Monge. But it is remarkable that in the hands of some of his pupils (I cite particularly Hachette and Olivier) it has undergone a strange and vexatious deviation to which I wish to call your attention.

One of the principal aims of descriptive geometry is to replace by drawings in a plane the purely ideal constructions of geometry of three dimensions. Until some physicist will have taught us how to *design in space,* the inverse process, namely that of replacing a planometric construction by a stereometric one, represents from the graphic point of view a veritable step backward. Nevertheless, in several works of the two geometers whom I have mentioned, one finds space constructions which have for their object the determination of tangents to particular curves—constructions which give the deplorable delusion of having solved problems which, on the contrary, are yet unsolved. I think, on the contrary, that descriptive geometry should have as a basis a complete and detailed knowledge of all of plane geometry. It is for this reason that I long for the publication of a comprehensive work of which Part I shall comprise all constructive geometry of plane figures, while Part II shall be a treatise on descriptive geometry. In order to render more striking the utility of such a fusion, I will cite an example.

In 1826, H. Garbinski, the apostle of descriptive geometry in Poland, published in the *Annales de Mathématiques* de Gergonne (T. XVI, pp. 167–172), an article on the conical spiral, a curve which he conceived in generalizing the definition of the ordinary cylindrical helix, but which B. Pascal had considered

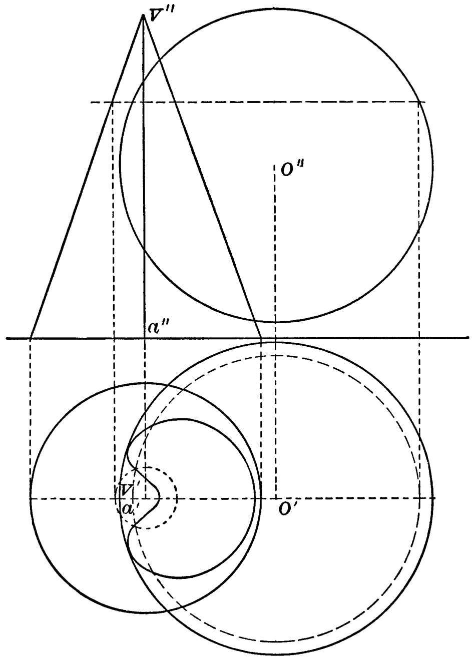

FIG. 1.

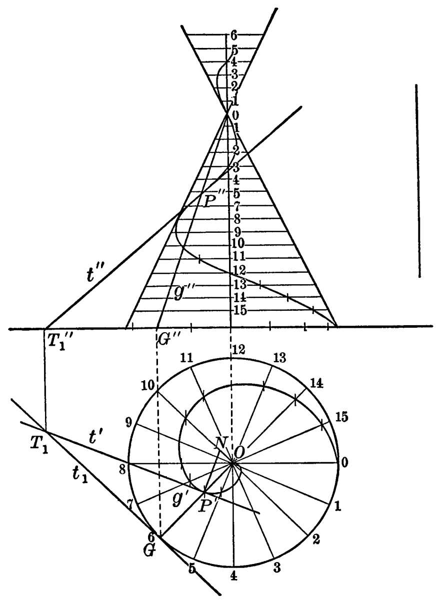

FIG. 2.

two centuries before. This curve is the trajectory described by a point, which, starting at the vertex of a right circular cone, describes with uniform motion a generatrix, which in turn moves around the surface of the cone with a constant velocity. From this definition the following properties may be deduced: (1) The orthographic projection of the conical spiral on a plane normal to the axis of the cone is a spiral of Archimedes. (2) The conical spiral lies on the surface of a right helicoid, which is coaxial with the cone of this spiral. In order to represent the curve in question by the method of Monge, one can use a process which is entirely analogous to that which is used to represent the cylindrical helix (see Fig. 2). The curve may be represented analytically by the following equations:

$$(1) \qquad x = \frac{r\omega}{2\pi}\cos\omega, \qquad y = \frac{r\omega}{2\pi}\sin\omega, \qquad z = l - \frac{l\omega}{2\pi},$$

in which l is the height of the cone, r is the radius of its base, and ω is a parameter. These equations prove that the orthographic projection Γ' of the spiral Γ on the plane xy is the spiral of Archimedes

$$(2) \qquad \rho = \frac{r\omega}{2\pi}$$

and that Γ lies upon the surface of the right helicoid

$$\frac{y}{x} + \tan\frac{2\pi z}{l} = 0. \tag{3}$$

Equation (2), on differentiation, yields the equation

$$\frac{d\rho}{d\omega} = \frac{r}{2\pi}, \tag{4}$$

and consequently the curve Γ' enjoys the remarkable property that its polar subnormal ON in any point whatever P' has the constant value $r/2\pi$. It is then, indeed, easy to construct the normal, and consequently the tangent t' to Γ' at the arbitrary point P', that is to say, the horizontal projection of the tangent t at a general point of the conical spiral. To complete the representation of the tangent t, let us remark (1) that t'' passes through P'', (2) that t lies in the plane τ which is tangent to the given cone along the generatrix g which passes through the point P. Now if G is the horizontal trace of g, the horizontal trace of the plane τ is the tangent t_1 at the point G to the circular base of the given cone. Then the intersection T_1 of t' and t_1 is the horizontal trace of the required tangent, while t'' is the line which joins T_1'' and P''.

Now in place of having recourse to this simple solution, Garbinski, whose knowledge of the spiral of Archimedes seemed rather limited, made use of the fact that the curve Γ lies on the helicoid (3), and therefore that the tangent sought lies in the corresponding tangent plane to this surface. He thus constructed this plane by the aid of a related hyperbolic paraboloid. From another point of view this last artifice was not indispensable, for as Vallès remarked at once,[1] it is most simple to make use of one of the cylindrical helices lying on the helicoid. As a consequence of all of these considerations, Garbinski and Vallès gave constructions for the tangent to the spiral of Archimedes which were extremely more complicated than the construction which is a result of the constancy of the subnormal.

I wish to add that the preceding construction of the tangent to the conical spiral may serve also an analogous end for the *conical helix* (that is the curve which is now called, on account of one of its remarkable properties, the helical cylindrical-conic). This curve has the property of projecting itself orthographically on the plane of the base of the cone into a logarithmic spiral. Now the tangents of this curve make constant angles with the corresponding radii vectores. Consequently nothing is easier than to construct the horizontal projection of the tangent to the conical helix. Then the vertical projection is obtained by the aid of the tangent plane to the given cone precisely as in the case of the conical spiral.

III.

Although descriptive geometry may have been conceived by Monge as a practical science, I have not been able to find in the publications of this great

[1] *Annales de Mathématiques*, T. XVI, pp. 376–377.

geometer any word relative to the necessity for obtaining the *most simple solutions possible,* that is to say, those that are at the same time the most economical and the most exact. The criterion of simplicity for the choice among several solutions of the same problem is entirely modern; to be sure it has not yet been applied to its full extent, with the result that researches in this direction make it possible, if I do not deceive myself, to penetrate farther into several domains which are now regarded by some writers as having been exhaustively studied. To prove to you that I do not allow myself to be deceived, I am going to return to a classic problem, namely, to the construction of the tangent t at a general point P (Fig. 4) of the cylindrical helix, ordinarily represented by the method of Monge. It is known[1] that to effect this construction one begins by drawing the horizontal projection t' of the tangent t (that is, the tangent t' at the point P' of the circular base of the given cylinder). After determining the horizontal trace T_1 of t, the second projection t'' of t is found by connecting T_1'' with P''. But in order to determine T_1, it is necessary to find a rectilinear segment equal in length to the arc of the circular base which is comprised between P' and the point where the helix pierces the horizontal plane. Now the rectification of a circular arc is a very delicate transcendental operation which may be performed by a simple approximation process,[2] and which must be repeated for each point of the curve where the tangent is required. Consequently I flatter myself that I may interest you in a process which holds for all points of the helix—a unique transcendental operation which is more simple than the rectification of the circular arc. In order to establish this result, I will make use of the following remark (to which, moreover, we will have recourse in several analogous questions): "If one considers two projections Γ' and Γ_0 (parallel or central) of a curve Γ on the same plane σ, the tangents to these curves at any two corresponding points P' and P_0 will intersect in the trace on the plane σ, of the tangent at the point P, to the objective curve Γ." To apply it to our question, let us recall[3] that if one projects the helix, represented by the equations

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \qquad z = \frac{H}{2\pi}\varphi, \tag{5}$$

from the point $(0, 0, c)$ on the plane xy, there results the hyperbolic spiral having for its polar equation

$$\rho\left(\frac{2\pi c}{H} - \omega\right) = \frac{2\pi c r}{H}, \tag{6}$$

from which by differentiation one obtains the equation,

$$\frac{1}{\rho}\frac{d\rho}{d\omega}\left(\frac{2\pi c}{H} - \omega\right) = 1. \tag{7}$$

On dividing member by member equation (6) by equation (7), one obtains the

[1] See *Vorlesungen,* Vol. II, p. 137.
[2] See *Vorlesungen,* Vol. II, p. 63.
[3] *Vorlesungen,* Vol. II, pp. 130–32.

relation

$$\rho^2 \frac{d\omega}{d\rho} = \frac{2\pi c r}{H} \tag{8}$$

which shows that the curve (6) possesses the property that its polar subtangent has a constant value $l = 2\pi cr/H$. In order to determine a segment equal in

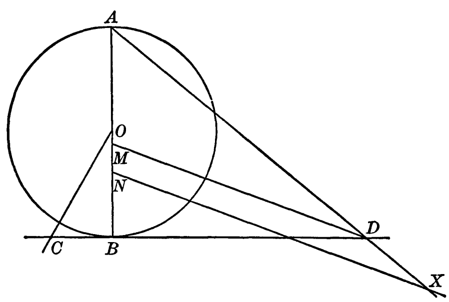

FIG. 3.

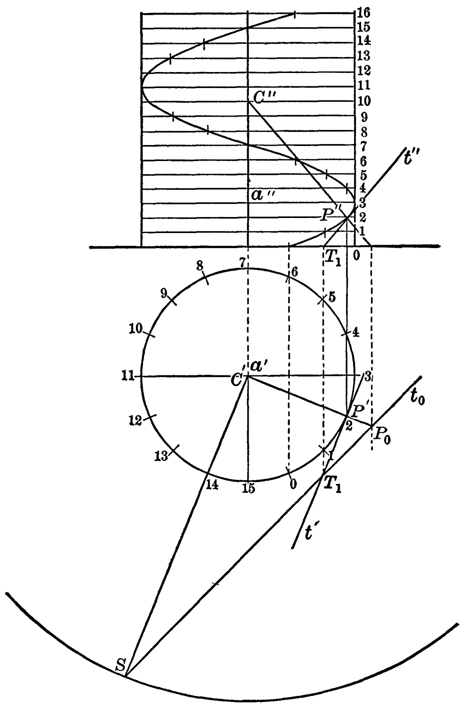

FIG. 4.

length to l (Fig. 3), let us construct by the method of Kochanski,[1] or by any other process, a line AD equal to the half circumference πr of the circle of diameter $AB = 2r$. Let us now lay off on AB the segments $AM = H/2$ and $AN = c$, and then join M to D and draw NX parallel to MD. Then

$$\frac{AX}{AN} = \frac{AD}{AM}, \quad \text{or indeed} \quad \frac{AX}{c} = \frac{\pi r}{H/2}, \quad \text{whence} \quad AX = \frac{2\pi c r}{H} = l.$$

We may now pass to Fig. 4, which represents the given helix. Let $P \equiv (P', P'')$ be any point of the helix and P_0 its projection from the center $C \equiv (C', C'')$ on the horizontal plane. Describe the circle of center a' and radius l and draw in this circle the radius $a'S$ making with the line $a'P'$ an angle of 90° in the negative sense of rotation. Then the line SP_0 will be the tangent at the point P_0 to the hyperbolic spiral, that is to say, the central projection, from the center

[1] *Vorlesungen*, Vol. II, p. 63.

C, of the tangent, at the point P, to the given helix. This tangent cuts the tangent t', which is drawn at the point P' to the circle Γ, in the horizontal trace T_1 of the required tangent. Etc.

GENOA, May–June, 1918.

DIFFERENCE QUOTIENTS.

By J. P. BALLANTINE, Cambridge, Mass.

The whole subject of interpolation, in the case of equally spaced ordinates, by means of algebraic polynomials is commonly based on the subject of finite differences. The method, however, fails for unequal spacing. The purpose of this paper is to show, that by a suitable extension of the conception of the difference quotient, the method can be generalized for the case when the ordinates are spaced according to any law.[1]

Let $f(x)$ be any single valued function. Plot the curve $y = f(x)$. Pick out on the curve the $(n+1)$ points whose x coördinates are $1, 2, \cdots, (n+1)$, respectively. By means of the method of finite differences, find the equation of the curve of the form $y = P_n(x)$ which passes through the $(n+1)$ points. For convenience, call this curve a secant of degree n, noticing that a secant of the first degree is an ordinary secant. It is easily verified that the nth derivative of this secant is precisely equal to the difference of order n of the ordinates of the $(n+1)$ points.

Now let us take our $(n+1)$ points on the curve at random. We have to define the difference quotient of order n at these $(n+1)$ points in such a way that when the values of $x_1, x_2, \cdots, x_{n+1}$ are $1, 2, \cdots, n+1$ respectively, it will reduce to a difference of order n.

We can define the secant through these $(n+1)$ points as the curve of the form $y = P_n(x)$ which passes through them. By way of a more explicit notation, $y = P_n(x;\ 1, n+1)$ will be the equation of the nth degree secant at the points $(x_1, y_1), (x_2, y_2) \cdots (x_{n+1}, y_{n+1})$. It follows that

$$P_n(x_i;\ 1, n+1) = y_i, \qquad 1 \leq i \leq n+1.$$

DEFINITION. The difference quotient of order n, taken at the $(n+1)$ points $(x_1, y_1), (x_2, y_2) \cdots (x_{n+1}, y_{n+1})$ is the nth derivative of the nth degree secant through those points. It is denoted thus $Q^n_{1,n+1}$:

$$Q^n_{1,n+1} = \frac{d^n}{dx^n} P_n(x;\ 1, n+1).$$

Consider the special polynomial and associated equation

$$P_{n+1}(x;\ 1, n+2) - P_n(x;\ 1, n+1),$$

$$(1) \quad P_{n+1}(x_i;\ 1, n+2) - P_n(x_i;\ 1, n+1) = y_i - y_i = 0, \ 1 \leq i \leq n+1.$$

[1] Cf. A. A. Markoff, *Differenzenrechnung*, 1896, p. 10.

From the factor theorem for polynomials, we may say that:

$$(2)\quad P_{n+1}(x;\ 1,\ n+2) - P_n(x;\ 1,\ n+1) = K(x - x_1)(x - x_2)\cdots(x - x_{n+1}).$$

Now, determine K by $(n + 1)$ differentiations:

$$(3)\qquad Q_{1,\,n+2}^{n+1} = K(n+1)!.$$

Using this value of K in (2), and transposing, we have

$$(\mathrm{I})\quad P_{n+1}(x;\ 1,\ n+2) = P_n(x;\ 1,\ n+1) + \frac{Q_{1,\,n+2}^{n+1}}{(n+1)!}(x - x_1)(x-x_2)\cdots(x-x_{n+1}).$$

In this way a secant of degree $(n + 1)$ is expressed in terms of one of degree n and a difference quotient. We know how to write the equation of a secant of degree one or zero, and so our problem of writing the equation of a secant of the nth degree at $(n + 1)$ arbitrary points is solved, except for determining the difference quotient.

In the same way as we inferred the truth of (2), we may show that

$$(2')\quad P_{n+1}(x;\ 1,\ n+2) - P_n(x;\ 2,\ n+2) = K(x - x_2)(x - x_3)\cdots(x - x_{n+2}).$$

It is not obvious that the two arbitrary constants must be the same, until one notices that in each case K must be equal to the coefficient of x^{n+1} in the polynomial, $P_{n+1}(x;\ 1,\ n+2)$.

Subtracting (2′) from (2), we have

$$(4)\quad K(x_{n+2}-x_1)(x-x_2)(x-x_3)\cdots(x - x_{n+1}) = P_n(x;\ 2,\ n+2) - P_n(x;\ 1,\ n+1).$$

Differentiating n times, we obtain

$$(5)\qquad K(x_{n+2} - x_1)(n!) = Q_{2,\,n+2}^{n} - Q_{1,\,n+1}^{n}.$$

Then by means of the value of $Q_{1,\,n+2}^{n+1}$ as obtained in (3), we have

$$(\mathrm{II})\qquad Q_{1,\,n+2}^{n+1} = \frac{Q_{2,\,n+2}^{n} - Q_{1,\,n+1}^{n}}{\dfrac{x_{n+2} - x_1}{n+1}}.$$

We are now in a position to form a table of difference quotients. Formula (II) tells us, that besides forming a difference, we must divide by a denominator, which is simply the average of the $(n + 1)$ intervals of x between the $(n + 2)$ points at which $Q_{1,\,n+2}^{n+1}$, the entry we are at the time computing, is taken. It may be denoted thus:

$$(6)\qquad \Delta_k x_i = \frac{x_{k+i} - x_i}{k}.$$

Applying the definition of the difference quotient of order zero, we notice that the secant of order zero through the point $(x_i y_i)$ is given by the equation: $y = y_i$. Without differentiation, we find: $Q_i^0 = y_i$.

The table of difference quotients may be arranged as follows:

$$
\begin{array}{cccc|l ccccccc}
 & & & x_1 & y_1 = Q_1^0 & & & & & & \\
 & & \Delta_1 x_1 & & & \Delta Q_{1,2}^0 & Q_{1,2}^1 & & & & \\
 & \Delta_2 x_1 & & x_2 & y_2 = Q_2^0 & & & \Delta Q_{1,3}^1 & Q_{1,3}^2 & & \\
\Delta_3 x_1 & & \Delta_1 x_2 & & & \Delta Q_{2,3}^0 & Q_{2,3}^1 & & & \Delta Q_{1,4}^2 & Q_{1,4}^3 \\
 & \Delta_2 x_2 & & x_3 & y_3 = Q_3^0 & & & \Delta Q_{2,4}^1 & Q_{2,4}^2 & & \\
 & & \Delta_1 x_3 & & & \Delta Q_{3,4}^0 & Q_{3,4}^1 & & & & \\
 & & & x_4 & y_4 = Q_4^0 & & & & & &
\end{array}
$$

The elements x_i and y_i are assigned.

All the elements to the left of these can be computed by relation (6) above, and those on the right by the following two relations:

$$\Delta Q_{i,\,i+k+1}^k = Q_{i+1,\,i+k+1}^k - Q_{i,\,i+k}^k, \tag{7}$$

$$Q_{i,\,i+k}^{k+1} = \frac{\Delta Q_{i,\,i+k+1}^k}{\Delta_{k+1} x_i}. \tag{8}$$

One way of writing the equation of the secant through the four points $(x_1 y_1)$, $(x_2 y_2) \cdots$ would be:

$$y = y_1 + \frac{Q_{1,2}^1}{1!}(x - x_1) + \frac{Q_{1,3}^2}{2!}(x - x_1)(x - x_2) + \frac{Q_{1,4}^3}{3!}(x - x_1)(x - x_2)(x - x_3), \tag{9}$$

obtained by a repeated application of equation (I).

In the general case the formation of a table of difference quotients involves a few laborious divisions; but these can be avoided by a proper choice of the spacing of the intervals of x. If, for instance, all of the intervals of x are unity, then $\Delta_k x_i$, the average of any k of them, will also be unity, and the rule for the formation of the difference quotients reduces to that of the formation of a table of differences. In that case the formulæ of Newton and Gauss are special cases of form (I).

These are by no means the only important special cases. Let us first consider the limiting case when one point, (x_j, y_j), approaches the point, (x_i, y_i). The limiting value of $Q_{i,j}^1$ would be $f'(x_i)$, if such a derivative existed, where $f(x) = y$ is the path along which the first point approached the second. The derivative may be regarded as a special case of the difference quotient, by extending the conception of the secant, as defined earlier, to take into account multiplicity. Whether regarded as a special case or as a limiting case, it may be used in formula (I) and on the right hand side of (II). Equation (II) does not hold in the special case when $Q_{1,n+2}^{n+1} = f^{(n+1)}(x_1)$, for if all the $(n + 2)$ points are regarded as coincident, then the denominator is zero. In that case, however, $f^{(n+1)}(x_1)$ is part of the data and does not have to be computed.

Let us illustrate the case of two coincident points by a simple example. To form a polynomial which, when x has the value 1, has the value 2 and its derivative the value 3; and when x has the value 4, has the value 5. The table of difference quotients is below.

$\Delta_2 x$	$\Delta_1 x$	x	y	Δy	Q^1	ΔQ^1	Q^2
		1	2				
	0			0	3		
$\frac{3}{2}$		1	2			-2	$-\frac{4}{3}$
	3			3	1		
		4	5				

Then

$$y = 2 + (x - 1)3 + (x - 1)^2\left(-\frac{4}{3}\right)\left(\frac{1}{2!}\right).$$

In the special case when all the points are coincident, the difference quotients are derivatives, and the polynomial one obtains from equation (I), is exactly the polynomial one would obtain using the same number of terms in the Taylor expansion.

Suppose we had an indefinite integral tabulated to a large number of decimal places, and wished to interpolate. For the sake of argument, suppose the table of differences would not converge rapidly enough. It often happens that we are able to compute the derivative of a function represented by an indefinite integral quite readily. It appears intuitively, that if we could in some way use to advantage the values of the derivative the process would converge more rapidly.

In the case under supposition, we know the values of a function and its derivative for equally spaced values of the argument. It is as if we knew the value of the function for equally spaced pairs of points, the two points of each pair being coincident. The difference quotient at any pair of coincident points is given to us as the derivative, and all the other difference quotients may be found by formula (II). There are so many simplifications in the process, that I will submit an example.

To find the value of $\log_{10} 31.029971$ to ten places. First we notice that it is easy to compute the derivative:

$$\frac{d}{dx}\log_{10} x = \frac{1}{x}\log_{10} e = \frac{0.4342944819}{x}.$$

We tabulate a few nearby logarithms, and the corresponding values of $\dfrac{0.4342944819}{x}$, as follows:

x	$\log x$	derivative
29	1. 46239 79979	0. 01497 56718
30	1. 47712 12547	0. 01447 64827
31	1. 49136 16938	0. 01400 94994
32	1. 50514 99783	0. 01357 17026
33	1. 51851 39399	0. 01316 04389.

The table may be arranged as follows:

x	$y = \log x$	Q^1	$\frac{1}{2}Q^2$	$\frac{1}{2}\Delta Q^2$
29.	1. 46239 79979	0. 01497 56718	$-0.$ 00025 24150	0. 00000 56409
29.	1. 46239 79979	1472 32568	24 67741	1 07305
30.	1. 47712 12547	1447 64827	23 60436	51039
		1424 04391	23 09397	97248

x	$y = \log x$	Q^1	$\frac{1}{2}Q^2$	$\frac{1}{2}\Delta Q^2$
31.	1. 49136 16938	0. 01400 94994	−0. 00022 12149	0. 00001 46330
		1378 82845	21 65819	88409
32.	1. 50514 99783	1357 17026	20 77410	42183
		1336 39616	20 35227	
33.	1. 51851 39399	1316 04389		

$\frac{1}{3}Q^3$	$\frac{1}{6}Q^4$	$\frac{1}{6}\Delta Q^4$	$\frac{1}{5}Q^5$
0. 00001 12818	−0. 00000 05513	0. 00000 00286	0. 00000 00858
1 07305	5227	396	792
1 02078	4830	243	729
97248	4588	336	672
92660	4251	207	621
88409	4043		
84366			

The entries in the column headed y are the data as copied from some table. Notice that the second entry is the same as the first, for they are the values of the function for two coincident values of x. This repetition is superfluous, and may be omitted. The column headed Q^1 contains the difference quotients at every two consecutive points. The first two values of x are coincident, so the first difference quotient is copied from the table of derivatives. The second and third values of x are a unit apart, so the difference quotient is the difference. And so on down the column, alternate entries are copied from the table of derivatives, and the others are obtained as differences of y.

In order to obtain Q^2, formula (II) tells us to divide the difference of two consecutive entries in the column of first order difference quotients by $\frac{1}{2}$. If you have a set of intervals, alternate ones zero and one, the average of any two consecutive ones of them will be $\frac{1}{2}$. Let us omit dividing by this factor, and thereby obtain $\frac{1}{2}Q^2$ instead of Q^2. To obtain the entries in the column headed $\frac{1}{2}\Delta Q^2$, we difference those of the column before. Formula (II) says that in order to obtain Q^3 we must divide the first ΔQ^2 by 1/3, and the second by 2/3. The differences of x are 0, 1, 0, 1, etc., and the average of three consecutive intervals will be 1/3 and 2/3 alternately. In order to avoid labor, we multiply every other one by 2, for they look smaller than the others, and thereby in each case obtain $\frac{1}{3}Q^3$ instead of $\frac{1}{2}Q^3$. Absorb these factors into the difference quotients, denoting the pseudo difference quotients by $Q_1{}^n$. The factorial divisors in the terms of the polynomial will be changed, as follows:

$$F(t) = y_1 + \frac{tQ_1}{(0!)^2(1!)} + \frac{t^2Q_1{}^2}{(0!)^2(1!)^2} + \frac{t^2(t-1)Q_1{}^3}{(1!)^2(2!)} + \frac{t^2(t-1)^2Q_1{}^4}{(1!)(2!)^2} + \cdots$$

$$(10) \qquad + \frac{t^2(t-1)^2(t-2)^2\cdots(t-n+1)^2(t-n)}{(n!)^2(n+1)!}Q_1{}^{2n+1}$$

$$+ \frac{t^2(t-1)^2(t-2)^2\cdots(t-n)^2}{(n!)((n+1)!)^2}Q_1{}^{2n+2}, \qquad (t = x - x_1).$$

We obtain in this particular example the polynomial:

$$y = 1.4913616938 + 0.0140094994t - 0.000221212149t^2$$
$$+ \frac{t^2(t-1)}{2} 0.0000092660 - \frac{t^2(t-1)^2}{4} 0.0000004251 + \cdots.$$

To obtain log 31.029971, set $t = 0.029971$; adding the various terms of the polynomial, we obtain the result

$$\begin{array}{r} 1.\ 49136\ 16938 \\ 41\ 98787 \\ -1995 \\ -41 \\ -1 \\ \hline 1.\ 49178\ 13688 \end{array} = \log 31.029971.$$

Although the definition given above for the secant is geometrical, it could just as well be stated analytically. The work which I have carried out for reals could be extended into the domain of imaginaries.

When one has obtained the interpolatory polynomial, the secant, it may be asked, "How near will this come to giving the true value of the function throughout a certain interval?"

In equation (I), set $x = x_{n+2}$, and replace $P_{n+2}(x_{n+2};\ 1, n+2)$ by its equivalent value $f(x_{n+2})$. Considering x_{n+2} as the independent variable, and calling it x for simplicity, we get the equation:

$$\text{(III)} \quad f(x) \equiv P_n(x;\ 1, n+1) + \{Q^{n+1}_{1, n+1}(x)\}(x - x_1)(x - x_2)\cdots(x - x_{n+1})\frac{1}{(n+1)!}$$

where $Q^{n+1}_{1, n+1}(x)$ is the difference quotient taken for the $(n+1)$ fixed values $x_1, x_2, \cdots, x_{n+1}$, and the one variable value, x. It is a function of x.

A *remainder* is that which has to be added to an approximation in order to give the true result.

Let

$$f(x) = P_n(x;\ 1, n+1) + R(x).$$

Then

$$\text{(IV)} \qquad R(x) = \frac{1}{(n+1)!}\{Q^{n+1}_{1, n+1}(x)\}(x - x_1)(x - x_2)\cdots(x - x_{n+1}).$$

We have determined $R(x)$ in terms of $Q^{n+1}_{1, n+1}(x)$, and so let us study the latter. Let us examine its derivative. By the use of formula (II), we have

$$\left.\frac{dQ^{n+1}_{1, n+1}(x)}{dx}\right|_{x=x_0} = \lim_{x_{n+2}\to x_0} \frac{Q^{n+1}_{1, n+2} - Q^{n+1}_{0, n+1}}{x_{n+2} - x_0} = \lim_{x_{n+2}\to x_0} \frac{Q^{n+2}_{0, n+2}}{n+2}.$$

In general there is no limit, but let us suppose $f'(x)$ exists at $x = x_0$. Then the secant at the points $x_0, x_1, \cdots, x_{n+2}$ approaches a limiting secant as x_{n+2} approaches x_0, namely the secant at the points $x_0, x_1, \cdots, x_{n+1}$, which has contact

at x_0. Its derivative of order $(n+2)$, the difference quotient, has a limiting value also, and is denoted thus: $Q^{n+2}_{0,0,n+2}$. Then we have

$$\text{(V)} \qquad \left.\frac{dQ^{n+1}_{1,n+1}(x)}{dx}\right|_{x=x_0} = \frac{Q^{n+2}_{0,0,n+2}}{n+2}.$$

Proceeding in this way one can determine the properties of $Q^{n+1}_{1,n+1}(x)$ from those of $f(x)$, and in that way can obtain the oscillation of the remainder for any interval. If the problem is to determine the derivative of $f(x)$, one differentiates in formula (III), and obtains the expression for the remainder in that case in terms of $Q^{n+1}_{1,n+1}(x)$ and its derivative, which are already known.

In this way, the difference quotient enables us to expand an arbitrary function, analytic or merely continuous, real or complex, at points to suit the peculiar needs of the problem. Moreover, in any special case, in which a simpler and less elastic method of expansion can be used, the expansion in terms of difference quotients reduces automatically to that method. The remainder can be expressed explicitly in terms of the difference quotient, not as in the case of a remainder in terms of a derivative at some unknown point, but taken at the points about which we are expanding and the one additional point x, the point at which we want the remainder.

A THEOREM IN THE GEOMETRY OF THE TRIANGLE.

By J. W. CLAWSON, Ursinus College.

It is the purpose of this paper to state a general theorem of which the well-known Wallace (or Simson[1]) line theorem is a very special case; to point out the result of a polar reciprocation; to mention some special cases of the theorem; and to give an application to the geometry of the quadrilateral.

1. *Theorem.*[2] $A_1A_2A_3$ is a triangle, a_i the side opposite A_i, $(i = 1, 2, 3)$, P a point on the circumference of the circumscribed circle[3] C, Q any point in the plane. Let A_iQ cut the circle[3] at U_i; PU_i intersects a_i at L_i. Then L_1, L_2, L_3, Q are collinear on l. Let us call this line the "P-transversal of Q" with respect to the triangle $A_1A_2A_3$.

The theorem follows at once from Pascal's Theorem for, since $A_2A_3A_1U_1PU_2$, $A_1A_2A_3U_3PU_1$ are inscribed hexagons, L_1, L_2, Q and L_3, L_1, Q are collinear.

Cor. 1.—If a transversal cut the side a_i of a triangle at L_i, and P be any point on the circumcircle, and if PL_i cut the circle at U_i, then A_1U_1, A_2U_2, A_3U_3 are concurrent at Q, a point on the transversal.

Cor. 2.—If a transversal cut the side a_i of a triangle at L_i, if Q is any point

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[1] For references, see this MONTHLY, 1916, p. 61.

[2] It would be strange if this simple theorem were new; yet the writer has not found it recorded. A special case is given in Hatton's *Projective Geometry*, 1913, Ex. p. 164.

[3] Or conic.

on the transversal, and if A_iQ cut the circumcircle at U_i, then U_1L_1, U_2L_2, U_3L_3 are concurrent at P, a point on the circumcircle.

2. Now let the polar reciprocal of the figure be found with respect to any circle having P for center (Fig. 1). The sides of the triangle $A_1A_2A_3$ or T reciprocate into the vertices of a triangle $A_1'A_2'A_3'$ or T', which is inversely similar to T; and A_1', A_2', A_3', P lie on a circle C'. For[1]

$$\measuredangle A_1'PA_3' = \measuredangle A_3A_2A_1 = \measuredangle A_3PA_1 = \measuredangle A_1'A_2'A_3'.$$

If $A_1'U_1'$ is the reciprocal of L_1, $\measuredangle A_3'A_1'U_1' = \measuredangle A_2PL_1 = \measuredangle A_2A_1U_1$. Similarly, if $A_2'U_2'$ is the reciprocal of L_2, $\measuredangle A_1'A_2'U_2' = \measuredangle A_3PL_2 = \measuredangle A_3A_2U_2$. Hence Q', the reciprocal of l, is the point in T' isogonally conjugate to the point which corresponds to Q in the inverse similarity of two triangles.

It is clear that A_iU_i reciprocates into L_i', the point where PU_i' intersects a_i'; and that Q reciprocates into l', the P-transversal of Q' with respect to T'.

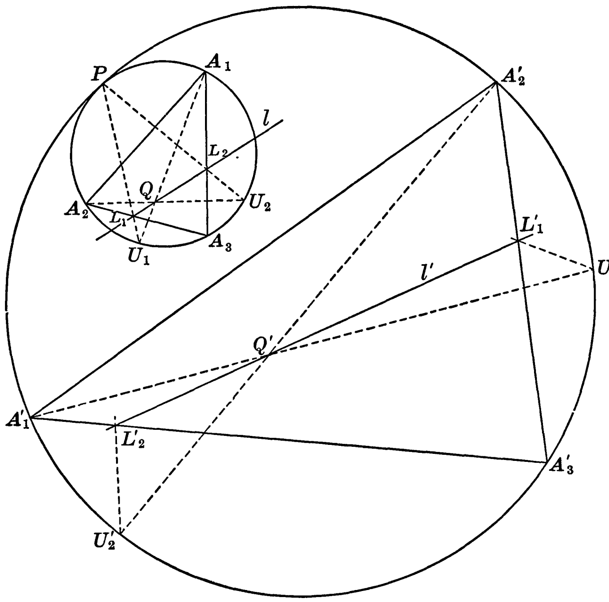

FIG. 1.

In particular, if Q is at the in-center or at one of the ex-centers of T, its reciprocal is the P-transversal of the in-center or the corresponding ex-center of T'.

3. If the point Q is at infinity,[2] it is easily shown by elementary geometry, remembering that A_1U_1, A_2U_2, A_3U_3 are parallel, that $\measuredangle PL_1A_2 = \measuredangle PL_2A_3 = \measuredangle PL_3A_1$. The line is in this case one of the family of lines described by

[1] The notation is that of R. A. Johnson for "directed angles" (this MONTHLY, 1917, p. 101), $\measuredangle BAC$ meaning the positive angle through which the line AB, taken as a whole, must be rotated, to coincide with the line AC, taken as a whole.

[2] This is the case given in Hatton, *loc. cit.*

Poncelet in his *Propriétés Projectives*, 1822, § 468, as generalizations of the Wallace line.[1]

Cor. If l is a transversal cutting the side a_i of a triangle in L_i and if A_iU_i is drawn parallel to it cutting the circumcircle at U_i, then L_1U_1, L_2U_2, L_3U_3 concur at a point P on the circumcircle. Moreover $\measuredangle PL_1A_2 = \measuredangle PL_2A_3 = \measuredangle PL_3A_1$.[2]

Now in the general case, $\measuredangle PL_1A_2 = \measuredangle Q'A_1'P$ and $\measuredangle PL_2A_3 = \measuredangle Q'A_2'P$. Hence, in the special case under consideration, $\measuredangle Q'A_1'P = \measuredangle Q'A_2'P$. Hence Q' is on the circle C', l' being in fact the chord PQ'.

Since the reciprocals of L_1, L_2, L_3 concur at a point on the circle C', the line l touches a parabola, focus P, which touches the sides of the triangle T. Hence Poncelet's lines envelope this parabola.[3]

In particular, if Q' is the point diametrically opposite to P in C', l will evidently be the pedal (or Wallace) line of the point P with respect to the triangle T. That is, if Q is at infinity in the direction of the directrix of the parabola of Steiner, l is the pedal line.

Since, in this last case, Q' and the circumcenter of T' are collinear with P, their reciprocals, the pedal line of P and the P-transversal of H, where H is the orthocenter of T, are parallel, as may easily be proved directly. This P-transversal of H is then the directrix of Steiner's parabola.

Again, if Q is the point isogonally conjugate to P with respect to T, since

$$\measuredangle A_2A_3A_1 = \measuredangle A_2PA_1 = \pi - \measuredangle A_1A_2P - \measuredangle PA_1A_2 = \pi - \measuredangle QA_2A_3 - \measuredangle A_3A_1Q,$$

Q is at infinity.[4]

Now $\measuredangle A_3A_1Q = \measuredangle PA_1A_2 = \measuredangle PU_1A_2$. But $\measuredangle U_1A_1A_3 = \measuredangle U_1A_2A_3$. Moreover, if l is the pedal line, $\measuredangle PU_1A_2$ and $\measuredangle U_1A_2A_3$ are complementary. Hence A_1Q is perpendicular to A_1U_1. But in this case (in fact for any of Poncelet's lines) A_1U_1 is parallel to l. Hence A_1Q is perpendicular to the pedal line of P.[5] Then PQ is the axis of the Steiner parabola when Q is the point isogonally conjugate to P. It follows that the P-transversal of Q in this case, a tangent to the Steiner parabola at infinity, must be the line at infinity. This can be proved independently by showing that in this case PU_i is parallel to a_i.

4. Let A_{12}, A_{34}; A_{13}, A_{24}; A_{14}, A_{23} (Fig. 2), be opposite vertices of a complete quadrilateral. Let P be its Wallace point,[6] common to the circumcircles of the four triangles T_1, T_2, T_3, T_4, where T_1 is $A_{23}A_{34}A_{24}$. It is evident that the polar reciprocal of the quadrilateral with respect to any circle, center P, is a cyclic quadrangle whose vertices L_1', L_2', L_3', L_4' are the reciprocals of the four sides of the quadrilateral. The circle C', circumscribing the quadrangle, passes through P.

[1] Mackay, *Proc. Edin. Math. Soc.*, 1890–1891.

[2] Boyman (*Archiv der Math. u. Phys.*, 1849, 13, p. 364) gives a theorem equivalent to this corollary.

[3] Steiner, Gergonne's *Annales*, XIX, 1828.

[4] Casey, *Sequel to Euclid*, 1886, Ex. 4, p. 167.

[5] Lachlan, *Modern Pure Geometry*, 1893, Ex. 4, p. 68.

[6] Mackay, *loc. cit.*

Now it is easily proved[1] that the 16 in- and ex-centers of the four triangles formed by taking the vertices of the quadrangle in threes lie four by four on four lines parallel to a bisector of the angle between $L_1'L_3'$ and $L_2'L_4'$ and also four by four on four lines perpendicular to these. In the figure, J_1', J_{21}', J_{34}', J_4' are shown lying on one of these lines.

By reciprocation, remembering the fact asserted in the last paragraph of 2, the 16 P-transversals of the in- and ex-centers of the four triangles T_i ($i = 1, 2, 3, 4$), are concurrent in fours in four points which all lie on a line through P

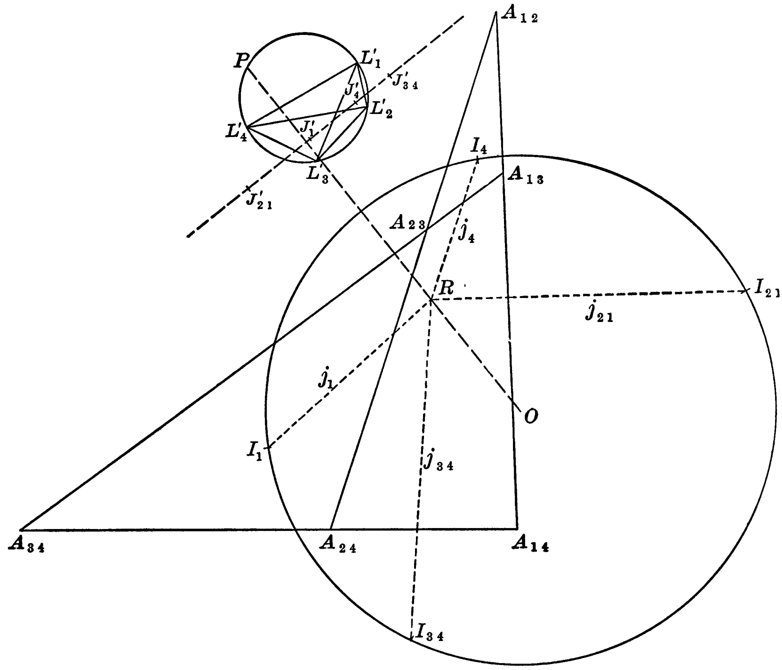

FIG. 2.

bisecting the angle $A_{12}PA_{34}$. One of these points, at which meet the P-transversals of I_1, I_{21}, I_{34}, I_4 is shown in the figure at R.

It may be added that these two lines (PR and the line perpendicular to it at P) are the lines otherwise defined by Steiner,[2] as determined each by the centers of four circles on each of which lie four of the 16 in- and ex-centers of the triangles T. One of these circles, determined by I_1, I_{21}, I_{34}, I_4, Center O, is shown in the figure. The writer has been able to prove this fact only by making use of a long, circuitous proof based on that given by Mention in *Nouv. Ann. de Math.*, 1862, p. 76, proving that the line OP is also a bisector of the angle $A_{12}PA_{34}$. It seems as though a simpler proof might be found.

[1] Neuberg, *Mathesis*, 6, 1906, p. 14. This MONTHLY, 1917, p. 430.

[2] Gergonne's *Annales*, 1828, XVIII, p. 302.

ON AN ELEMENTARY PROBLEM OF CLOSURE ON AN EQUILATERAL HYPERBOLA.

By ARNOLD EMCH, University of Illinois.

In a recent number of the *Nouvelles Annales de Mathématiques*[1] Professor P. Appell proposes the following problem concerning a system of quadruples of points on an equilateral hyperbola:

Let A_1, B_1, C_1, D_1 be four points on an equilateral hyperbola; the altitudes of the triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$ intersect respectively in their ortho-centers A_2, B_2, C_2, D_2, which also lie on the curve. Thus, from the four points A_1, B_1, C_1, D_1, four new points A_2, B_2, C_2, D_2, are derived; in the same manner from these four other points A_3, B_3, C_3, D_3 are obtained, . . . and so forth. Under what conditions does the quadruple $A_nB_nC_nD_n$ coincide entirely, or in part, with $A_1B_1C_1D_1$?

According to Appell's suggestion one may choose $xy = 1$ as the equation of the equilateral hyperbola and represent it parametrically in the form

$$x = e^t, \qquad y = e^{-t}, \tag{1}$$

with t as the parameter. If b_1, c_1, d_1, a_2 are the abscissas of A_1, C_1, D_1 and the orthocenter of $A_1C_1D_1$, respectively, then, as is well known, $a_2b_1c_1d_1 = -1$. Hence, when α_1, β_1, γ_1, δ_1; α_2, β_2, $\cdots$ are the parameters of A_1, B_1, C_1, D_1; A_2, B_2, $\cdots$, we have, when the congruence sign $\equiv$ refers throughout to modulo $2i\pi$ (where $i = \sqrt{-1}$),

$$\begin{aligned} \alpha_{p+1} + \beta_p + \gamma_p + \delta_p &\equiv i\pi, \\ \beta_{p+1} + \gamma_p + \delta_p + \alpha_p &\equiv i\pi, \\ \gamma_{p+1} + \delta_p + \alpha_p + \beta_p &\equiv i\pi, \\ \delta_{p+1} + \alpha_p + \beta_p + \gamma_p &\equiv i\pi. \end{aligned} \tag{2}$$

Putting $S_n = \alpha_n + \beta_n + \gamma_n + \delta_n$, from (2) is found

$$S_{p+1} + 3S_p \equiv 0, \tag{3}$$

and

$$\begin{aligned} \alpha_{p+1} - \alpha_p + S_p &\equiv i\pi, \\ \beta_{p+1} - \beta_p + S_p &\equiv i\pi, \\ \gamma_{p+1} - \gamma_p + S_p &\equiv i\pi, \\ \delta_{p+1} - \delta_p + S_p &\equiv i\pi. \end{aligned} \tag{4}$$

This is as far as Appell's suggestion goes. From (3) and (4) by recurrence

[1] Vol. XVIII, pp. 41–42 (February, 1918).

α_{n+1} may be expressed in terms of α_1 and S_1 as

$$\alpha_{n+1} = \alpha_1 - \frac{1-(-3)^n}{4} S_1 + n' i\pi, \tag{5}$$

where n' is even or odd with n.

Similar formulas are obtained for the parameters β_{n+1}, γ_{n+1}, δ_{n+1} of the points B_{n+1}, C_{n+1}, D_{n+1}. The points A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1} coincide in the same order with A_1, B_1, C_1, D_1, when

$$-\frac{1-(-3)^n}{4} \cdot S_1 + n' i\pi \equiv 0, \tag{6}$$

which, in case of real points, is possible only when either

I $$S_1 = \alpha_1 + \beta_1 + \gamma_1 + \delta_1 \equiv i\pi,$$

or

II $$S_1 = \alpha_1 + \beta_1 + \gamma_1 + \delta_1 \equiv 0.$$

Clearly, in the first case, the quadruple $A_2B_2C_2D_2$ coincides in the same order with $A_1B_1C_1D_1$, and is therefore closed in itself. For case II, from (4) we find $\alpha_2 = \alpha_1 - S_1 + (2\lambda + 1)i\pi$, or $\alpha_2 = \alpha_1 + (2\lambda + 1)i\pi$; similarly

$$\beta_2 = \beta_1 + (2\lambda + 1)i\pi, \qquad \gamma_2 = \gamma_1 + (2\lambda + 1)i\pi, \qquad \delta_2 = \delta_1 + (2\lambda + 1)i\pi.$$

From this follows that the abscissas of A_2, B_2, C_2, D_2 are in the same order opposite in sign to those of A_1, B_1, C_1, D_1, *i. e.*, that the quadrangles $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are congruent and in central symmetry with respect to the center of the hyperbola.

Since the condition for four concyclic points A_1, B_1, C_1, D_1,

$$\begin{vmatrix} e^{2\alpha_1} + e^{-2\alpha_1} & e^{\alpha_1} & e^{-\alpha_1} & 1 \\ e^{2\beta_1} + e^{-2\beta_1} & e^{\beta_1} & e^{-\beta_1} & 1 \\ e^{2\gamma_1} + e^{-2\gamma_1} & e^{\gamma_1} & e^{-\gamma_1} & 1 \\ e^{2\delta_1} + e^{-2\delta_1} & e^{\delta_1} & e^{-\delta_1} & 1 \end{vmatrix}$$

$$= \left\{ e^{-2(\alpha_1+\beta_1+\gamma_1+\delta_1)} - e^{-(\alpha_1+\beta_1+\gamma_1+\delta_1)} \right\} \cdot \begin{vmatrix} e^{3\alpha_1} & e^{2\alpha_1} & e^{\alpha_1} & 1 \\ e^{3\beta_1} & e^{2\beta_1} & e^{\beta_1} & 1 \\ e^{3\gamma_1} & e^{2\gamma_1} & e^{\gamma_1} & 1 \\ e^{3\delta_1} & e^{2\delta_1} & e^{\delta_1} & 1 \end{vmatrix} = 0$$

is evidently satisfied in case II, the original quadrangle $A_1B_1C_1D_1$, and consequently also the quadrangle $A_2B_2C_2D_2$ is concyclic. This follows also from the fact that the product of the abscissas of four concyclic points on the hyperbola (1) is $+1$. From (4) follows further that

$$\alpha_3 = \alpha_1 + 2k\pi, \qquad \beta_3 = \beta_1 + 2k\pi, \qquad \gamma_3 = \gamma_1 + 2k\pi, \qquad \delta_3 = \delta_1 + 2k\pi,$$

so that $A_3B_3C_3D_3$ coincides in the same order with $A_1B_1C_1D_1$. Hence in case II the closed series contains two quadruples. Outside of I and II there are no other cases of closed series of *real* quadruples.

The result of case II may be stated in the

THEOREM: *Four concyclic points A_1, B_1, C_1, D_1 on an equilateral hyperbola form four triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$, whose orthocenters A_2, B_2, C_2, D_2 are also concyclic, and in central symmetry with $A_1B_1C_1D_1$, with respect to the center of the hyperbola. The orthocenters of $B_2C_2D_2$, $C_2D_2A_2$, $D_2A_2B_2$, $A_2B_2C_2$ coincide in the same order with A_1, B_1, C_1, D_1.*

The fact that A_2, B_2, C_2, D_2 are concyclic is independent of the fact that the points $A_1B_1C_1D_1$ are on an equilateral hyperbola. Necessary and sufficient condition is that A_1, B_1, C_1, D_1 are concyclic. It is always possible to pass an equilateral hyperbola through the points of a proper quadrangle. In fact, the pencil of conics through such a quadrangle intersects the line at infinity in an involution. Any finite point joined to this involution determines an involutory pencil, which contains at least one rectangular pair whose directions are parallel to the asymptotes of the corresponding hyperbola of the pencil. This hyperbola is therefore equilateral.

The foregoing theorem concerning a concyclic quadrangle is not new. It was first stated without proof by Steiner in a foot-note.[1] A purely geometrical proof was first given by Heinen.[2] Later on an analytic proof was published by Greiner.[3]

The question whether A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1} may coincide in the same order, say, with $B_1D_1A_1C_1$ must be answered in the negative, if only real solutions are considered. According to (5) and the three similar equations such a condition would give for α_1, β_1, γ_1, δ_1 fractional values of $i\pi$, and, therefore no proper quadrangle on the hyperbola. Coincidences of less than four points may occur, but as they seem not of sufficient geometric interest, they will not be considered in this note.

REMARKS ON A PREVIOUS ARTICLE.

By NATHAN ALTSHILLER, University of Oklahoma.

In connection with my article "On the I-centers of a Triangle"[1] Prof. J. W. Clawson kindly calls my attention to the fact that the propositions (11) and (13) are known. They were proved in 1906 by the well-known Belgian mathematician, Prof. J. Neuberg, of the University of Liege.[5] Prof. Neuberg states in his article that these propositions were published before without proof.[5]

I arrived at these results in the early summer of 1917 while giving a course in "College Geometry" at the University of Oklahoma, Summer Session. The library facilities at my command were inadequate for a satisfactory biblio-

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[1] *Annales de Mathématiques Pures Appliquées*, Vol. 19, p. 43 (1828).

[2] *Journal für die reine und angewandte Mathematik*, Vol. 3, p. 291 (theorem 9) (1828).

[3] *Archiv der Mathematik und Physik*, Vol. 60, p. 184 (1877).

[4] This MONTHLY, June, 1918, pp. 241–246.

[5] *Mathesis*, 1906, pp. 14–17.

[6] These theorems were also discovered independently by J. V. Morley and published as problems in Volume 24 of this MONTHLY, pages 124 and 430.

graphical investigation. It would seem however that these properties are not well known, since just recently the theorem (11) has been proposed for proof in as serious a journal as the *Nouvelles Annales de Mathématiques*.[1]

It is noteworthy that the sides of the rectangle (11) are proved by Prof. Neuberg to be parallel to the bisectors of the angles formed by the diagonals of the quadrilateral; V. Thébault states that these sides are equally inclined on the sides of the given quadrilateral, while in my paper they are shown to be parallel to the lines joining the mid-points of the arcs subtended by the opposite sides of the given quadrilateral on its circumcircle (compare sections 10, 11). Prof. Clawson adds that they are parallel to the bisectors of the angles formed by any pair of opposite sides of the given quadrilateral.

Thus comes to light an interesting and rather involved property of the inscriptible quadrilateral, worthy perhaps of a direct proof.

BOOK NOTICES.

Edited by W. H. Bussey, University of Minnesota.

A circular advertising the new book on *Unified Mathematics* by Karpinski, Benedict and Calhoun says that "particular attention is paid to problems dealing with projectiles, and the 'mil,' the artillery unit of angular measurement, is carefully explained." But the reader will look in vain for the word "mil" in the index of the book; and he will look in vain in the paragraphs on angles and angular measurement where the degree and radian are defined. However if he is bound to know what a "mil" is and searches further he will be rewarded when he finds Ex. 11 on page 114 which reads: "In the artillery service angles are measured in 'mils'; a 'mil' is defined as 1/6,400 of a complete revolution. Compute the value in radians of one mil." Of course all this is in no sense a real criticism of the book, which was not written primarily for men of the S. A. T. C. The book contains 522 pages. It is supposed to be a course in elementary mathematics adapted to the needs of the freshmen students in the ordinary college or technical school course. According to the preface the material includes the work commonly covered in the past in separate courses in college algebra, trigonometry and analytical geometry. But there is no chapter on Permutations, Combinations and the more simple elements of Probability, and there is no mention of these topics in the index. As one might expect from the fact that Prof. Karpinski is known to be interested in the history of mathematics, the book abounds in historical notes and references. The book is published by D. C. Heath and Co.

Unified mathematics seems to be making its way. In addition to the book just mentioned, several other books on correlated mathematics have recently been published or are about to be published.

[1] V. Thébault, *N. A. M.*, August, 1917, p. 319.

The McGraw-Hill Book Co. has published a new and revised edition of the well known *Elementary Mathematical Analysis* written by Professor CHARLES S. SLICHTER of the University of Wisconsin.

Another unified mathematics book is *Freshmen Mathematics* by Professor WILLIAM R. RANSOM, of Tufts College, published by Longmans, Green and Co. It is a book of 285 pages, much less extensive than the two books just mentioned. "The chief features of the book are its brevity, the breadth and simplicity of its methods, its selection of subject matter for utility and interest rather than for mathematical completeness, and the careful preparation of problem material." Among other novel features it contains a short chapter entitled "Trigonometry in three dimensions" which begins with an explanation of the theory of the sun-dial and ends with longitude problems.

Ginn and Company have just published an *Introduction to the Elementary Functions* by R. B. MCCLENON and WILLIAM J. RUSK, of Grinnell College. It is an attempt to solve the problem of the first year collegiate course in mathematics. The idea of functionality is the unifying principle. The last chapter is a 28 page introduction to the differential calculus. There is no work on integral calculus, although the authors "firmly believe that this topic should eventually be included in the first year course."

Ginn and Co. are about to publish a book called *General Mathematics*, by Mr. RALEIGH SCHORLING, of the Lincoln School, Teachers' College, Columbia University, and Mr. W. D. REEVE, of the University High School, University of Minnesota. It is the first of a series of four volumes on mathematics for high school students. The first volume is for first year high school students.

At the annual meeting of the Association in December, 1917, many mathematicians who had never studied descriptive geometry were awakened to an interest in the subject by Professor Roever's paper on descriptive geometry. Those who have not yet followed that inspiration to study the subject may be interested in the fifth edition of *Descriptive Geometry* by W. L. AMES and CARL WISCHMEYER, of Rose Polytechnic Institute, recently published by the McGraw-Hill Book Company. It is a small book of 112 pages.

There has recently been issued by the Bureau of Education at Washington a Bulletin on *The Training of Teachers of Mathematics for the Secondary Schools of the Countries Represented in the International Commission on the Teaching of Mathematics*. This Bulletin has been prepared by Professor R. C. ARCHIBALD, of Brown University. It is a work of nearly three hundred pages, giving in great detail the requirements set by the various governments for a teacher of secondary mathematics. The Bureau of Education has a limited number of copies of this Bulletin which it can send to those who are particularly interested in the work. After this limited number has been exhausted, copies can be obtained from the Superintendent of Documents, Government Printing Office at Washington, D. C., at 30 cents per copy.

Contents—Introduction, pp. 3–4; chapter I: Australia, 5–14; II: Austria, 15–27; III: Belgium, 28–38: IV: Denmark, 39–44; V: England, 45–60, VI: Finland, 61–65, VII: France, 66–

QUESTIONS AND DISCUSSIONS.

Edited by U. G. Mitchell, University of Kansas, Lawrence.

Three questions which, among others, have been standing for some time without having been answered are republished below in the hope that some of our readers may thereby be stimulated to send in suitable replies.

15. In the *Proceedings of the Royal Society of Edinburgh*, Vol. VII, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

(A partial reply to the above has been received showing that if $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$. Can some one show how the "easy proof" then follows?)

21. For the diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$$\begin{array}{lcccccccc} x = & 3, & 4, & 5, & 9, & 23, & 282, & 375, & 378661, \\ y = & -2, & -1, & 2, & 4, & 8, & 43, & 52, & 5234. \end{array}$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

32. In a discussion of the Peaucellier[1] Cell by analytic methods the following equations are obtained:

$$\begin{array}{llll} (1) & (x_2 - x_1)^2 + (y_2 - y_1)^2 - b^2 = 0; & (2) & (x_3 - x_1)^2 + (y_3 - y_1)^2 - b^2 = 0; \\ (3) & (x_2 - X)^2 + (y_2 - Y)^2 - b^2 = 0; & (4) & (x_3 - X)^2 + (y_3 - Y)^2 - b^2 = 0; \\ (5) & x_2^2 + y_2^2 - K^2 = 0; & (6) & x_3^2 + y_3^2 - K^2 = 0; \end{array}$$

$$(7) \quad x_1^2 + y_1^2 - 2cx_1 = 0.$$

The result of eliminating $x_1, y_1, x_2, y_2, x_3, y_3$ gives an equation of the first degree, which establishes that the linkage will trace a straight line. There are various ways of effecting this elimination.

1. What element of the situation is left unused by the following procedure in the elimination?

(*a*) From equations (1), (3), (5) eliminate x_2 and y_2 and obtain an equation

$$(8) \quad f_1(x_1, y_1) = 0.$$

(*b*) From equations (2), (4), (6) eliminate x_3 and y_3 and obtain an equation

$$(9) \quad f_2(x_1, y_1) = 0.$$

(*c*) From equations (7), (8), (9) eliminate x_1 and y_1 and obtain the desired equation.

2. How should this procedure be supplemented to secure the result?

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[1] If reference is made to the article on "Linkages" in the December, 1915, Monthly, by Mr. Leavens, the following coördinates may be applied to his figure: $O(0, 0)$; $C(c, 0)$; $P_1(x_1, y_1)$; $M(x_2, y_2)$; $M_1(x_3, y_3)$; $P_2(X, Y)$.

NEW QUESTION.

36. A number of Discussions have been published in this department relating to cubic and biquadratic equations (cf. Vol. XXXV, p. 29, pp. 268–269 and 343–347; Vol. XXIV, pp. 136–137 and 436–439; Vol. XXIII, pp. 314–315). Below are given a number of questions sent in by Professor Harris Hancock of the University of Cincinnati which relate to the cubic and biquadratic and might, perhaps, more properly be proposed as problems were it not for the advantage to be gained, if possible, by treating them all, or at least several of them, in one discussion.

1. For what values of n can $\cos 2\pi/n$ be expressed in the form $(a + \sqrt{b})/c$ where a, b and c are integers?

2. Write the biquadratic in the form $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$. Show that its reducing cubic may be expressed by means of a determinant of the third order which when expanded is

$$4y^3 - g_2y - g_3 = 0,$$

where $g_2 = ae - 4bd + 3c^2$, and

$$g_3 = \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = ace - ad^2 - eb^2 - c^3 + 2bcd.$$

3. For the same biquadratic show that

$$a^3(x_0 + x_1 - x_2 - x_3)(x_0 + x_2 - x_1 - x_3)(x_0 + x_3 - x_1 - x_2) = 32(3abc - a^2d - 2b^3)$$

without making any use of symmetric functions.

4. If x_0, x_1, x_2 are the roots of a cubic and D its discriminant, show that x_1 is a rational function of x_0 and $\sqrt{D}$. Derive a much simpler relation than that given in Serret, *Cours d'Algèbre Supérieure*, 5th ed., No. 511.

5. If x_0, x_1, x_2, x_3 are the roots of a biquadratic, D its discriminant, e_1, e_2, e_3 the roots of the reducing cubic, show that x_1 is a rational function of x_0, e_1, e_2, e_3 and consequently also of x_0, e_1 and $\sqrt{D}$.

6. If the biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0 \tag{1}$$

has a double root, show that the reducing cubic (cf., for example, Burnside and Panton's *Theory of Equations*)

$$t^3 + 3Ht^2 + \left(3H - \frac{a^2I}{4}\right)t - G^2/4 = 0 \tag{2}$$

has a root in common with the cubic

$$8t^3 + 12Ht^2 + G^2 = 0; \tag{3}$$

and conversely, if (2) and (3) have a common root, then (1) has a double root.

DISCUSSIONS.

Concerning Haversines in Plane Trigonometry.

By G. W. Evans, Charlestown High School, Boston, Mass.

The work for the S. A. T. C. in American colleges has called attention to the use of haversines. It appears not to be generally known that plane trigonometry can be simplified, much as to teaching and somewhat as to computation, by the use of these ratios.

The haversine is defined as half the versed sine; that is,

$$\text{hav } A = \frac{1 - \cos A}{2} = \sin^2 \frac{A}{2}.$$

For the case where a right triangle is given by the hypotenuse c and one side, b, nearly equal to c, there formerly appeared in elementary text-books the formula

$$\sin \tfrac{1}{2}A = \sqrt{\frac{c - b}{2c}}$$

but its application seemed to be not much insisted upon. In the form

$$\text{hav } A = \frac{c - b}{2c}$$

it is an obvious deduction from the definition of cos A and there is no square root to be found.

In oblique triangles the case where two sides and the included angle are given and the case where the three sides are given have formulæ that require a considerable amount of detail in the way of proof. With haversines the proofs would be direct and simple.

From the fundamental formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

we obtain

$$\frac{1 - \cos A}{2} = \frac{a^2 - (b - c)^2}{4bc}.$$

From this equation, for the case where two sides and the included angle are given, we have the formula

I. $$a^2 - (b - c)^2 = 4bc \text{ hav } A.$$

For the case where the three sides are given, we have, from the same equation, the formula

II. $$\text{hav } A = \frac{(s - b)(s - c)}{bc}.$$

The following arrangement of work is suggested:

Example 1. Given $A = 94.39°$, $b = 55.12$, $c = 39.90$.

Formulæ: $a^2 - (b - c)^2 = 4bc \text{ hav } A$; $\sin B = \dfrac{b \sin A}{a}$.

$A = 94° \, 23.4'$.

		Logarithms.			
$b =$	55.12			b	1.7414
$c =$	39.90			c	1.6010
$b - c =$	15.22		1.1824	hav A	9.7310
$(b - c)^2 =$	231.6		2.3648	4	0.6021
$a^2 - (b - c)^2 =$	4737.				3.6755
$a^2 =$	4969.		3.6963		
$a =$	70.50		1.8482		
$A =$	94.39°	sin A	9.9987	sin A	9.9987
$B =$	51.24°	$\frac{1}{a}$	8.1518	$\frac{1}{a}$	8.1518
$C =$	34.36°	b	1.7414	c	1.6010
Check:	179.99°	sin B	9.8919	sin C	9.7515

Example 2. Given $a = 70.50$, $b = 55.12$, $c = 39.90$.

$$\text{Formula: hav } A = \frac{(s-b)(s-c)}{bc}.$$

Logarithms.

$a =$ 70.50	1.8482			$\frac{1}{a}$	8.1518	$\frac{1}{a}$	8.1518
$b =$ 55.12	1.7414	$\frac{1}{b}$	8.2586			$\frac{1}{b}$	8.2586
$c =$ 39.90	1.6010	$\frac{1}{c}$	8.3990	$\frac{1}{c}$	8.3990		
$2s =$ 165.52							
$s =$ 82.76							
$s - a =$ 12.26	1.0885				1.0885		1.0885
$s - b =$ 27.64	1.4415		1.4415				1.4415
$s - c =$ 42.86	1.6321		1.6321		1.6321		
	log hav	A	9.7312	B	9.2714	C	8.9404

Check: $94^\circ\ 25' + 51^\circ\ 13' + 34^\circ\ 21' = 179^\circ\ 59'$.

The work in example 2 very closely resembles the work in spherical trigonometry for the solution of the triangle in which three sides are given. The work in example 1 is of the same character as for the corresponding problem in spherical trigonometry; for example, where the latitude, longitude, and declination are used to compute the altitude for locating a Sumner line. Aside, however, from the advantage that this method of treatment may have in preparing for the work in spherical trigonometry, it has immediate advantage here in the fact that it bases the work in plane triangles on the sine formula for the first two cases, and on the cosine formula for the last two. The use of one or the other of the two haversine formulæ for checking the work in the first two cases makes the check somewhat more laborious than the original computation, but that is no reason why the check should not be insisted upon. On the other hand, the check in the last two cases is simple.

Five place tables of natural haversines and of logarithmic haversines, values for every 15″, are published in Bowditch's *American Practical Navigator* and in publication No. 200 of the U. S. Hydrographic Office; five-place tables for every minute in H. Jacoby's *Navigation* (Macmillan, 1918). The Harvard University Press printed, in the fall of 1918, a four-place table of logarithms of haversines,[1]

[1] Prepared by Mr. Evans, and occupying two of the four pages of a leaflet.

The term haversine was introduced by James Inman (1776–1859) in the third edition (1835) of his *Navigation and Nautical Astronomy for the Use of British Seamen.* A table of logarithms of haversines is there given. Inman was professor of navigation and nautical science in the Royal Naval College, Portsmouth. See *Dictionary of National Biography.* This note constitutes the reply to Question 2311 (by C. Wargny of Valparaiso) in *L'Intermédiaire des Mathématiciens*, April, 1902.—Editor.

from which the entries in the above examples are taken. It is to be hoped that such tables will be included in text-books, and made available at the examinations of the College Entrance Board.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS AND SOLUTIONS TO B. F. FINKEL, Springfield, Mo.

2742. Proposed by S. A. COREY, Des Moines, Iowa.

Establish the following algebraic identity without actually performing the indicated operations:

$$
\begin{aligned}
2(t_1t_2 + c_1t_3t_4 + c_2t_5t_6 + c_1c_2t_7t_8)(r_1r_2 + c_1r_3r_4 + c_2r_5r_6 + c_1c_2r_7r_8)& \\
= (r_1t_1 - c_1r_3t_3 - c_2r_5t_5 + c_1c_2r_7t_7)(r_2t_2 - c_1r_4t_4 - c_2r_6t_6 + c_1c_2r_8t_8)& \\
+ (r_1t_2 - c_1r_3t_4 - c_2r_5t_6 + c_1c_2r_7t_8)(r_2t_1 - c_1r_4t_3 - c_2r_6t_5 + c_1c_2r_8t_7)& \\
+ c_1(r_1t_3 + r_3t_1 - c_2r_5t_7 - c_2r_7t_5)(r_2t_4 + r_4t_2 - c_2r_6t_8 - c_2r_8t_6)& \\
+ c_1(r_1t_4 + r_3t_2 - c_2r_5t_8 - c_2r_7t_6)(r_2t_3 + r_4t_1 - c_2r_6t_7 - c_2r_8t_5)& \\
+ c_2(r_1t_5 + c_1r_3t_7 + r_5t_1 + c_1r_7t_3)(r_2t_6 + c_1r_4t_8 + r_6t_2 + c_1r_8t_4)& \\
+ c_2(r_1t_6 + c_1r_3t_8 + r_5t_2 + c_1r_7t_4)(r_2t_5 + c_1r_4t_7 + r_6t_1 + c_1r_8t_3)& \\
+ c_1c_2(r_1t_7 - r_3t_5 + r_5t_3 - r_7t_1)(r_2t_8 - r_4t_6 + r_6t_4 - r_8t_2)& \\
+ c_1c_2(r_1t_8 - r_3t_6 + r_5t_4 - r_7t_2)(r_2t_7 - r_4t_5 + r_6t_3 - r_8t_1).&
\end{aligned}
$$

By assuming special relations between the constants involved show that the product of the sum of four squares by the sum of four squares equals the sum of four squares.

2743. Proposed by DANIEL KRETH, Wellman, Iowa.

In the right angle triangle ABC, right angle C, we have given on the hypotenuse the segments $AD = 15$, $DE = 10$, and $EB = 15$; and the angle DCE equal to the angle ECB. Find the angle DCE, and the sides AC and BC.

2744. Proposed by J. B. REYNOLDS, Lehigh University.

The vertices of a triangle are $(0, 0)$, $(2a, 0)$, and $(2x, 2y)$. Where are the vertices of the triangle of least area having its vertices on the perpendicular bisectors of the sides of the given triangle and the same center of gravity as the given triangle.

2745. Proposed by C. N. SCHMALL, New York City.

In the parabola, $y^2 = 4ax$, two normals to the curve are drawn at the ends of a focal chord. Show that the area between these normals and the curve is $20a^2/3 \sin^3 2\phi$, where ϕ is the angle between one of the normals and the x-axis.

2746. Proposed by A. CAMPBELL, St. Johnsbury, Vt.

Given the base, the sum of the sides of the triangle, and the difference of the base angles, to construct the triangle.

2747. Proposed by ENOS W. WITMER, Sophomore in Franklin and Marshall College.

Investigate the problem of solving the equation, $x^4 + ay^4 = w^2 + av^2$. Carmichael's *Diophantine Analysis*, problem 18, page 54.

2748. Proposed by ROGER E. MOORE, University of Wisconsin.

Test for convergence, the series $\sum_{n=1}^{\infty} a_n$, in which

$$a_n = \left[\frac{1\cdot 3\cdot 5 \cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n}\right]^2.$$

2749. Proposed by A. M. HARDING, University of Arkansas.

Through a point, P, within a circle, draw a chord which will be trisected at P.

2750. Proposed by J. W. LASLEY, JR., University of North Carolina.

Given $\bar{x} = \tan^{-1}\dfrac{z}{\sqrt{x^2+y^2}}$, $\bar{y} = \tan^{-1}\dfrac{y}{x}$, and $\bar{z} = \log\sqrt{x^2+y^2+z^2}$, solve for x, y, and z in terms of $\bar{x}$, $\bar{y}$, and $\bar{z}$.

2751. Proposed by J. L. RILEY, Stephenville, Texas.

Every number whose square is the sum of the squares of two consecutive integers is equal to the sum of the squares of three integers of which two, at least, are consecutive.

2752. Proposed by PAUL CAPRON, U. S. Naval Academy.

Given a parallelogram with center O, vertices $PQP'Q'$, mid-points of sides $ABA'B'$ (cyclic order $PAQBP'A'Q'B'$). Let K be any point of OA. Draw KLH parallel to $Q'Q$ cutting AQ at L, and draw BLM, meeting OA produced at M. Draw MH, parallel to PP', to meet KLH at H, and draw $B'KE$ to meet BL at E. Repeat, changing A, B, P, Q to A', B', P', Q', respectively, and *vice versa*. Repeat each of the foregoing, changing P, P', B, B' to Q, Q', B', B, respectively, and *vice versa*.

What are the loci of E and H? Show that EH passes through A' when K is a point of OA, through A when K is a point of OA'. Consider the effect of interchanging the rôles of A and B.

(This construction, as commonly given, is specialized in these particulars: the parallelogram is rectangular, the divisions of OA are equal, and the locus of H is not found.)

SOLUTIONS OF PROBLEMS.

385 (Calculus). Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If $f(x)$ is continuous between a and x, show that

$$\lim_{n \doteq \infty} \frac{\underline{|n}}{(x-a)^n} \int_a^x \cdots \int_a^x f(x)dx^n = f(a).$$

SOLUTION BY LOUIS WEISNER, Student, College of the City of New York.

Let $f(x) = \phi^{(n)}(x)$, n being finite.

Hence,

$$\int_a^x f(x)dx = \phi^{(n-1)}(x) - \phi^{(n-1)}(a),$$

$$\int_a^x \int_a^x f(x)dx^2 = \phi^{(n-2)}(x) - \phi^{(n-2)}(a) - (x-a)\phi^{(n-1)}(a),$$

and

$$\int_a^c \int_a^x \int_a^c f(x)dx^3 = \phi^{(n-3)}(x) - \phi^{(n-3)}(a) - (x-a)\phi^{(n-2)}(a) - \frac{(x-a)^2}{2}\phi^{(n-1)}(a).$$

Integrating n times, we have

$$\int_a^c \cdots \int_a^c f(x)dx^n = \phi(x) - \phi(a) - (x-a)\phi'(a) - \frac{(x-a)^2}{\underline{|2}}\phi''(a) - \cdots - \frac{(x-a)^{n-1}}{\underline{|n-1}}\phi^{(n-1)}(a).$$

But

$$\phi(x) = \phi(a) + (x-a)\phi'(a) + \frac{(x-a)^2}{\underline{|2}}\phi''(a) + \cdots + \frac{(x-a)^{n-1}}{\underline{|n-1}}\phi^{(n-1)}(a) + \frac{(x-a)^n}{\underline{|n}}\phi^{(n)}(a)$$
$$+ \frac{(x-a)^{n+1}}{\underline{|n+1}}\phi^{(n+1)}(a) + \frac{(x-a)^{n+2}}{\underline{|n+2}}\phi^{(n+2)}(a) + \cdots.$$

Hence,

$$\int_a^x \cdots \int_a^x f(x)dx^n = \frac{(x-a)^n}{\lfloor n} \phi^{(n)}(a) + \frac{(x-a)^{n+1}}{\lfloor n+1} \phi^{(n+1)}(a) + \frac{(x-a)^{n+2}}{\lfloor n+2} \phi^{(n+2)}(a) \cdots$$

$$= \frac{(x-a)^n}{\lfloor n} f(a) + \frac{(x-a)^{n+1}}{\lfloor n+1} f'(a) + \frac{(x-a)^{n+2}}{\lfloor n+2} f''(a) + \cdots.$$

Hence,

$$\frac{\lfloor n}{(x-a)^n}\int_a^x \cdots \int_a^x f(x)dx^n = f(a) + \frac{x-a}{n+1} f'(a) + \frac{(x-a)^2}{(n+2)(n+1)} f''(a) + \cdots$$

which when n increases without limit, becomes,

$$\underset{n \doteq \infty}{\text{limit}} \frac{\lfloor n}{(x-a)^n}\int_a^x \cdots \int_a^x f(x)dx^n = f(a).$$

425 (Calculus). Proposed by O. S. ADAMS, U. S. Coast and Geodetic Survey, Washington, D. C.

Show that the infinite product

$$(1-z)(1+\tfrac{1}{2}z)(1-\tfrac{1}{3}z)(1+\tfrac{1}{4}z) \cdots = \frac{\sqrt{\pi}}{\Gamma(1+\frac{1}{2}z)(\frac{1}{2}-\frac{1}{2}z)}.$$

Solution by C. F. Gummer, Kingston, Ont.

The right member should evidently have been printed

$$\frac{\sqrt{\pi}}{\Gamma(1+\frac{1}{2}z)\Gamma(\frac{1}{2}-\frac{1}{2}z)},$$

since this equals

$$\frac{\Gamma(\frac{1}{2})}{\Gamma(1+\frac{1}{2}z)\Gamma(\frac{1}{2}-\frac{1}{2}z)},$$

$$= \underset{n\to\infty}{\text{Lim}} \left[\left(\frac{n^{\frac{1}{2}}}{\frac{1}{2}} \prod_{i=1}^{n} \frac{i}{i+\frac{1}{2}} \right) \left(\frac{1+\frac{1}{2}z}{n^{1+\frac{1}{2}z}} \prod_{i=1}^{n} \frac{i+1+\frac{1}{2}z}{i} \right) \left(\frac{\frac{1}{2}-\frac{1}{2}z}{n^{\frac{1}{2}-\frac{1}{2}z}} \prod_{i=1}^{n} \frac{i+\frac{1}{2}-\frac{1}{2}z}{i} \right) \right]$$

$$= \underset{n\to\infty}{\text{Lim}} \left[\frac{(1-z)(1+\frac{1}{2}z)}{n} \prod_{i=1}^{n} \left\{ \left(1-\frac{z}{2i+1}\right)\left(1+\frac{z}{2i+2}\right)\frac{i+1}{i} \right\} \right]$$

$$= \underset{n\to\infty}{\text{Lim}} \left[(1-z)(1+\tfrac{1}{2}z) \prod_{i=1}^{n} \left\{ \left(1-\frac{z}{2i+1}\right)\left(1+\frac{z}{2i+2}\right) \right\} \right]$$

$$= \prod_{i=0}^{\infty} \left\{ \left(1-\frac{z}{2i+1}\right)\left(1+\frac{z}{2i+2}\right) \right\},$$

as required.

440 (Calculus). Proposed by J. B. REYNOLDS, Lehigh University.

If t is the differential vector joining two consecutive points on a space curve, and R is the radius of curvature at that point prove that

$$R^2 = \frac{(t \cdot t)^3}{(t \times \dot{t}) \cdot (t \times \dot{t})}.$$

I. Solution by Louis Weisner, Student, College of the City of New York.

Let $r = f(\tau)$ be the vector equation of the curve, s its length measured from a fixed point, and K the directed curvature.

Then,

$$K = \frac{d^2r}{ds^2} = \frac{\frac{d}{d\tau}\left[\frac{\frac{dr}{d\tau}}{\frac{ds}{d\tau}}\right]}{\frac{ds}{d\tau}} = \frac{\frac{d}{d\tau}\left[\frac{t}{\frac{ds}{d\tau}}\right]}{\frac{ds}{d\tau}},$$

n which

$$\left.\begin{aligned} t &= r' - r_{h \doteq 0} \\ t' &= r'' - r'_{h \doteq 0} \\ t'' &= t' + t \end{aligned}\right\} \tag{4}$$

Expanding (2) to the second powers of h, we find

$$t = ha\left\{f'(u) + \frac{h}{2}f''(u)\right\}i + bh\left\{\varphi'(u) + \frac{h}{2}\varphi''(u)\right\}j + ch\left\{\psi'(u) + \frac{h}{2}\psi''(u)\right\}k_{h \doteq 0},$$

$$t' = ha\{f'(u) + \tfrac{3}{2}hf''(u)\}i + bh\{\varphi'(u) + \tfrac{3}{2}h\varphi''(u)\}j + ch\left\{\psi'(u) + \frac{3h}{2}\psi''(u)\right\}k_{h \doteq 0},$$

$$t'' = 2ha\{f(u) + hf''(u)\}i + 2bh\{\varphi'(u) + h\varphi''(u)\}j + 2ch\{\psi'(u) + h\psi''(u)\}k_{h \doteq 0}.$$

$$t \times t = h^3ab\{f'(u)\varphi''(u) - f''(u)\varphi'(u)\}i + h^3bc\{\varphi'(u)\psi''(u) - \varphi''(u)\psi'(u)\}j + h^3ac\{\psi'(u)f''(u) - \psi''(u)f'(u)\}k_{h \doteq 0}.$$

Substituting these values in (3), we find as $h = 0$

$$R^2 = \frac{(t \cdot t)^3}{(t \times t)\cdot(t \times t)}.$$

In the case of the helix,

$$r = a \cos ui + a \sin uj + cuk; \qquad t \cdot t = (a^2 + c^2)h^2_{h \doteq 0};$$

and

$$t \times t = h^3\{a^2 i + ac \sin uj - ac \cos uk\}_{h \doteq 0}.$$

Hence,

$$R = \frac{a^2 + c^2}{a}.$$

NOTE.—The dots over the t's in the denominator of the problem as it appeared in the MONTHLY are misprints, according to a statement of the Proposer. We are publishing Mr. Weisner's solution of the problem as it appeared in print and also Professor Reynolds's solution of the problem as he wished it to appear.—EDITOR OF PROBLEMS AND SOLUTIONS.

322 (Mechanics). Proposed by FRANK R. MORRIS, Glendale, California.

A pole of uniform size and weight throughout its length stands in a vertical position. The height of the pole is h and its weight w. It hinges at the base and falls, passing through a horizontal position. At the moment it reaches the horizontal position, how far from the base is the maximum vertical force tending to break the pole? How great is this force? What is the horizontal force at the same position in the pole?

I. SOLUTION BY C. F. GUMMER, Kingston, Ontario.

Let the rod coincide initially with the y-axis and later with the x-axis. Let θ be its inclination to the latter at time t; P, Q, and N the radial and transverse stresses and the breaking couple at distance r from the origin.

The equations of motion for the element dr are

(1) $$\frac{\partial P}{\partial r}\cos\theta - \frac{\partial Q}{\partial r}\sin\theta = \frac{wr}{gh}\left(\frac{d}{dt}\right)^2\cos\theta,$$

(2) $$\frac{\partial P}{\partial r}\sin\theta + \frac{\partial Q}{\partial r}\cos\theta = \frac{wr}{gh}\left(\frac{d}{dt}\right)^2\sin\theta + \frac{w}{h},$$

and

(3) $$Q + \frac{\partial N}{\partial r} = 0.$$

The boundary conditions are

(4) $$P = Q = N = 0 \qquad \text{when} \qquad r = h, \text{ and}$$

(5) $$N = 0 \qquad \text{when} \qquad r = 0.$$

The initial condition is

6) $$\frac{d\theta}{dt} = 0 \qquad \text{when} \qquad \theta = \frac{\pi}{2}.$$

By integrating (1) and (2) for r, and using (4), we get

$$P = \frac{w}{2gh}(h^2 - r^2)\left(\frac{d\theta}{dt}\right)^2 - \frac{w}{h}(h - r)\sin\theta,$$

$$Q = -\frac{w}{2gh}(h^2 - r^2)\frac{d^2\theta}{dt^2} - \frac{w}{h}(h - r)\cos\theta,$$

and

$$N = -\int_0^r Q\,dr = \frac{w}{2gh}\left(h^2 r - \frac{r^3}{3}\right)\frac{d^2\theta}{dt^2} + \frac{w}{h}\left(hr - \frac{r^2}{2}\right)\cos\theta.$$

Hence, by (5),

$$2h\frac{d^2\theta}{dt^2} = -3g\cos\theta$$

and hence, by (6),

$$h\left(\frac{d\theta}{dt}\right)^2 = 3g(1 - \sin\theta).$$

It follows that, when $\theta = 0$,

$$P = \frac{3w}{2h^2}(h^2 - r^2), \qquad Q = \frac{w}{4h^2}(h - r)(3r - h), \qquad \text{and} \qquad N = \frac{w}{4h^2}r(h - r)^2.$$

Therefore, the shear, Q, is a maximum for $r = 2h/3$, which makes $P = 5w/6$, $Q = w/12$, and $N = hw/54$.

It may be added that N is a maximum for $r = h/3$, which gives $P = 4w/3$, $Q = 0$, and $N = hw/27$.

This problem is discussed in E. J. Routh's *Elementary Rigid Dynamics*, § 151, 7th edition.

II. Solution by J. B. Reynolds, Lehigh University, South Bethlehem, Pa.

At any instant, let the pole make an angle θ with the vertical and let O be the point about which it is turning. Taking moments about O, we have

$$w\frac{h}{2}\sin\theta = \frac{w}{g}\frac{h^2}{3}\ddot{\theta},$$

or

$$\omega\frac{d\omega}{d\theta} = \ddot{\theta} = \frac{3g}{2h}\sin\theta, \tag{1}$$

whence

$$\omega^2 = \dot{\theta}^2 = \frac{3g}{h}(1 - \cos\theta), \tag{2}$$

ω being the angular velocity of the pole.

When in a horizontal position, let any differential mass of the pole of mass k per unit length be acted upon by a vertical stress force F and a horizontal force T. If this differential of mass is at distance r from O, its accelerations will be $r\ddot{\theta}$ vertically and $r\dot{\theta}^2$ horizontally.

Resolving vertically, we have

$$F + g\,dm = r\ddot{\theta}\,dm, \tag{3}$$

in which $dm = k\,dr$.

Then if M is the moment of the couple tending to break the pole at a point distant u from O, we have

$$M = kg\int_u^h F(r - u) = \int_u^h (r\ddot{\theta} - g)(r - u)\,dm, \text{ by (3), or by (1) for } \theta = 90°,$$

$$M = kg\int_u^h (3r/2h - 1)(r - u)\,dr = w/2h^2\int_u^h (3r - 2h)(r - u)\,dr, \ = w/4h^2\{h^2u + u^3 - 2hu^2\}$$

since $kgh = w$.

For this to be a maximum $dM/du = 0$, or $3u^2 - 4hu + h^2 = (3u - h)(u - h) = 0$; whence $u = 1/3h$, giving the point at which there is the greatest tendency to break. At this point, $M = wh/27$.

Resolving horizontally for a differential mass, we have $T + dT - T = r\dot{\theta}^2 dm$, so that by (2) for $\theta = 90°$,

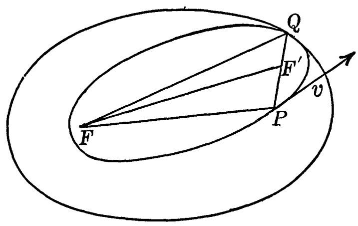
$$T = \frac{3gk}{h}\int_{1/3h}^h r\,dr = \frac{3w}{h^2}\cdot\frac{r^2}{2}\Big]_{1/3h}^h = \tfrac{4}{3}w.$$

328 (Mechanics). Proposed by J. B. REYNOLDS, Lehigh University.

Find the envelope of all possible trajectories when a particle is projected with a constant velocity, v, from a fixed point at a distance a from the center of attraction under the law of gravitation.

I. Solution by the Proposer.

For parabolic or hyperbolic velocities there is evidently no envelope for all directions of projection as any point in space may be reached. For velocities giving elliptic orbits we have



$$v^2 = \mu\left(\frac{2}{a} - \frac{1}{A}\right) \tag{1}$$

in which A is the semi-major axis of the orbit; whence

$$A = \frac{\mu a}{2\mu - av^2} \tag{2}$$

In the figure, let P be the point of projection, F and F' the foci of the path and Q the point where PF' extended cuts the orbit. We will prove that the locus of Q is an ellipse.

Now

$$\begin{aligned} FQ + F'Q &= 2A \\ a \quad + F'P &= 2A. \end{aligned} \tag{1}$$

Adding, we get

$$FQ + F'Q + F'P = 4A - a = \frac{2\mu a + v^2a^2}{2\mu - av^2} = a \cdot \frac{2\mu + av^2}{2\mu - av^2}. \tag{2}$$

or

$$FQ + PQ = \text{const.} \tag{3}$$

Hence, the locus of Q is an ellipse. Furthermore, at Q both ellipses have the same normal so that the inner ellipse does not cross the outer. The ellipsoid of revolution generated by revolving the outer ellipse about FP is the envelope of all the trajectories in question.

If the major axis of the generating ellipse is $2a'$ and its eccentricity e', we have by (2)

$$2a' = a\frac{2\mu + av^2}{2\mu - av^2}. \tag{4}$$

Suppose the particle projected directly away from F. Then since $vdv/dr = -\mu/r^2$, we have, when the particle comes to rest at distance r from F,

$$-v^2 = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right) \quad \text{or} \quad r = \frac{2\mu a}{2\mu - av^2}.$$

This value of r is evidently $a'(1 + e')$; so that

$$a'(1 + e') = \frac{2\mu a'}{2\mu - av^2} \cdots. \tag{5}$$

Now (4) and (5) give

$$e' = \frac{2\mu - av^2}{2\mu + av^2},$$

and the envelope is consequently determined.

II. Solution by C. F. Gummer, Kingston, Ontario.

The origin being the center of attraction, $(a, 0)$ the initial position, and μ/ρ^2 the attraction, the equations of motion will be

(1) $$\ddot{\rho} - \rho\dot{\theta}^2 = -\mu\rho^{-2},$$

and

(2) $$2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} = 0.$$

By (2), $\rho^2\dot\theta = A$, so that (1) gives

$$\ddot\rho - A^2\rho^{-3} = -\mu\rho^{-2},$$

or

$$\dot\rho = \sqrt{-A^2\rho^{-2} + 2\mu\rho^{-1} + B}.$$

The speed is

$$\sqrt{\dot\rho^2 + \rho^2\dot\theta^2} = \sqrt{2\mu\rho^{-1} + B}.$$

Hence,

$$B = v^2 - 2\mu a^{-1}.$$

Hence,

$$\frac{d\rho}{d\theta} = \frac{\dot\rho}{\dot\theta} = \rho^2 A^{-1}\sqrt{v^2 - 2\mu a^{-1} + 2\mu\rho^{-1} - A^2\rho^{-2}}$$

and therefore,

$$\theta = -\int_{a^{-1}}^{\rho^{-1}} A^{-1}\sqrt{v^2 - 2\mu a^{-1} + 2\mu z - A^2 z^2}\, dz;$$

that is,

$$\rho^{-1} = \mu A^{-2} - (\mu A^{-2} - a^{-1})\cos\theta + \sqrt{v^2 A^{-2} - a^{-2}}\cdot\sin\theta,$$

which is the equation of the trajectory.

Writing this in the form,

$$\mu^2 a^2(1-\cos\theta)^2 + \{2\mu a(1-\cos\theta)(\cos\theta - a\rho^{-1}) - a^2v^2\sin^2\theta\}A^2 + \{(\cos\theta - a\rho^{-1})^2 + \sin^2\theta\}A^4 = 0,$$

and applying the condition for equal roots in A^2, we get for the envelope

$$\frac{4\mu a^2v^2}{4\mu^2 - a^2v^4}\frac{1}{\rho} = 1 - \frac{2\mu - av^2}{2\mu + av^2}\cos\theta,$$

the equation of an ellipse.

349 (Mechanics). Proposed by S. A. COREY, Albia, Iowa.

A 9-pound weight is attached to a string which passes over a smooth fixed pulley. The other end of the string is fastened to and supports a smooth pulley P_1 of weight 1 pound over which passes a second string, one end attached to a 3-pound weight and the other end attached to and supporting another smooth pulley P_2 of weight 1 pound. Over the pulley P_2 passes a third string supporting weights 2 pounds and $3\frac{1}{3}$ pounds.

If the system is acted upon by gravity alone show that the acceleration of the 9-pound weight, $3\frac{1}{3}$-pound weight, and pulley P_2 are 0, $\frac{1}{2}g$, and $\frac{1}{3}g$, respectively.

Determine the motion of the weights when pulleys are not smooth, that is, when friction is present.

II. Solution by William Hoover, Columbus, Ohio.

Calling the fixed pulley P and taking it as the origin of coördinates, we have for the x-coordinate of m_1, x_1; of P_1, $l_1 - x_1$; of m_3, $x_2 + l_1 - x_1$; of P_2, $l_1 + l_2 - (x_1 + x_2)$; of m_4, $x_3 + l_1 + l_2 - (x_1 + x_2)$; of m_5, $l_1 + l_2 + l_3 - (x_1 + x_2 + x_3)$, where m_1, m_3, m_4 and m_5 are the masses of the 9-, 3-, 2-, and $3\frac{1}{3}$-pound weights, respectively.

Under the hypothesis that there is no friction, the equation of motion of system is

$$\begin{aligned} T = \tfrac{1}{2}m_1\dot x_1^2 + \tfrac{1}{2}P_1\dot x_1^2 + \tfrac{1}{2}m_3(\dot x_2 - \dot x_1)^2 + \tfrac{1}{2}(\dot x_1 + \dot x_2)^2 + \tfrac{1}{2}m_4(\dot x_3 - \dot x_1 - \dot x_2)^2 + \tfrac{1}{2}m_5(\dot x_1 + \dot x_2 \\ + \dot x_3)^2 = m_1 g x_1 + P_1 g(l_1 - x_1) + m_3 g(x_2 + l_1 - x_1) + P_2 g\{l_1 + l_2 - (x_1 + x_2)\} \\ + m_4 g\{l_1 + l_2 + x_3 - (x_1 + x_2)\} + C = V. \qquad (i) \end{aligned}$$

Using Lagrange's equations of type

$$\frac{d}{dt}\frac{dT}{d\dot x} - \frac{dT}{dx} = \frac{dV}{dx} \qquad (ii)$$

there are

$$\begin{aligned} &(a) \quad \tfrac{58}{3}\ddot x_1 + \tfrac{10}{3}\ddot x_2 + \tfrac{4}{3}\ddot x_3 = g(m_1 - P_1 - m_3 - P_2 - m_4 - m_5) = -\tfrac{4}{3}g, \\ &(b) \quad \tfrac{10}{3}\ddot x_1 + \tfrac{28}{3}\ddot x_2 + \tfrac{4}{3}\ddot x_3 = g(m_3 - P_2 - m_4 - m_5) = -\tfrac{10}{3}g, \\ &(c) \quad \tfrac{4}{3}\ddot x_1 + \tfrac{4}{3}\ddot x_2 + \tfrac{16}{3}\ddot x_3 = g(m_4 - m_5) = -\tfrac{4}{3}g, \end{aligned}$$

giving $\ddot x_1 = 0$, $\ddot x_2 = -\dfrac{g}{3}$, $x_3 = -\dfrac{g}{6}$.

[In the previously printed solution nothing is given to show *how* the *literal* terms on right of (a), (b), (c) arise.]

In case of friction of the movable pulleys, account generally would have to be taken of the *vis viva* of rotary motion of P_1 and P_2, there being, then, two additional terms in the energy equation $T = V$, namely,

$$\frac{k_1^2}{2}P_1\dot{\varphi}_1^2 \quad \text{and} \quad \frac{k_2^2}{2}P_2\dot{\varphi}_2^2,$$

k_1 and k_2 being the radii of gyration of the pulleys P_1, P_2, supposed to be cylinders of radii, say, r_1, r_2, and so $k_1^2 = r_1^2/2$, $k_2^2 = r_2^2/2$, and $\dot{\varphi}_1$, $\dot{\varphi}_2$, the angular velocities of P_1, P_2.

There were not stated any dimensions of P_1, P_2; permitting us to take r_1 and r_2 indefinitely small. This would, though, be a special condition.

In taking r_1, r_2 of finite value, the energy equation would be

$$T = \tfrac{1}{2}(m_1 + P_1 + m_3 + P_2 + m_4 + m_5)\dot{x}_1^2 + \text{etc.} + (- m_3 + P_2 + m_4 + m_5)\dot{x}_1\dot{x}_2 + \text{etc.}$$
$$+ \tfrac{1}{2}k_1^2\dot{\varphi}_1^2 + \tfrac{1}{2}k_2^2\dot{\varphi}_2^2 = m_1gx_1 + \text{etc.} + C = V. \quad (d)$$

Regardless of signs, $r_1\dot{\varphi}_1 = \dot{x}_1$, $r_2\varphi_2 = x_2$; and substituting $\dot{\varphi}_1$, $\dot{\varphi}_2$ from these equations in (d), we have an equation differing in form from (i), but to which (ii) may be applied as before, and generally with different values for $\ddot{x}_1$ etc.

So far the fixed pulley has been thought of as smooth. If this pulley be regarded as rough, the energy equation would still further be modified so as to take account of work done against friction, or of energy due to its rotation and some assigned mass

350 (Mechanics). Proposed by J. B. REYNOLDS, Lehigh University.

If an elastic tube filled with liquid under pressure doubles in length in what ratio will the radius be increased?

SOLUTION BY THE PROPOSER.

Let the initial length and radius of the tube be l_0 and r_0, the final lengths l and r and the coefficient of elasticity of a strip of the material one unit wide be λ.

If P is the pressure per square unit we have, since the length is doubled,

$$P\pi r^2 = 2\pi r\frac{\lambda}{l_0}l_0; \qquad \text{whence} \qquad P = \frac{2\lambda}{r}. \quad (1)$$

Again, if T is the peripheral tension per unit length, in order to balance the internal pressure

$$T = \lim_{\Delta\theta \doteq 0}\frac{Pr\Delta\theta}{2\sin\Delta\theta/2} = Pr. \quad (2)$$

Also,

$$T = \frac{\lambda}{2\pi r_0}(2\pi r - 2\pi r_0) = \frac{\lambda}{r_0}(r - r_0). \quad (3)$$

By (1), (2), and (3) we get,

$$Pr = 2\lambda = \frac{\lambda}{r_0}(r - r_0);$$

whence $r = 3r_0$.

356 (Mechanics). Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore, Md.

A ray of light enters a prism having vertex angle α. If the angle between the incoming and outgoing directions is defined as the angle of deviation, at what angle must the ray enter the prism in order that the angle of deviation may be a minimum?

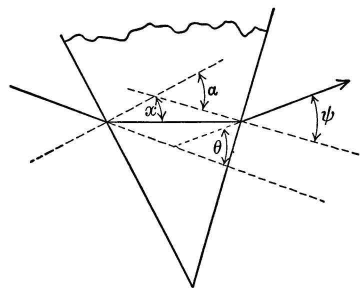
SOLUTION BY J. B. REYNOLDS, Lehigh University.

We have

$$\sin\varphi = \epsilon\sin x, \qquad \sin\psi = \epsilon\sin(\alpha - x),$$

ϵ being the index of refraction. If θ is the angle of deviation, we have, by geometry, $\theta = \psi + \varphi - \alpha$ or

$$\theta = \sin^{-1} \epsilon \sin (\alpha - x) + \sin^{-1} \epsilon \sin x - \alpha.$$



For θ to be a minimum

$$\frac{d\theta}{dx} = \frac{-\epsilon \cos (\alpha - x)}{\sqrt{1 - \epsilon^2 \sin^2 (\alpha - x)}} + \frac{\epsilon \cos x}{\sqrt{1 - \epsilon^2 \sin^2 x}} = 0;$$

from which $x = \frac{1}{2}\alpha$ and therefore $\varphi = \psi = \sin^{-1} \epsilon \sin \frac{1}{2}\alpha$. That is, the angle of deflection or deviation is a minimum when the ray in the glass makes equal angles with the faces of the prism.

357 (Mechanics). Proposed by J. B. REYNOLDS, Lehigh University, South Bethlehem, Pa.

Two beads each of mass m connected by a string of length $2l$ and carrying a mass m' at its middle point are threaded symmetrically with respect to the major axis which is vertical on a smooth ellipse of eccentricity e and latus rectum $2l$. The string is held taut and horizontal, then released; find the velocities of the beads when the end ones impinge.

SOLUTION BY THE PROPOSER.

There are two cases: I, when the end beads are at the extremities of the upper latus rectum II, when the end beads are at the extremities of the lower latus rectum. In either case when the end beads impinge their velocities will be equal and the velocity of m' will be zero. If a is the semi-major axis we have by the principle of work,

Case I

$$m'g\{l - a(1 - e)\} - 2mga(1 - e) = \frac{2m}{2} v^2;$$

or since

$$l = a(1 - e^2),$$

$$mv^2 = ag(1 - e)(m'e - 2m);$$

or

$$v^2 = \frac{gl}{m(1 + e)} \{m'e - 2m\}.$$

For the beads to impinge in this case $m' > 2m/e$.

Case II

$$m'g\{l + a(1 - e)\} + 2mga(1 - e) = \frac{2m}{2} v^2,$$

whence as before

$$v^2 = \frac{gl}{m(1 + e)} \{(2 + e)m' + 2m\}.$$

260 (Number Theory). Proposed by ALBERT A. BENNETT, University of Texas.

Let $\binom{n}{r}$ denote, as usual, the binomial coefficient $n!/[r!(n - r)!]$, where $\binom{n}{0} = 1$, but where n, r, $(n - r)$ are always to be supposed to be positive integers or zero. Let us define $k_i(m, n)$ as $\Sigma_i \binom{m-i+j}{i-j}\binom{n-j}{j}$. Prove that the following recursion formula is consistent:

$$\Sigma(-1)^i k_i(m, n) C_{m+n-i} = \binom{m+n}{m}$$

and determine $C_0 = 1, C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, C_5 = 42, C_6 = 132, C_7 = 429, C_8 = 1{,}430$, etc. Prove also that these quantities satisfy the following relations, as well:

$$\sum_i (-1)^i C_{m-n-i} \binom{m-1}{i} = 0$$

for each n where $2n \leq m$.

SOLUTION BY C. F. GUMMER, Kingston, Ontario.

Consider the sum $\Sigma_i \Sigma_m \Sigma_n (-1)^i k_i(m, n)x^m y^n z^i$, where x, y, z are small enough to ensure absolute convergence. It equals $\Sigma_i \Sigma_j \Sigma_m \Sigma_n (-1)^i \binom{m-i+j}{i-j}\binom{n-j}{j} x^m y^n z^i$. It is not difficult to sum this in the order of writing, beginning with n. The result is that $(-1)^i k_i(m, n)$ is the coefficient of $x^m y^n z^i$ in $(1 - x + x^2 z)^{-1}(1 - y + y^2 z)^{-1}$, as may also be directly verified. It will then be sufficient to show that a power series $F(z) = C_0 + C_1 z + C_2 z^2 + \cdots$, independent of x and y can be found so that the coefficient of $x^m y^n z^{m+n}$ in $F(z)\cdot(1 - x + x^2 z)^{-1}(1 - y + y^2 z)^{-1}$ equals the corresponding coefficient in $\Sigma_m \Sigma_n \binom{m+n}{m} x^m y^n z^{m+n}$, that is, in $(1 - xz - yz)^{-1}$. This does not mean that these generating functions are to be identical, since the first contains terms not of the form $Ax^m y^n z^{m+n}$. To escape this difficulty, write $x = u/z$, $y = v/z$. Then we have to find $F(z)$ so that the part independent of z in $F(z)\cdot(1 - u/z + u^2/z)^{-1}(1 - v/z + v^2/z)^{-1}$ is identical with $(1 - u - v)^{-1}$, remembering that we are now to have positive and negative powers of z in the expansion of the former.

But

$$(1 - u/z + u^2/z)^{-1}(1 - v/z + v^2/z)^{-1} = (u - v)^{-1}(1 - u - v)^{-1}\{(1 - u/z + u^2/z)^{-1} - (1 - v/z + v^2/z)^{-1}\};$$

so that, on expanding this in negative powers of z, and multiplying by $F(z)$, we find that $\Sigma_r C_r\{(u - u^2)^{r+1} - (v - v^2)^{r+1}\}$ must be identical with $u - v$. This can be satisfied only by having

$$\sum_r C_r(u - u^2)^{r+1} = u + A,$$

that is by making $zF(z) - A$ the expansion of u in ascending powers of z where $u^2 - u + z = 0$. Therefore,

$$zF(z) = A + \tfrac{1}{2} \pm \tfrac{1}{2}(1 - 4z)^{1/2};$$

whence,

$$A = -\tfrac{1}{2} \mp \tfrac{1}{2},$$

and

$$F(z) = \pm\{1 - (1 - 4z)^{1/2}\}/(2z);$$

and the upper sign must be taken, since that alone makes $u + v < 1$, which is necessary for the convergence of $(1 - u - v)^{-1}$. It follows that $C_r = 2(2r - 1)!/[(r - 1)!(r + 1)!]$ when $r > 0$, and $C_0 = 1$. The numerical values may then be found, as stated.

As regards the second part of the problem, we observe that $\Sigma_i (-1)^i \binom{m-i}{i} C_{m-n-i}$ is the coefficient of $x^m z^{m-n}$ in the expansion of $F(z)\cdot\{1 - (x - zx^2)\}^{-1}$ for small x and z; and by using partial fractions in terms of x, we find that this coefficient is the coefficient of z^{m-n} in

$$\{1 - (1 - 4z)^{1/2}\}(2z)^{-1}\cdot[\{\tfrac{1}{2} + \tfrac{1}{2}(1 - 4z)^{1/2}\}^{m+1} - \{\tfrac{1}{2} - \tfrac{1}{2}(1 - 4z)^{1/2}\}^{m+1}](1 - 4z)^{-1/2}.$$

The second of the terms in square brackets has no powers of z below z^{m+1}, and may be omitted. The remaining part reduces to $\{\tfrac{1}{2} + \tfrac{1}{2}(1 - 4z)^{1/2}\}^m(1 - 4z)^{-1/2}$. For a similar reason, for powers of z below z^m, this may be replaced by

$$[\{\tfrac{1}{2} + \tfrac{1}{2}(1 - 4z)^{1/2}\}^m - \{\tfrac{1}{2} - \tfrac{1}{2}(1 - 4z)^{1/2}\}^m](1 - 4z)^{-1/2} = 2^{-m+1}\left\{\binom{m}{1} + \binom{m}{3}(1 - 4z) + \binom{m}{5}(1 - 4z)^2 + \cdots\right\}$$

which has a degree less than $m/2$. The coefficient considered therefore vanishes when $0 < 2n \leq m$. It is equal to 1 when $n = 0$, since

$$\sum_i (-1)^i \binom{m-i}{i} C_{m-i} = \sum_i (-1)^i \binom{m-1-i}{i} C_{m-1-i} + \sum_i (-1)^i \binom{m+1-i}{i} C_{m-i}$$

$$= \sum_i (-1)^i \binom{m-1-i}{i} C_{m-1-i} = \sum_i (-1)^i \binom{m-2-i}{i} C_{m-2-i} = \cdots$$

$$= \sum_i (-1)^i \binom{1-i}{i} C_{1-i} = 1.$$

NOTES AND NEWS

Edited by E. J. Moulton, Northwestern University, Evanston, Illinois.

The editor will appreciate the coöperation of readers in reporting news to him concerning teachers of collegiate mathematics and others interested in advanced mathematics.

Dr. H. A. Kingston has been promoted to an assistant professorship of mathematics and astronomy at the University of Manitoba.

Professor H. M. Showman, of the Colorado School of Mines, is studying at Harvard University as Shattuck Scholar, holding also an appointment as lecturer on physics.

Assistant Professor L. A. H. Warren has been appointed acting professor of mathematics and astronomy at the University of Manitoba, in the absence of Professor N. B. MacLean, who is major in the Canadian artillery service in France.

Professor Clyde S. Atchison, of Washington and Jefferson College, has returned to his college work after spending the summer as cost accountant and auditor in the service of the U. S. Shipping Board at Bath, Maine.

Dr. Artemas Martin, computer in the Division of Tides in the U. S. Coast and Geodetic Survey, a charter member of the Association, and well known mathematician, died in Washington City on November 7, in the eighty-fourth year of his life. Until fifty years of age, he devoted his time to wood-chopping, oil-well drilling, farming and teaching (four winters), his leisure time being spent in mathematical study. In 1878 he began the publication of the *Mathematical Visitor*, of which the last number, volume 2, number 4, dated January, 1894, appeared in October, 1895. This periodical was followed, in 1882, by the *Mathematical Magazine* which continued at irregular intervals to volume 2, number 12, part 2, September, 1910. Dr. Martin was not a college graduate but his mathematical attainments secured for him honorary academic distinction. Yale conferred an honorary A.M. upon him in 1877, Rutgers granted him the Ph.D. in 1882, and Hillsdale College the LL.D. in 1885. Dr. Martin possessed one of the finest private collections of mathematical books in America; this collection was given to the American University, shortly before his death, to found the Artemas Martin Library. He also endowed the Artemas Martin Lectureship in mathematics and physics at the same university.

Lieutenant W. L. Hart of the Coast Artillery has been advanced to the rank of major and is serving in France.

Dr. C. H. Yeaton has been appointed instructor in mathematics at the University of Minnesota.

Dr. M. G. Gaba, of Cornell University, has been appointed associate professor of mathematics at the University of Nebraska.

At the University of Saskatchewan, Assistant Professor L. L. Dines has been promoted to a full professorship of mathematics.

Lieutenant T. R. Hollcroft has been released from the teaching staff of the school of artillery at Camp Taylor, and has resumed the duties of his professorship of mathematics at Wells College. During his absence the position was filled by Mrs. Hollcroft.

At the University of Missouri, Mr. F. Duncan, Miss Z. Ferguson, and Mr. A. Grossman have been appointed instructors in mathematics.

Professor E. R. Hedrick, of the University of Missouri, has recently sailed for France, where he is to take charge of mathematical work for American soldiers in the Y. M. C. A. educational system.

At the University of Illinois, Dr. Elizabeth B. Grennan and Dr. Josephine Glasgow have been appointed instructors in mathematics and Mr. L. E. Yaeger assistant in mathematics. Mr. C. H. Richardson, assistant in mathematics, has been appointed professor of mathematics in Georgetown College, Kentucky.

At Cornell University, Mrs. D. Naylor, Mrs. Helen B. Owens, and Mr. H. L. Smith have been appointed instructors in mathematics. Instructor E. P. Fraleigh resigned to engage in the ordnance work at the Aberdeen Proving Ground. Seven members of the faculty were temporarily transferred from other departments to assist in the mathematical instruction last autumn.

Mr. H. W. Powell, tutor in mathematics in the College of the City of New York, died July 23, 1918.

Professor H. G. Keppel, head of the department of mathematics in the University of Florida, died October 5, at the age of fifty-two years. Dr. T. M. Simpson, of the University of Wisconsin, has been appointed his successor as professor and head of the department.

Col. E. W. Bass, professor of mathematics at West Point from 1878 to 1898, died November 6 at the age of seventy-six years.

Dr. R. M. Winger, of the University of Oregon, has been appointed professor of mathematics at the University of Washington.

Dr. A. H. Norton, professor of mathematics at Elmira College, has returned from a year's work in the Y. M. C. A. with the American Army in France.

In *The Scientific Monthly* we read of the death of the astronomer A. N. SKINNER, professor of mathematics in the U. S. Navy from 1898 to the time of his retirement from active service, with rank of Commander U. S. N., in 1907.

Professor C. CARATHÉODORY, of the University of Göttingen, has been appointed professor of mathematics at the University of Berlin, as successor to the late Professor G. FROBENIUS.

The Rev. Dr. G. M. SEARLE, superior general of the Paulist Fathers from 1904 to 1909, and previously professor of mathematics and director of the astronomical observatory of the Catholic University at Washington, died on July 8, 1918 at the age of 79 years.

Mr. ARTHUR RAMSEY, of the department of mathematics in Grove City College has been promoted to an assistant professorship.

Mr. J. W. BALDWIN, who has been an instructor in mathematics in Michigan State Normal College, has been appointed to a like position in Detroit Junior College.

Professor OTTO HÖLDER has been chosen rector of the University of Leipsic.

Mr. HERMAN BETZ, formerly of the University of Michigan, instructed S. A. T. C. classes at Hobart College, after having served in the ordnance department of the army.

Professor H. L. SLOBIN, of the University of Minnesota, was engaged in Jewish welfare work in the various army camps of the United States.

Dr. GUY W. SMITH, of Beloit College, has been appointed instructor in mathematics at the University of Kentucky.

Professor OTTO STAUDE, of the University of Rostock, has been chosen rector of the University for the year 1918–19.

Professor F. B. WILLIAMS has returned to his work in Clark University after a year spent in Y. M. C. A. work in France.

Among a party of college and university men taking the officers' course in coast artillery at Fort Monroe the past summer were the following members of the Association: Professor J. J. LUCK, of the University of Virginia, H. B. PHILLIPS, of the Massachusetts Institute of Technology and D. T. WILSON, of Case School of Applied Science.

Professor D. E. SMITH, of Columbia University, was one of the two principal speakers at the annual meeting of the Association of Teachers of Mathematics

in New England, held in Boston, December 7, 1918. At this meeting Professor W. R. RANSOM, of Tufts College, was elected President of the Association. On February 7–8, Professor Smith addressed the Association of Teachers of Secondary Mathematics of North Carolina in Greensboro, N. C. He is to speak before the Rhode Island Circle of Mathematics Teachers at Providence, R. I., March 15.

At the recent election Professor D. A. ROTHROCK, of Indiana University, was elected a member of the State Legislature. He is relieved of duties at the University during the legislative session beginning in January.

The following deaths are announced: Professor E. R. NEOVIUS, of the University of Helsingfors, on September 26, 1917, at the age of sixty-seven years; Professor J. H. GRAF, of the University of Bern, at the age of sixty-six years; Professor C. STEPHANOS, of the University of Athens, at the age of sixty years; and Dr. O. REICHEL, professor of mathematics at the Landwirtschaftliche Hochschule, Berlin, at the age of eighty-three years.

Professor MAX NOETHER, of the University of Erlangen, who is now seventy-five years of age, was relieved from lecturing after April 1, 1918.

Professor A. v. BRILL, of the University of Tübingen is also about to retire after fifty-one years of teaching.

The University of Frankfort has recently conferred the degree of honorary doctor on Professor L. KÖNIGSBERGER, of the University of Heidelberg, who was characterized as "the highly deserving and many sided investigator, the brilliant and model teacher, the intelligent interpreter of the scientific life-work of Helmholtz and Jacobi."

Dr. G. M. GREEN, instructor in mathematics at Harvard University, and one of the most promising young geometers of the country, died at Cambridge on January 25, in the twenty-ninth year of his age. He graduated from College of the City of New York, B.Sc. 1911, Columbia University, M.A. 1912, Ph.D. 1913. His thesis was entitled *Projective differential geometry of triple systems of surfaces* (Lancaster, Pa., 1913, 28 pp.). His mathematical papers appeared in: *Transactions of the American Mathematical Society*, 1914–17; *Bulletin of the American Mathematical Society*, 1917–18; *Annals of Mathematics*, 1918; and *Proceedings of the National Academy of Sciences*, 1915–18.

At Wesleyan University C. L. STEARNS has been appointed instructor in mathematics and assistant in the Van Vleck observatory.

Professor E. G. BILL, on leave of absence from Dartmouth College, has recently been appointed Assistant to Director of Military Service Branch, Department of Justice, Ottawa.

Major WILLIAM D. MACMILLAN has received his discharge from the Ordnance Department and has resumed his duties as associate professor of astronomy at the University of Chicago.

Professor W. R. LONGLEY, of Yale University, has been on leave of absence, engaged in the ballistic division of the Dupont Company in Delaware.

Professor M. J. GOLDEN, emeritus professor of practical mechanics in Purdue University, and, since 1884, a member of the faculty, died on December 18, aged fifty-eight years.

To the list (published in this MONTHLY last September) of those who received the Doctorate in Mathematics in American universities, should be added the name of C. P. PAINE, Wisconsin. The title of his thesis was: "Modes of air motion and the equations of the general circulation of the earth's atmosphere."

Professor E. J. OGLESBY, after having served as instructor in gunnery, orientation and matériel at the Coast Artillery Training Camp, Fort Monroe, Va., with the rank of captain, resumed on January first the chair of mathematics at William and Mary College.

New instructors in mathematics at Northwestern University this year are Miss JESSICA M. YOUNG, Mr. T. DOLL, Dr. M. G. SMITH, and Mr. P. E. HEMKE. During the fall term five new assistants were also teaching mathematics. During last year Professor C. H. YEATON, Dr. C. E. WILDER, Mr. A. D. CAMPBELL and Mr. I. ROMAN left the department for military service.

At the University of Illinois, Assistant Professor J. E. MCATEE, of the mathematics department, died of pneumonia, following an attack of influenza, on December 1. His place has been filled by the appointment of Dr. F. W. REED, formerly connected with the astronomy department, and more recently with the instruction in the ground aviation school at the University. Mr. JOSEPH ROSENBACH, assistant and graduate student in mathematics, is just recovering from a severe attack of pneumonia and will be absent on sick leave until April 1.

Professor WARREN WEAVER, of Throop College of Technology, has been promoted to second lieutenant in the air service; he was recently engaged in experimental research work in the Science and Research Division.

From *Science* we learn that the departments of descriptive geometry and mechanical drawing and of mechanism and machine design at Stevens Institute of Technology have been combined to form a new department of machine design, of which FRANKLIN DER. FURMAN is professor and head. LEWIS E. ARMSTRONG, instructor at the Institute has been promoted to be an assistant professor of mathematics.

At the University of Chicago, the Students Army Training Corps, together with the Naval Unit and the Naval Auxiliary special students from the Municipal pier, numbered about two thousand. The surveying classes included about 350 students and there were about 600 students in navigation and trigonometry classes.

During November and December Professor J. W. YOUNG was on leave of absence from Dartmouth College, while serving as director of instruction in mathematics for the National War Work Council of the Young Men's Christian Associations of the United States. Professor F. M. MORGAN administered the affairs of the department of mathematics during his absence.

Professor PAUL HEEGAARD, of the University of Copenhagen has been appointed professor of mathematics at the University of Christiania on the understanding that he should lecture on geometry. His inaugural address, delivered in January, 1918, was entitled "Topics in the history of geometry."

Last autumn Professors E. H. MOORE, W. B. FITE, R. C. ARCHIBALD, H. G. KEPPEL and W. A. MANNING were appointed by the National War Work Council of the Young Men's Christian Association as a national committee to make a survey of the mathematical instruction given under the auspices of the Y. M. C. A. at the various Naval Stations, representing respectively the Central, Eastern, Northeastern, Southeastern and Western Departments. The object of this committee, working under the chairmanship of Professor G. A. MILLER, was to report on mathematical instruction given at these stations and to make suggestions tending to improving the instruction. As one result of reports thus made, a series of text-books in elementary mathematics is being prepared.

Commissions as second lieutenants were given to the following university teachers of mathematics, at Camp Zachary Taylor in November: Mr. R. W. BARNARD and Professor T. H. HILDEBRANDT, Michigan; Dr. L. R. FORD, Harvard; Mr. J. D. ESHLEMAN, Rochester; Dr. T. R. HOLLCROFT, Columbia; Dr. C. E. WILDER, Northwestern; Mr. H. E. WOLFE, Indiana.

Mr. Frank E. WOOD, of the meteorological section, Division of Science and Research, Signal Corps, and former professor of mathematics at the University of New Mexico, has been in charge of a group of computers engaged in the study of meteorological questions in the theory of exterior ballistics at the Aberdeen Proving Ground.

Two meetings of an Inter-Allied Scientific Conference have been held to consider international coöperation in science. The first meeting was in London under the auspices of the Royal Society, the second in Paris under the auspices of the Academy of Sciences of the Institute of France. The American delegates were: Dr. H. A. BUMSTEAD, Colonel J. J. CARTY, Professor W. F. DURAND, Dr. SIMON FLEXNER, Dr. GEORGE E. HALE, and Professor A. A. NOYES.

At Brown University the following appointments were made for the year 1918–19: Dr. A. B. FRIZELL as lecturer in mathematics, and Mr. T. A. CORNELL and Mr. W. R. BURWELL as instructors in mathematics. Mr. C. R. ADAMS, who was also an instructor for the first term, has been awarded the Grand Army Fellowship and is continuing graduate work in mathematics.

Professor K. O. E. LAMPE, of the Technische Hochschule, Berlin, died September 4, 1918, in his seventy-eighth year. He was appointed to the Hochschule staff in 1889 after twenty-four years of experience as teacher in Berlin secondary schools. He was an editor of *Jahrbuch über die Fortschritte der Mathematik* since 1885, and of *Archiv der Mathematik und Physik*, since 1900.

The list of mathematicians who spent the autumn working in coöperation with Captain VEBLEN at the Aberdeen Proving Grounds, on problems in ballistics, includes the following: Professor H. F. BLICHFELDT, Stanford University; Professor G. A. BLISS, University of Chicago; Professor W. C. GRAUSTEIN, Rice Institute; Dr. T. H. GRONWALL; Professors C. N. HASKINS and C. R. DINES, Dartmouth College; Professor H. H. MITCHELL, University of Pennsylvania; and Professor W. H. ROEVER, Washington University. Professors BLISS, HASKINS, and ROEVER have returned to their university work.

On account of conditions arising out of the war there will be no meeting of the American Mathematical Society in February this year.

This year for the first time in over twenty years Professor F. N. COLE, of Columbia University, secretary of the American Mathematical Society, took a vacation at Christmas time and did not perform the duties of secretary at the annual meeting of the Society.

At the meeting of the American Association of University Professors, held in Baltimore, December 27 and 28, Dr. A. O. LOVEJOY, professor of philosophy in the Johns Hopkins University, was elected president.

H. Y. BENEDICT, professor of mathematics, and dean of the College of Arts and Sciences at the University of Texas since 1911, was the compiler of the eight hundred and fifty page *Source Book relating to the History of the University of Texas: legislative, legal, bibliographical, statistical,* published in 1918 by the university, as Bulletin 1757.

Captain A. A. BENNETT, of the University of Texas, First Lieutenant T. BUCK, of the University of California, and First Lieutenant W. E. MILNE, of Bowdoin College, have finished their work with Major F. R. MOULTON in the Ordnance Department at Washington, D. C., and have again taken up academic occupations. Major Moulton expects to return to the University of Chicago in April.

At the regular meeting of the American Mathematical Society held at Chicago on Friday and Saturday, December 27 and 28, 1918, about eighty persons attended, including sixty members. On Friday afternoon President L. E. DICKSON delivered his retiring address on "Mathematics in War Perspective." Following this address there was a joint session with The Mathematical Association of America, which is to be reported fully in the next issue of this MONTHLY. During the Saturday sessions twenty-six papers were presented, the titles and abstracts of which will be found in the *Bulletin of the American Mathematical Society*. At a business meeting there was an election of officers, resulting as follows: President, Professor FRANK MORLEY; Vice-Presidents, Professors G. D. BIRKHOFF and FLORIAN CAJORI; Secretary, Professor F. N. COLE; Treasurer, Professor J. H. TANNER; Librarian, Professor D. E. SMITH; Committee of Publication, Professors F. N. COLE, VIRGIL SNYDER and J. W. YOUNG; Members of the Council, Professors H. E. HAWKES, W. A. HURWITZ, A. C. LUNN, and C. N. MOORE. At a meeting of the Council of the Society it was decided to postpone the Colloquium proposed for 1919 at Chicago until 1920. Further, it was voted that the incoming president be requested to appoint a committee of three to report to the Council at the April meeting on "steps which should be taken to organize or promote the publication, in America or elsewhere, of an adequate and comprehensive survey of the current mathematical literature of the world." The following committee was later appointed: Professors E. W. BROWN, G. A. MILLER, and R. C. ARCHIBALD (chairman).

P. PIAZETTI, professor of geodesy and celestial mechanics at the University of Pisa, died April 14, 1918, aged 58 years.

T. BONNESEN has been appointed professor of descriptive geometry at the Ecole Polytechnique, Copenhagen.

S. I. LATTÈS, for some time maître de conference at the University of Montpellier, but latterly a member of the faculty of sciences of the University of Toulouse, died July 5, 1918, aged 45 years. He received his doctorate from the University of Paris in 1906, and his thesis, *Sur les équations fonctionnelles qui definissent une courbe ou une surface invariante par une transformation*, occupies 137 pages of *Annali di matematica*, volume 13, series 3.

The American Mathematical Monthly

OFFICIAL ORGAN OF

The Mathematical Association of America

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Surveys of the contents of recent books and periodicals constitute a valuable guide to current mathematical literature.

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The Notes and News cover a wide range of interest and information both in this country and in foreign countries.

The Problems and Solutions hold the attention and activity of a large number of persons who are lovers of mathematics for its own sake.

There are other journals suited to the Secondary field, and there are still others of technical scientific character in the University field: but the MONTHLY is the only journal of Collegiate grade in America suited to the needs of the non-specialist in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, Ohio.

VOLUME XXVI MARCH, 1919 NUMBER 3

THE AMERICAN MATHEMATICAL MONTHLY

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THE MATHEMATICAL ASSOCIATION OF AMERICA

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PUBLISHED BY THE ASSOCIATION

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

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VOLUME XXVI MARCH, 1919 NUMBER 3

THIRD ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The third annual meeting[1] of the Association was held at the University of Chicago on Friday, December 27, 1918, in conjunction with the annual meeting of the American Mathematical Society, which was held on Friday and Saturday of that week. There were 86 in attendance at the sessions, including the following 73 members of the Association:

MARY ANDERSON, Illinois Woman's College.
R. C. ARCHIBALD, Brown University.
C. S. ATCHISON, Washington and Jefferson College.

R. P. BAKER, University of Iowa.
I. A. BARNETT, Washington University.
A. A. BENNETT, Aberdeen Proving Ground.
H. F. BLICHFELDT, Aberdeen Proving Ground.
HENRY BLUMBERG, University of Illinois.
J. W. BRADSHAW, University of Michigan.
H. T. BURGESS, University of Wisconsin.

W. D. CAIRNS, Oberlin College.
J. A. CAPARÓ, University of Notre Dame.
R. D. CARMICHAEL, University of Illinois.
C. C. CARTER, Chapin, Ill.
E. H. CARUS, La Salle, Ill.
H. E. COBB, Lewis Institute.
A. B. COBLE, University of Illinois.
C. E. COMSTOCK, Bradley Polytechnic Institute.
D. R. CURTISS, Northwestern University.

L. E. DICKSON, University of Chicago.
E. L. DODD, University of Texas.

E. B. ESCOTT, Peoples Life Insurance Company, Chicago, Ill.

W. J. FERGUSON, Ginn and Co., Chicago, Ill.
J. A. FOBERG, Crane Junior College, Chicago, Ill.
W. B. FORD, University of Michigan.
TOMLINSON FORT, University of Alabama.
A. F. FRUMVELLER, Marquette University.

O. E. GLENN, University of Pennsylvania.
ALICE BACHE GOULD, University of Chicago.
CORNELIUS GOUWENS, Graduate School, University of Chicago.

J. O. HASSLER, Crane Junior College, Chicago, Ill.
C. T. HAZARD, Purdue University.
OLIVE C. HAZLETT, Mount Holyoke College.
P. E. HEMKE, Northwestern University.
G. W. HESS, Shurtleff College.
T. H. HILDEBRANDT (University of Michigan), F. A. C. O. T. S. Instructor, Camp Taylor.

L. C. KARPINSKI, University of Michigan.
A. M. KENYON, Purdue University.
J. M. KINNEY, Hyde Park High School, Chicago, Ill.

KURT LAVES, University of Chicago.
SOLOMON LEFSCHETZ, University of Kansas.
A. C. LUNN, University of Chicago.

[1] The Association was founded in December, 1915.

W. D. MacMillan, University of Chicago, Major, Ordnance Department, Washington, D. C.
Gertrude I. McCain, Oxford College for Women.
Malcolm McNeill, Lake Forest College.
Bessie I. Miller, Rockford College.
G. A. Miller, University of Illinois.
C. N. Moore, University of Cincinnati.
E. H. Moore, University of Chicago.
E. J. Moulton, Northwestern University.

C. A. Nelson, Graduate School, University of Chicago.

S. E. Rasor, Ohio State University.
H. L. Rietz, University of Iowa.
W. J. Risley, James Millikin University.
W. H. Roever, Aberdeen Proving Ground.

Ida M. Schottenfels, Chicago, Ill.
A. R. Schweitzer, Chicago, Ill.
J. B. Shaw, University of Illinois.
H. E. Slaught, University of Chicago.
G. W. Smith, University of Kentucky.
E. B. Stouffer, University of Kansas.
Mary C. Suffa, Beloit College.

W. H. Taylor, Carbondale, Ill.
E. L. Thompson, High School, Joliet, Ill.
E. J. Townsend, University of Illinois.

R. N. Van Horne, Morningside College.
J. N. Van der Vries, U. S. Chamber of Commerce, Chicago, Ill.
L. G. Weld, Free School of Manual Training, Pullman, Ill.
E. J. Wilczynski, University of Chicago.
C. E. Wilder, Evanston, Ill.
F. B. Wiley, Denison University.
R. E. Wilson, Northwestern University.

C. H. Yeaton, F. A. C. O. T. S. Instructor, Camp Taylor.

Noteworthy and in keeping with the character of the program of the meeting was the presence of Major MacMillan (who had just finished work at Washington under Major F. R. Moulton), and of four men from the Aberdeen Proving Ground, namely, Captain A. A. Bennett of the University of Texas, First Lieutenant P. L. Alger, Professor H. F. Blichfeldt of Leland Stanford Junior University, and Professor W. H. Roever of Washington University. Aside from these there were two members present from Pennsylvania, one each from Alabama, Kentucky, Massachusetts, Rhode Island and Texas.

The committee on local arrangements, Professor H. E. Slaught, chairman, provided the usual comfortable and convenient plans which characterize Chicago meetings. Members and visitors took lunch each day at the Quadrangle Club. The joint dinner of the two organizations was held on Friday evening at the same place with an attendance of 68, the dinner being followed by a brief discussion of the plans of the National Research Council by Professor Moore, and of a plan proposed by Professor Hedrick whereby a group of universities and colleges might send a purchasing agent to Europe for the acquisition of books and periodicals. A telegram was read during the meeting from Major E. V. Huntington, president of the Association, who is still engaged in the statistics branch of the general staff of the war department. The telegram indicated the impossibility of his attending the meetings owing to the continued rush in that bureau and threw a vivid light on his experiences in certain lines of statistical activity by the remark that "there is a great future for mathematics in this country, especially *accurate arithmetic*."

By appointment of the Council Professor R. D. Carmichael presided at all the sessions except the joint session of the Society and the Association at which President Dickson presided. The members of the Association were invited to hear the presidential retiring address on the Society program Friday afternoon, President Dickson speaking on "Mathematics in War Perspective." The great key-note of the meetings was thus the general effect of the war upon mathematics

and the contribution made by mathematics in the prosecution of the war. The interest evoked by the papers on the joint program showed that a further half day might have been profitably devoted to these papers and their discussions.

The program, accompanied by numbered abstracts is grouped in two parts. Professor E. R. Hedrick was chairman of the program committee.

I. Conference on War Time Experiences.

"Deductions from War Time Experiences with respect to the Teaching of Mathematics." A conference participated in by representatives of various colleges and universities in which Students' Army Training Corps were located, including (1) H. E. Slaught, University of Chicago; (2) R. P. Baker, University of Iowa; (3) W. D. Cairns, Oberlin College; (4) A. R. Crathorne, University of Illinois; (5) D. R. Curtiss, Northwestern University; (6) W. B. Ford, University of Michigan; (7) A. M. Kenyon, Purdue University.

(8) "The Ensign School," Professors E. J. Moulton and R. E. Wilson, Northwestern University.

(9) "An Experiment in Supervised Study," Professor D. R. Curtiss, Northwestern University.

(10) Discussion by Professor H. L. Rietz, University of Iowa.

(1) Professor Slaught urged the importance of our deriving in this conference the greatest gain for the future from our recent experiences. He stated as one deduction from the experiences with the war classes the need of more concrete and practical applications and less of the complicated and abstract manipulation in all elementary courses. In this connection he called attention to the need of revising the definitions of the preparatory "units" and formulated the following statement on this subject:

"In view of the fact that there is a widespread desire among the secondary teachers of mathematics, as well as among many college teachers, for a reconsideration of the definitions of the 'Units' in preparatory courses in mathematics as formulated by a committee of the American Mathematical Society in 1903 and used as the basis of the examination questions set by the College Entrance Examination Board and by the Regents of the State of New York, and, further, in view of the fact that the Mathematical Association of America has a standing committee on Mathematical Requirements consisting of Professor J. W. Young, chairman, Professor D. E. Smith, Professor E. H. Moore, Professor C. N. Moore, Professor A. R. Crathorne, and Professor H. W. Tyler, all of whom are members of both the Society and the Association, which committee has been enlarged by associating with it a representative from each of the large secondary associations in New England, the Middle States, and the Middle Western States, thus making it truly representative and national in character, it would seem to be most appropriate for the Council of the Society and the Council of the Association to join in formally referring the whole matter to this National Committee for consideration." A resolution embodying this action was later passed by the Councils of both organizations.

(2) Professor Baker reported upon the teaching of trigonometry under the S. A. T. C. He pointed out specific instances where the necessary shortening of the course resulted in the students filling the gaps in the theory outside of class; this raised the question of the wisdom of such abbreviation. The net result seemed to be that the capable students overcame the added difficulties but that the total number acquiring the subject satisfactorily was considerably reduced.

(3) Professor Cairns described a course in map sketching and map reading which he supervised in the S. A. T. C. in Oberlin College, fulfilling the requirements called for by the Committee on Education and Special Training. Somewhat more than one hundred twenty-year-old men were given familiarity and practice in the field methods of pacing, obtaining bearing and azimuth, plane table mapping, profile leveling, measuring angles with transit, measurement of slopes, sketching and reading topographic maps, with a modicum of trigonometry. Working with men of that age who had, in large part, had no college training after graduation from the high school, one obtained a new and evidently valuable revision of one's previous notion of what constituted mathematics, and the necessity of joining the students' college mathematics and his preceding courses more skillfully, was impressed upon one as never before.

(4) Professor Crathorne reported that at the University of Illinois some 2100 students were registered in trigonometry which was given in two courses, one three hours a week for the non-engineering S. A. T. C. students and one four hours a week for the S. A. T. C. students registered in the College of Engineering. The extra hour in the latter course was devoted to the slide rule and the applications of trigonometry to military and nautical problems, given to the student in sheet and pamphlet form. The military authorities arranged the supervised study; this was not wholly a success due to the discomforts and noise in the barracks. The conflicts of military duties with academic studies together with the influenza epidemic caused a great many cuts from classes. Many students were absent for weeks at a time and naturally became discouraged. To help out in this situation a week's review was given about the middle of the term and much private help afforded to students asking it.

Changes in the University of Illinois mathematical curriculum due to war experience will doubtless be slight, perhaps more in the attitude of the teacher toward practical applications than in anything else. There seemed to be a general feeling among the instructors that the applied side of mathematical teaching had at least not been over-emphasized. The teaching force was augmented by instructors from other departments particularly from those in the college of engineering and one result of the experience will be a better understanding of the problem of teaching mathematics to freshmen by men who will have these students later in their engineering course.

At present it is not expected that there will be any marked change in the character of advanced courses in mathematics due to the military experience. Some advanced courses in special topics of applied mathematics like "Ballistics" and "Aërodynamics" may be given by the mathematics department or some

other department. There has grown up in other departments of the University a better feeling towards the teaching of mathematics and a respect for its usefulness, for example, several departments requiring mathematics for prerequisites have asked for more mathematics, whereas formerly there was a tendency to hold the mathematics to a minimum.

While not concerned with mathematics, a very important result of our war experience is the continuation of our "war aims" course into a course of training for citizenship to be open to the students of all the colleges of the university.

(5) Professor Curtiss stated that at Northwestern University little attempt had been made to change the usual subject matter of mathematical courses, but that new methods had been tried. The chief innovation he reserved for discussion in his subsequent paper. Seven different freshman courses (including two combining Trigonometry and Navigation) were given, and students who could not carry a course were put back at any time that seemed advisable into the next lower course. Thus every man was kept in some course for which he was presumably fitted, and the fluctuating character of the attendance was partially offset. A number of men had not had a complete course in plane geometry, but had finished a year and a half of algebra. It was noted that these men made better average records in trigonometry than those who presented a year of geometry and a year of algebra.

(6) Professor Ford pointed out that there is a slow but ever-increasing tendency in both college and secondary education to replace mathematics and the other so-called disciplinary subjects by subjects alleged to possess the same disciplinary value yet having the added merit of being closer to the world of affairs and hence of being more "practical." Among the colleges, for example, it is not difficult to find the teacher of economics who will argue that his subject is fully as logical, being founded upon definite laws, as is mathematics, and hence that it may serve equally well to impart disciplinary values, while withal the content of economics bears immediately upon all human affairs. As regards secondary education, there is little or no scruple against even replacing the older disciplinary studies by those having no such value at all, as is instanced by the common and increasing practice of introducing "vocational training" into the public schools. These tendencies, which were already well defined before the war began, will doubtless be intensified in the period lying directly ahead and hence it is an opportune moment for teachers of the older subjects, especially mathematics, to question what their attitude in the matter may well be; in particular, are we to regard the situation as based upon a sound educational policy, notwithstanding it may affect adversely our own interests? The answer is to be found in the often repeated but equally often forgotten principle of psychology which declares that in youth the mind is open and indeed eager to receive idealistic rather than material truth, and that in so far as the individual is deprived of what is thus natural to him at this period his later life is in corresponding measure thrown out of balance. The central duty of education is, or should be, to furnish a background upon which the later routine existence may at least be made tolerable, and, whatever be the

tendencies in vogue, it is a fundamental error to focus the mind prematurely upon matters essentially mundane.

(7) At Purdue University the courses in mathematics given to S. A. T. C. students were 1) Plane trigonometry, 5 hours, including the solution of right and oblique triangles by the usual methods, viz., the laws of sines, cosines, and tangents, and the half-angle formulas, emphasis being put upon computation with and without logarithms; 2) Differential calculus, 3 hours, treating the elements of differentiation and applications to mechanics.

While there were many interruptions caused by sickness and military interference, the effort was made to keep in mind that the primary purpose of the organization was to make soldiers, and in spite of difficulties the results were fairly satisfactory.

The main questions which have been raised by the experience are: (*a*) What topics, if any, in mathematics courses can be omitted without serious loss to technical students? (*b*) Which is the best method: to teach fundamental general methods with drill exercises and then go to the applications, or to begin with applications and treat methods as the need for their use arises? (*c*) Is supervised study desirable under ordinary academic conditions?

(8) In May, 1918, a special course was offered at Northwestern University to students of that institution who had enlisted in the U. S. Naval Auxiliary Force at the Municipal Pier, Chicago. This course consisted of instruction in mathematics, navigation, signalling, infantry drill, swimming, and a study of the *Blue Jacket's Manual.* Eight hours daily was devoted to the above subjects as follows: Mathematics, two hours; navigation, two hours; signalling, two hours; swimming and infantry drill combined, two hours. Selected chapters in the *Blue Jacket's Manual* were assigned to be read and examinations were held to insure that this reading was carefully done. This course, four weeks in length, was continued from May until December 20, a new course beginning every two weeks. All of the work of the course, except signalling, was handled by members of the university faculty. The Navy provided instructors in signalling. The instruction in each subject consisted of lectures approximately one hour in length and supervised study for an additional hour under the supervision of the instructor, who assigned problems and exercises based on the lecture.

Classes as large as 150 students were handled in this manner with satisfactory results. The content of the course in mathematics comprised a short review of algebra, particularly the solution of simple equations and theory of exponents; plane trigonometry, especially the solution of right triangles; logarithms; use of traverse tables and haversine tables of Bowditch; spherical trigonometry, particularly the solution of oblique spherical triangles.

Approximately one thousand took this course. About 90 per cent. of the men had completed plane trigonometry before they entered the course, and the work in mathematics was, for many, merely a review. Practically all the men carried the course satisfactorily. This may be explained on the ground that the men were eager to make as good a showing as possible in order to win promotion

in the Navy in the shortest time possible. The men in the course were detailed to Professor Wilson, who had charge of the course at Northwestern University; and since the course was almost entirely scholastic rather than military almost none of the difficulties of the S. A. T. C. arose. The course proved that fairly satisfactory results in mathematics may be obtained in classes as large as 150 students if there is a strong incentive on the part of the students to master the subject, and if the instructor has sufficient physical strength and nervous energy. It is, however, a task that no one would care to undertake except in time of crisis such as our country was facing during the past year.

(9) The scheme of supervised study followed at Northwestern University and described by Professor Curtiss involved class sessions lasting two and one half clock hours with one ten-minute intermission. The student had one such period each morning, afternoon, and evening, five times a week, and thus carried three subjects, of which one was mathematics. In general fifty minutes were devoted to lecture or recitation, and the rest of the period to study. The same instructor was with the class during the whole two and one half hours. Through excessive absences, lack of interest as demobilization approached, and other causes, this scheme could hardly be said to have had a fair trial. Instructors who tried to carry two large classes found themselves seriously overworked. In spite of these drawbacks, surprisingly good results were obtained. Professor Curtiss discussed the advantages of this method under ordinary academic conditions. He believed that, with some modifications, it should have further trial.

(10) Professor Rietz spoke as follows: The association is certainly indebted to Professor Curtiss for bringing before us the methods and results of this experiment. It is particularly interesting to me that, although supervised study was conducted very differently at the University of Iowa from that at Northwestern, still our experiments left very similar impressions in regard to the possibility of improving our teaching of freshman and sophomore mathematics by a certain amount of supervision of study. Perhaps I should explain that at the University of Iowa the liberal arts students had three large study centers at each of which there were instructors from various departments. There was no assignment of a particular subject to be studied at a particular time. On the other hand, the engineering freshmen and sophomore students were assigned to study mathematics from 7:30 to 9:30 p.m. daily. It was a matter of general comment that the liberal arts students called for more assistance in mathematics than in other subjects. One instructor per hundred students was kept very busy answering questions. I wish to emphasize the point made by Professor Curtiss that through obtaining the answer to some minor question the student was sometimes enabled to make progress and finish the preparation of the lesson, whereas otherwise he would have been entirely unprepared.

The fundamental question that arises in my mind from this experience can be stated as follows: Is it a wise division of time in freshman and sophomore mathematics to make provision for one hour of work by the student in the presence of the teacher to correspond to two hours of outside work? If we could afford to

give one hour to class work of the usual kind and a half hour to consultation and supervision of study, and leave the student one and one half hours for outside work, it seems to me that we would get much better results.

II. Joint Session of the Association with the Society.

(1) "Some Mathematical Features of Ballistics," Captain A. A. Bennett (University of Texas), Ord., Washington, D. C.

(2) "How the Map Problem was met in the War," Professor Kurt Laves, University of Chicago.

(3) "Notes concerning Recent Books on Navigation," Alice Bache Gould, University of Chicago.

(4) "Statistical Methods in Preparation for Service in Statistical Sections of the War Department," Professor H. L. Rietz, University of Iowa.

(5) "Ordnance Problems," Major W. D. MacMillan (University of Chicago), Ord., Washington, D. C.

(6) "Practical Exterior Ballistics," Lieutenant P. L. Alger, Ord., Aberdeen Proving Ground, Md.

(7) "The Effect of the Earth's Rotation and Curvature on the Path of a Projectile," Professor W. H. Roever (Washington University), Ord., Aberdeen Proving Ground, Md.

(8) "On Low Velocity High Angle Fire," Professor H. F. Blichfeldt (Stanford University), Ord., Aberdeen Proving Ground, Md.

(1) Captain Bennett spoke in a more or less informal manner on the subject of projectiles, pointing out some of the most obvious physical and mathematical problems suggested by them. During the passage of the projectile through the bore of the rifle, mechanical questions concerning the uniform or variable pitch of the rifling are presented. On emerging from the muzzle problems of nutation and precession arise at once. The question of air resistance presents numerous meteorological problems, and some of the known facts were briefly outlined. Dynamical questions of an experimental nature are offered by the notion of center of resistance, and a problem in calculus of variations is that of determining the ogive of least resistance. That the distribution of pressure on the projectile in flight may be a complicated one was suggested by pointing out as one factor the rôle of the velocity of sound as affecting the dissipation of energy. A few descriptive remarks on the terminology and nomenclature of a projectile and a trajectory, with data on the German long range gun, concluded the paper. The rôle of the theory of probability, statistical methods, mechanical quadrature and graphical methods of solution were merely hinted at in passing.

(2) The pre-war maps of France are based on the Bonne conical projection with the meridian of Paris as primary meridian. Since this is an "equivalent" but not a "conform" representation, the distortions of angles and of distances at the Eastern frontier amount to 18′ and 1/379 respectively. Such errors are far from negligible for present-day gunnery.

The Lambert conical map representation is conform. This is an ideal pro-

jection for artillery fire, since the angles are preserved and the distortion in distances amounts at most to only 1/2037 for the maps of northern France. Moreover the Lambert representation permits of unlimited extension in longitude east or west of the primary meridian (meridian near Trier, Germany). On the map the "kilometer grid" with its Y-axis parallel to the primary meridian is printed. The x and y coördinates of any terrestrial mark are readily taken off the map. The "orientation" officer determines the x and y coördinates of the prospective position of the battery by measuring the horizontal angles between the lines from the battery B to three known terrestrial marks (three point problem). The "Lambert North" is obtained by a "round of the horizon." The Y-azimuth of the target T on which fire is to be opened and the distance BT are easily found from simple formulas of coördinate geometry.

(3) Miss Gould (daughter of the late Benjamin Apthorp Gould, the astronomer), who has been an instructor in the navigation courses at the University of Chicago, gave the results of a comprehensive study of the available books in navigation. Since the revival and extension of courses in navigation in American universities and colleges due to the war has made this a subject of interest to so many mathematicians, it is our hope that Miss Gould's discriminating criticisms may be made useful to a larger circle of readers through a fuller presentation of this paper in the columns of the MONTHLY.

(4) Professor Rietz called attention to the fact that there is simply an appalling amount of statistical work in the War Department, and the question that arises is not in regard to the magnitude of the statistical projects involved, but in regard to whether the statistical problems to be solved are of such a nature that a department of mathematics should give them special attention. During a reorganization of the Quartermaster Corps last February and March, the speaker had an opportunity to investigate the statistical methods in use in the War Department. This experience convinced him that a course might well be given in a department of mathematics that would have a useful place in the preparation of men for this branch of the service. At the University of Iowa during the past quarter, there has been given a very elementary course in statistics with special reference to the purpose just indicated. The course was taken by seventy-three men. The subject matter had to be adapted to the preparation of the average S. A. T. C. student, and to a time schedule of six clock hours per week. The time allotted to the course was divided between one class period and three hours per week in a statistics laboratory.

A notion can perhaps be obtained of the subject matter of the course as given the past quarter by the following general divisions of the ground covered: (1) tabulation of data, (2) frequency distributions and elements of probability, (3) graphical methods, and (4) averages. Along with the presentation in class of the meanings of simple, double, treble and quadruple tabulation, real applied problems for tabulation were given as laboratory exercises. The idea of a frequency distribution was next developed and the first laboratory exercise was to prepare a frequency distribution from the monthly rainfalls at Iowa City for

each month of the past twenty-five years. The student was required to write the answers to the following questions about resulting frequency distribution:

(1) Give reasons for the selection of your interval of sub-classes rather than some other interval.
(2) What is your estimate from this frequency distribution of the rainfall that is most frequent at Iowa City?
(3) What is your estimate of the relative frequency of a monthly rainfall in excess of three inches at Iowa City?

The meaning of probability was next taken up and a few of the elementary propositions leading up to the probabilities for repeated trials were developed. After this excursion of two or three days into probability, a return, by analogy, was made to frequency distribution of rainfall with a new light that at least suggests an explanation of the character of the frequency distribution of rainfall. Graphical methods of presentation of data were next discussed. The laboratory exercises required for the best presentations a rather large variety of curves and diagrammatic forms. In this connection, the graphing of a few simple mathematical functions was also given.

The extensive use of logarithmic paper in the statistical section of the Quartermaster Corps impressed upon the speaker the desirability of emphasizing the purposes for which this paper is adapted. The preparation of a sheet of logarithmic paper was given as a laboratory exercise. This was prepared without the use of a logarithmic table.

A start was made on the meanings and functions of different kinds of averages but there was no treatment of average in the laboratory work.

Professor Reilly, Mr. Taylor and the speaker did the teaching and prepared the material for this course. With regard to the results of the experiment, all feel that the laboratory has been very successful, considering the irregularities of attendance. The students were interested and did the work with enthusiasm. It is perhaps unnecessary to say that a course in the elements of statistics can be given much better when the students have at least freshman mathematics as preparation than under the conditions met in connection with this course. However, the work as a course in elementary statistics, with its war title deleted, is to continue for the remainder of this school year. The speaker then concluded as follows:

"To give some notion of the subject matter of the remainder of the course, let me say that it will include a treatment of different kinds of averages, and the functions used to describe dispersion in a frequency distribution. This will necessarily require a knowledge of logarithms and of interpolation at least by proportional parts. We shall certainly give more of probability theory than is contained in a good freshman text-book and draw illustrations from statistics, including the meaning of the normal probability curve. In the laboratory, we shall give exercises to test how nearly the normal curve fits some real frequency distributions of statistics. The course will include the preparation of index numbers and the rationale of different methods of averaging them. Finally, the course may include something of the meaning of correlation, and of the use and limitations of the correlation coefficient."

(5) Because of the limited time at his disposal, Major MacMillan stated only one of a number of important ordnance problems, this being the particular problem with which he has been occupied: A plane or dirigible is at an unknown height, is moving in an unknown direction with an unknown velocity, the force and direction of the wind and some other such data are incompletely known. The artillery officer has six seconds within which to make the necessary observations and compute the required direction and elevation of fire. He stated that with the best efforts of artillerists to improve the methods the guns hit these targets once in 15,000 shots. Then, commenting good-humoredly on the slender results which have followed from all the efforts of expert mathematicians and others, he added that a practiced gunner will without the use of these methods hit such targets once in about three or four thousand shots.

(6) The development of the methods of exterior ballistics in practical use was briefly sketched by Lieutenant Alger. In this development a series of schemes for taking account of air resistance have been devised, tested, and abandoned in turn. In general, a certain law of air resistance has been assumed, and tables and formulæ based thereon have been developed. In these formulæ a single constant, characteristic of the projectile, has been left to be determined by experiment. As long as this "constant" is invariant with changes in elevation or velocity within the limits of accuracy desired, a given method is useful. But each method in turn has come to a stage where the variations of the "constant" have been important. After vain attempts to derive laws for such variations, a new scheme has been adopted.

It was first assumed (by Newton) that the resistance of the air was proportional to the square of a projectile's velocity; on this basis the tables and methods of Otto and Euler were prepared. Better experimental knowledge led to the adoption of the cube law of Bashforth and later of the fourth power law of Zaboudski and Lardillon. Increased muzzle velocities necessitated more general laws, however, and after extensive experiments by Krupp, Mayevski and Hojel introduced a discontinuous law of resistance whereby different power laws are assumed to hold over several velocity ranges. In connection with these laws of resistance, various methods of solution of the differential equations of motion have been employed. The fact that, if the resistance is proportional to any power n of the velocity, all trajectories having the same initial inclination and initial retardation are similar, was used by Otto. Didion integrated the equation of the hodograph by taking a mean value α for the secant of the inclination assuming it constant throughout the integrations. Siacci made a better approximation by taking $\cos^{n-2}\theta/\cos^{n-1}\phi$ as constant and equal to a mean value throughout. This latter scheme in conjunction with the use of a pseudo velocity introduced by Siacci made a very satisfactory method for use in conjunction with Mayevski's laws. The method was reduced to a convenient basis by Braccialini and Ingalls, these having devised and tabulated secondary functions which could be computed by formal integration. These tables are still in use for rapid computation, but the modern use of high angle fire has rendered necessary new methods

for exact computation. This need has been met by the assumption of a smooth empirical law for air resistance and by the introduction of methods of solution of the equations by numerical integration. Such methods have long been in use but others have recently been developed in England by Littlewood and in America by Major F. R. Moulton. Improvements and extensions of the American method by G. A. Bliss, Capt. A. A. Bennett, and T. H. Gronwall have much reduced the labor of computation required. It is hoped the introduction of further improvements and the compilation of tables will enable the scheme to compare favorably with Ingalls's methods in rapidity as well as to exceed them in accuracy.

The methods of taking into account the variation of air density with altitude, the effects of wind, and the precession of a projectile, were not touched upon on account of lack of time.

(7) In order that a clear description of the problem might be possible, the following principle of relative motion was first stated by Professor Roever: If the motion of a particle with respect to axes fixed in space be known, the motion of the same particle with respect to a set of rotating axes can be determined. To the forces which account for the motion in the fixed system there must be added two forces in order to account for the motion in the rotating system. One of these forces acts along the perpendicular from the particle to the axis of rotation, its direction is away from the axis of rotation and its magnitude is $\omega^2 r$, where r is the distance of the particle from the axis and ω is the angular velocity of rotation of the second system with respect to the first. The second additional force is perpendicular to both the direction of motion of the particle (in the rotating system) and the axis of rotation; its magnitude is $2\omega v \cos \gamma$, where v is the velocity of the projectile along its path and γ is the angle which the direction of motion makes with a plane perpendicular to the axis of rotation. A rule for determining the sense of this force will be stated later.

The particle was then supposed to be the projectile of a gun. In the system which does not rotate with respect to the fixed stars, the only force which acts is that of the gravitational attraction of the earth for the particle, provided air resistance and wind effects are neglected. In the system which is at rest with respect to the rotating earth, the forces which must be brought into play in order to account for the observed motion are, in addition to the gravitational attraction just mentioned, the two forces referred to in the preceding paragraph. The resultant of gravitational attraction and the first additional force is a force whose magnitude is the weight. It is this resultant which determines the form of the level surfaces. To obtain the form of the path of the projectile (in the moving system) the second additional force must also be taken into consideration. When this force is not negelected we shall say that *rotation* has been taken into consideration. However, this force is usually assumed to be negligible. It is also usually assumed that the force, just defined, whose magnitude is weight, is constant in direction and magnitude in the region traversed by the projectile. We will call this the assumption *neglecting curvature*.

It was then shown how the trajectory (path of the projectile) which corre-

sponds to the assumptions neglecting rotation and curvature differs from that which takes into consideration rotation and curvature, for the case of both a vacuum and an atmosphere, the layers of constant density of which are level surfaces. Qualitatively the results may be stated as follows: The effect of rotation is the same as that of a wind which blows parallel to the plane of the equator and to the right (or left) of a person who is imagined to be walking on the north (or south) side of this plane along the path and in the direction of motion of the projection of the projectile on this plane. This effect is somewhat modified by air resistance. Since a local horizontal plane is, in general, not parallel to the plane of the equator, the effect of rotation, besides causing a drift of the trajectory, causes also a change in range in that it raises or lowers the trajectory. The effect of curvature is very slight. The differential equations of motion as well as those of the differential corrections due to rotation have been obtained independently by several investigators for the case where curvature is neglected. From the latter equations the drift and change in range were computed by a short arc process. From more general equations of motion obtained by the author, the differential corrections due to curvature were computed.

(8) The velocities encountered in trench mortar fire are not greater than 800 feet per second. The air resistance may accordingly be assumed proportional to the square of the velocity; and if we assume the flight of the projectile to take place in an atmosphere of uniform density, the various elements of the trajectory can be expressed in terms of quadratures, in which the integrands involve constants and the function $J = \int d\theta/\cos^3\theta$ algebraically if $\tan\theta$ be taken as the variable of integration. In order to integrate in simple terms we are obliged to find a convenient function of $\tan\theta$ to represent J with sufficient accuracy. For instance, Siacci's method is equivalent to writing $a \tan\theta$ for J in this case. Greater accuracy will be obtained by using the expression $a\tan\theta/(1 - b\tan\theta)$.

One of the important problems of ballistics is to determine the deviations of the trajectory due to a wind of uniform velocity and direction ("ballistic wind"). Now, it appears that the ratio of the change in distance along the range, to the deflection from the plane of fire (measured as a distance perpendicular to this plane), divided by the ratio of the corresponding wind-components, is very nearly a function of the angle of elevation φ of the gun only. The limiting value of this function is 7/3 and 1 for $\varphi = 0°$ and $\varphi = 90°$, and is in the neighborhood of 5/3 for $\varphi = 45°$.

Meeting of the Council of the Association.

The following twenty-one persons and one institution, on applications duly certified, were elected to membership:

To individual membership:

J. J. Arnaud, B.S. (Coll. of the City of New York). Master computer, Ord. Dept., Washington, D. C.

G. C. Autenrieth, A.M. (Columbia). Asst. prof., descr. geom. and drawing, Coll. of the City of New York.

MRS. KATHARINE D. BROWN, A.M. (Bucknell). Prof. of math., Drexel Inst., Philadelphia, Pa.

PIERCE BUTLER, Ph.D. (Hartford Theol. Sem.). Senior asst., The Newberry Library, Chicago, Ill.

MINNIE W. CALDWELL, A.M. (Missouri). Teacher of math., Marvin Coll., Fredericktown, Mo.

C. C. CARTER, Chapin, Ill.

H. S. EVERETT, A.M., Sc.M. (Bucknell). Asst. prof. of math., Bucknell Univ., Lewisburg, Pa.

W. J. FERGUSON, A.B. (Williams). With Ginn and Co., Chicago, Ill.

HAIG GALAJIKIAN, Ph.D. (Princeton). Range Firing Section, Aberdeen Proving Ground, Md.

H. M. GEHMAN, recently Aberdeen Proving Ground. Norristown, Pa.

ALICE BACHE GOULD, A.B. (Bryn Mawr). Instructor in navigation, Univ. of Chicago, Chicago, Ill.

A. E. LAMPEN, A.M. (Michigan). Prof. of math., Hope Coll., Holland, Mich.

E. P. LANE, Ph.D. (Chicago). Instr. in math., Rice Inst., Houston, Tex.

M. J. MCCUE, C.E., M.S. (Notre Dame). Prof. of civil engineering, Notre Dame Univ., Notre Dame, Ind.

JUSTIN NICOLET. Stud., extension dept., Univ. of Wisconsin. Chicago, Ill.

W. P. PARKER, A.M. (Davidson Coll.). Prof. of math., Union Chr. Coll., Pyengyang, Korea.

L. G. POOLER. Stud., Columbia Univ. New York, N. Y.

HARRIS RICE, B.S. (Worcester Polytech. Inst.). Instr. in math., Tufts Coll., Mass.

C. H. RICHARDSON, M.S. (Illinois). Prof. of math., Georgetown Coll., Georgetown, Ky.

R. B. STONE, A.M. (Harvard). Asst. prof. of math., Purdue Univ., West Lafayette, Ind.

C. W. WATKEYS, A.M. (Harvard). Prof. of math., Univ. of Rochester, Rochester, N. Y.

To institutional memership:

UNIVERSITY OF UTAH, Salt Lake City, Utah, to date from January, 1918.

On the unanimous recommendation of the Committee on Editor-in-Chief, the Council appointed PROFESSOR R. C. ARCHIBALD to this office, expressing regret that Professor Carmichael found it necessary to lay down the work which he has so ably carried during the past year and giving voice to the great satisfaction felt in being able to name Professor Archibald as his successor. Professor Archibald assumed the duties of his office with the January, 1919 issue of the MONTHLY.

As stated earlier in this report, the plan proposed by Professor Slaught for a committee to re-define the "units" in secondary mathematics was approved, and the question was referred to the standing Committee on Mathematical Requirements.

The Council transacted further business with respect to the *Annals of Mathematics* and the question of providing for certain expenses of the Committee on Mathematical Requirements which will be necessarily incurred in carrying out the important and far-reaching plans of that committee.

It was voted that the Council favor holding a summer meeting in conjunction with the American Mathematical Society.

The Council made the following appointments on the staff of the MONTHLY:

Committee on Publications:

R. C. ARCHIBALD, Editor-in-Chief.
W. A. HURWITZ.
H. E. SLAUGHT.

Associate Editors:

HENRY BLUMBERG,	B. F. FINKEL,	HELEN A. MERRILL,
DANIEL BUCHANAN,	D. N. LEHMER,	U. G. MITCHELL,
E. L. DODD,	R. B. MCCLENON,	E. J. MOULTON,
OTTO DUNKEL,	H. P. MANNING,	D. E. SMITH.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION.

The secretary-treasurer announced the names of those just elected to membership by the Council. He also reported the death of the following eight members during the past year, all but Dr. McAtee having been charter members of the Association, and almost all having been long-time subscribers to the MONTHLY:

A. T. G. APPLE, Director Scholl Observatory, Franklin and Marshall College.
E. W. DAVIS, Professor of mathematics, University of Nebraska.
C. E. FLANAGAN, Conservative Life Insurance Company, Wheeling, W. Va.
R. A. HARRIS, U. S. Coast and Geodetic Survey.
CHRISTIAN HORNUNG, Professor of mathematics, Heidelberg University.
H. G. KEPPEL, Professor of mathematics, University of Florida.
ARTEMAS MARTIN, U. S. Coast and Geodetic Survey.
J. E. MCATEE, Instructor in mathematics, University of Illinois.

In connection with the list of mathematicians in war service published by the secretary-treasurer in the MONTHLY for January, 1919, it was announced that the names of 110 members of the Association were known to the secretary-treasurer as having been enrolled in national service, including Y. M. C. A. and other noncombatant branches; this is a record of which the association may justly be proud. An instance worthy of mention is that of one of the members of the Association, J. W. DAPPERT of Taylorville, Illinois, in whose family are listed four lieutenants (one deceased), a sergeant in Europe, and a daughter in the War Risk Bureau.

The election of officers for the year 1919 was conducted by mail and in person at this meeting, as provided by the constitution. The tellers (A. B. Coble and E. B. Stouffer) appointed by the Council reported the result of the balloting as follows, 284 ballots having been cast, some of which were blank in part:

For President: H. E. Slaught, 219 votes; J. W. Young, 65 votes.

For Vice-President: H. L. Rietz, 153 votes; R. G. D. Richardson, 146 votes; D. E. Smith, 137 votes; Alexander Ziwet, 122 votes.

For additional members of the Council to serve until January, 1922: E. V. Huntington, 186 votes; E. H. Moore, 183 votes; L. P. Eisenhart, 148 votes; B. F. Finkel, 140 votes; H. F. Blichfeldt, 139 votes; W. H. Roever, 118 votes; D. A. Rothrock, 110 votes; J. N. Van der Vries, 99 votes.

The following were accordingly declared elected:

President, H. E. SLAUGHT, University of Chicago.

Vice-Presidents, R. G. D. RICHARDSON, Brown University, and H. L. RIETZ, University of Iowa.

Additional members of the Council to serve until January, 1922:

L. P. EISENHART, Princeton University,
B. F. FINKEL, Drury College,
E. V. HUNTINGTON, Harvard University,
E. H. MOORE, University of Chicago.

The secretary-treasurer made his financial report for the year, giving an account of all business transacted for the Association up to December 2, 1918. The report of the auditing committee (Mary E. Sinclair, H. E. Slaught, and C. N. Moore, chairman) was then made, and both reports were accepted and approved. The financial report is printed in full below.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 2, 1918.

RECEIPTS.

Balance Dec. 1, 1917		$3,485.47
1917 subscriptions	$ 5.70	
1917 indiv. dues	94.00	
1917 instit. dues	13.00	
1918 subscriptions	438.70	
1918 indiv. dues	2,803.38	
1918 instit. dues	267.25	
Initiation fees	142.00	
Sale copies of MONTHLY	14.09	
Sale reprints	13.96	
Advertising	631.25	
Exchange	.94	
Interest State Savgs. Bk.	85.29	
Interest Peoples Bk.	36.65	
Interest Liberty Bond	20.00	
Total 1918 receipts		4,566.21
Total receipts up to 1919 business		$8,051.68
Total expenditures		4,539.84
Balance on 1918 business		$3,511.84

EXPENDITURES.

Publisher's bills		$2,953.76
Membership list		232.76
Managing editor's office		79.76
Editor-in-chief's office		77.08
Other editors' postage		29.12
Committee on Membership		45.89
Com. on Math. Requirements		112.13
Com. on Libraries		2.61
Dept. Undergraduate Math. Clubs		16.89
Secretary-Treasurer's office:		
Postage	$161.17	
Bond	5.00	
Office supplies	14.15	
Typewriter	45.00	
Express, telegrams, freight, etc.	16.78	
Library expense	1.00	
Clerical work	217.62	
Printing	158.54	
Chicago meeting	36.59	
Dartmouth meeting	80.80	
Paid to sections from initiation fees	24.51	
Safety deposit rental	1.67	
		762.83
Annals subvention		225.00
Interest on Liberty Bond		2.01
Total expenditures		$4,539.84

Received on 1919 business	216.27	Cash on hand, not deposited	$ 37.85
		Checking account	545.44
		State Savgs. Bk. Co. account	1,564.77
		Peoples Bkg. Co. account	1,080.05
		Liberty Bond	500.00
Balance Dec. 2, 1918	$3,728.11	Bank balance Dec. 2, 1918	$3,728.11

Approved by the auditing committee,

C. N. Moore, *Chairman*,
Mary Emily Sinclair,
H. E. Slaught for the Council,
Committee on Finance.

December 28, 1918.

When the accounts were closed on December 2, 1918, as was necessary in order to furnish the auditing committee a complete record, there remained on the total business for the year 1918 the following items:

Bills Receivable.		Bills Payable (either paid in December or estimated).	
Advertising	$150.00	Publisher's bills, Sept.–Dec	$1,450.00
1918 dues unpaid	75.00	Sept. and Dec. *Annals* subvention	150.00
	$225.00	Init. fees due to sections	60.00
		President's office	10.00
		Manager's office	20.00
		Editor-in-Chief's office	30.00
		Other editors' postage	25.00
		Secretary-treasurer's office	150.00
		Printing annual ballot, program, etc.	75.00
		Additional postage	50.00
			$2,020.00

If to the balance on 1918 business shown in this report, $3,511.84, there be added the amount of bills receivable, $225.00, and there be subtracted the estimated amount of bills payable, $2,020.00, there results an estimated final balance on 1918 business of approximately $1,700.00. It will again be recalled that about $1,000 of this surplus was the amount turned over to the Association by the management of the Monthly when the Association was formed, a fund which the Council feels must be held intact as a reserve fund. The financial gain made the past year is very gratifying, in view of the fact that we must face the coming year a certain decrease in the returns from advertising, and an almost inevitable increase in the item of printing if the Monthly is to remain at its present status, or, as we hope, to develop still further in its inspiring task of stimulating and strengthening American mathematics.

W. D. Cairns, *Secretary-Treasurer*.

ON THE ENVELOPE OF THE WALLACE LINES OF AN INSCRIBED QUADRANGLE.

By DAVID F. BARROW, University of Georgia.

It has long been known that the Wallace lines of a triangle envelop a three cusped hypocycloid.[1] Steggall[2] has investigated the envelope of the Wallace lines of an inscribed polygon. He gives the equation of the Wallace line in the general case and some facts about regular polygons. All he does with the quadrangle is to give the condition that the envelope be a four-cusped hypocycloid. We shall examine this case in more detail.

If perpendiculars be dropped upon the sides of a triangle from a point on the circum-circle, the three feet are collinear on a line called the Wallace line of the point with regard to the triangle. If we take an inscribed quadrangle, and omit each vertex in turn, we obtain four triangles; and the feet of the perpendiculars dropped from any point of the circle upon its four Wallace lines with regard to these triangles lie on a line called the Wallace line of the point with regard to the quadrangle. The process is capable of indefinite extension.

Let A_1, A_2, A_3, A_4 be the vertices of a quadrangle inscribed in a circle of radius R, and P any point on the circle. With the origin at the center, let θ_i and θ denote the angles measured from the positive x-axis around the circumference to A_i and P respectively. Then the equation of the Wallace line of P with regard to the quadrangle is[3]

$$
\begin{aligned}
& x \cos \tfrac{1}{2}[\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2(\theta - \pi)] + y \sin \tfrac{1}{2}[\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2(\theta - \pi)] \\
(1) \qquad & \quad = R\{- \cos \tfrac{1}{2}[\theta_1 + \theta_2 + \theta_3 + \theta_4 - 4(\theta - \pi)] \\
& \qquad + 2 \sin \tfrac{1}{2}(\theta - \theta_1) \sin \tfrac{1}{2}(\theta - \theta_2) \sin \tfrac{1}{2}(\theta - \theta_3) \sin \tfrac{1}{2}(\theta - \theta_4)\}.
\end{aligned}
$$

Let the origin be moved to the point whose coördinates are

$$[\tfrac{1}{4}R(\cos \theta_1 + \cos \theta_2 + \cos \theta_3 + \cos \theta_4),\ \tfrac{1}{4}R(\sin \theta_1 + \sin \theta_2 + \sin \theta_3 + \sin \theta_4)].$$

This is the point of concurrence of the three lines joining mid-points of pairs of opposite connectors of the quadrangle. Furthermore let $\theta_1 + \theta_2 + \theta_3 + \theta_4 = \pi$, which may be done without loss of generality by a proper choice of the direction of the x-axis. All this causes the equation of the Wallace line to reduce to

$$
\begin{aligned}
(2) \qquad x \sin \theta + y \cos \theta = \tfrac{1}{4}R[3 \sin 2\theta & - \cos \tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3 - \theta_4) \\
& - \cos \tfrac{1}{2}(\theta_1 - \theta_2 + \theta_3 - \theta_4) - \cos \tfrac{1}{2}(\theta_1 - \theta_2 - \theta_3 + \theta_4)].
\end{aligned}
$$

[1] *Steiner*, "Gesammelte Werke," pp. 641–647, "Über eine besondere Curve dritter Classe (und vierten Grades)." Other articles have been written about it, but this is probably the first.

[2] J. E. A. Steggall, "On the envelope of the Simson line of a polygon." *Edinb. Math. Soc. Proceedings*, Vol. 14, p. 122–126 (1896).

[3] This equation is taken from Steggall, *loc. cit.*, but with a change of notation.

Now θ is the parameter in this equation, and on differentiating with regard to θ we obtain

$$x \cos \theta - y \sin \theta = \tfrac{3}{2}R \cos 2\theta. \tag{3}$$

Now (3) is a line which meets (2) at the point where (2) touches its envelope. Moreover (3) is evidently orthogonal to (2), hence (3) envelops the evolute of the envelope of (2). But we recognize[1] that (3) envelops a hypocycloid of four cusps.

THEOREM 1. *The envelope of the Wallace lines of an inscribed quadrangle is an involute of a four cusped hypocycloid whose cusps lie on a circle of radius three times that of the circum-circle.*

Next let an angle α be defined by the equation

$$\begin{aligned} -3 \sin 2\alpha = \cos \tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3 - \theta_4) \\ + \cos \tfrac{1}{2}(\theta_1 - \theta_2 + \theta_3 - \theta_4) + \cos \tfrac{1}{2}(\theta_1 - \theta_2 - \theta_3 + \theta_4). \end{aligned}$$

Equation (2) may then be written

$$x \sin \theta + y \cos \theta = \tfrac{3}{2}R \sin (\theta + \alpha) \cos (\theta - \alpha).$$

Transform to oblique axes by

$$x = x' \cos \alpha + y' \sin \alpha,$$

$$y = x' \sin \alpha + y' \cos \alpha,$$

which turns the x-axis through an angle α and the y-axis through an angle $-\alpha$. This gives

$$\frac{x'}{\tfrac{3}{2}R \cos (\theta - \alpha)} + \frac{y'}{\tfrac{3}{2}R \sin (\theta + \alpha)} = 1$$

The intercepts on the axes are read off from the denominators in this equation, and since the angle between the axes is $\pi/2 - 2\alpha$, we easily calculate the length intercepted on the line by the axes to be $\tfrac{3}{2} \cdot R \cos 2\alpha$.

THEOREM 2. *There are two Wallace lines of an inscribed quadrangle which intercept a constant segment on all the others. These two pass through the point of concurrence of the three lines joining the mid-points of pairs of opposite sides of the quadrangle.*

This gives a very neat generation of the Wallace lines by allowing a constant segment to move with its ends sliding along two oblique lines. If, however, $\alpha = \pm \pi/4$ this generation becomes illusory since the two oblique lines would coincide. But from it we can get another generation which is never illusory, as follows: If a circle rolls on the interior of another circle of twice its radius, any point on the circumference of the rolling circle traces a diameter of the fixed circle. Therefore a chord of the rolling circle will be a line of constant length

[1] See Williamson's Differential Calculus, seventh edition, New York (1889), page 347, formula (18).

moving with its end points on two oblique lines. If the chord becomes a diameter the oblique lines become perpendicular, and if the chord becomes tangent the oblique lines coincide.

THEOREM 3. *The Wallace lines of an inscribed quadrangle are the successive positions of a chord of a circle which rolls upon the interior of a circle of twice its radius.*

In trying to picture the curve imagine first a hypocycloid of four cusps, which is one extreme case corresponding to $\alpha = 0$. Then suppose two opposite arches to grow longer at the expense of the other two which are shortened, so that the cusps stand at the vertices of a rectangle. This will be an intermediate figure. To get the other extreme let the two opposite arches swell till they become tangent to each other at the center of the curve. This corresponds to $\alpha = \pm \pi/4$. If all the vertices of the quadrangle approach coincidence the figure approaches this shape, but this is not the only case giving such a shape. It should be noted that not every involute of a four-cusped hypocycloid can be the envelope of the Wallace lines of a quadrangle if we confine ourselves to real quadrangles.

We have seen that when $\alpha = 0$ the envelope is a four-cusped hypocycloid. In terms of the θ_i this condition is

$$\cos \tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3 - \theta_4) + \cos \tfrac{1}{2}(\theta_1 - \theta_2 + \theta_3 - \theta_4) + \cos \tfrac{1}{2}(\theta_1 - \theta_2 - \theta_3 + \theta_4) = 0.$$

A rotation of the axes has the effect of adding a constant to each θ_i, which obviously leaves this condition unchanged. We may therefore suppose that the axes have such a direction that $\theta_4 = 0$. Then the condition may be manipulated into the form

$$(4) \qquad 3 \cot \tfrac{1}{2}\theta_1 \cot \tfrac{1}{2}\theta_2 \cot \tfrac{1}{2}\theta_3 + \cot \tfrac{1}{2}\theta_1 + \cot \tfrac{1}{2}\theta_2 + \cot \tfrac{1}{2}\theta_3 = 0.$$

This looks familiar and reminds us of the following fact. If B_1, B_2, B_3 are the vertices of an equilateral triangle and Q any other point in the plane, and if φ_i denotes the angle made by the line QB_i with the bisector of the interior angle of the triangle at B_i, then

$$3 \tan \varphi_1 \tan \varphi_2 \tan \varphi_3 + \tan \varphi_1 + \tan \varphi_2 + \tan \varphi_3 = 0.$$

This shows that if we take $\theta_i = \pi - 2\varphi_i$ $(i = 1, 2, 3)$, condition (4) will be satisfied, which suggests how to solve the following

Problem. Given three points on a circle, to construct a fourth point thereon so that the Wallace lines of the quadrangle formed by the four points shall envelop a four cusped hypocycloid.

Call the given points A_2, A_3, and A_4; and measure the angles from A_4 to A_3 and from A_4 to A_2 around the circumference of the circle. Lay off half the supplements of these angles from the bisectors of two of the angles of an equilateral triangle. Join the third vertex to the intersection of the two lines thus constructed. Take the angle formed by this last line and the bisector of the interior

angle at the third vertex, and lay off twice its complement from A_4 around the circumference of the circle. This locates the desired point A_1 to complete the quadrangle.

Two special cases of quadrangles which yield four cusped hypocycloids deserve mention. We easily prove by condition (4)

THEOREM 4. *If an inscribed quadrangle is a square, or if three of its vertices form an equilateral triangle, then its Wallace lines envelop a four-cusped hypocycloid.*

Suppose we cause the point P to remian fixed while the quadrangle $A_1A_2A_3A_4$-revolves about the center of the circum-circle without changing its shape. This can be done if in equation (1) we let θ be constant and replace each θ_i by $\theta_i + \lambda$. Thus the θ_i will be initial values and λ the angle through which the quadrangle has rotated, λ being now the parameter. Then by methods similar to those we have used it is not difficult to prove

THEOREM 5. *If an inscribed quadrangle revolves about the center of its circumcircle, the Wallace line of a fixed point on the circumcircle envelops a curve whose evolute is in general a two-cusped epicycloid; but if the quadrangle is a rectangle the envelope is a circle, and if it is a square*[1] *the envelope is a point. The successive positions of the Wallace line can be described as the successive positions of a line rigidly attached to a circle which rolls upon the exterior of a fixed circle of equal radius.*

It seems a pity not to generalize some of these theorems to the case of an inscribed n-gon, but only theorem 4 seems easy to extend. Steggall has found the envelope in the case of a regular n-gon to be an n-cusped hypocycloid, which generalizes part of theorem 4; and the present writer has generalized the remainder of this theorem so that it reads:

THEOREM 6. *If an inscribed n-gon is regular, or if $n - 1$ of its vertices form a regular polygon, its Wallace lines envelop an n-cusped hypocycloid.*

GEOMETRIC EXPLANATION OF A CERTAIN OPTICAL PHENOMENON.[2]

By WM. H. ROEVER.

Description of the Phenomenon.—In the parcel checking-room of the new Union Station at Kansas City, Missouri, there is a counter covered with brass plates which have, during the course of time, received numerous scratches by the baggage which is moved around upon the counter. The scratches are not very deep and they seem to be of fairly uniform distribution in both density and direction, as one might expect them to be after the cause of their formation has been in operation for some time. The baggage room is lighted by large electric lamps which are not very close together, so that an observer near the counter may regard the illumination in his immediate neighborhood as being due to a

[1] Steggall proves that for any regular n-gon the envelope is a point, *loc. cit.*

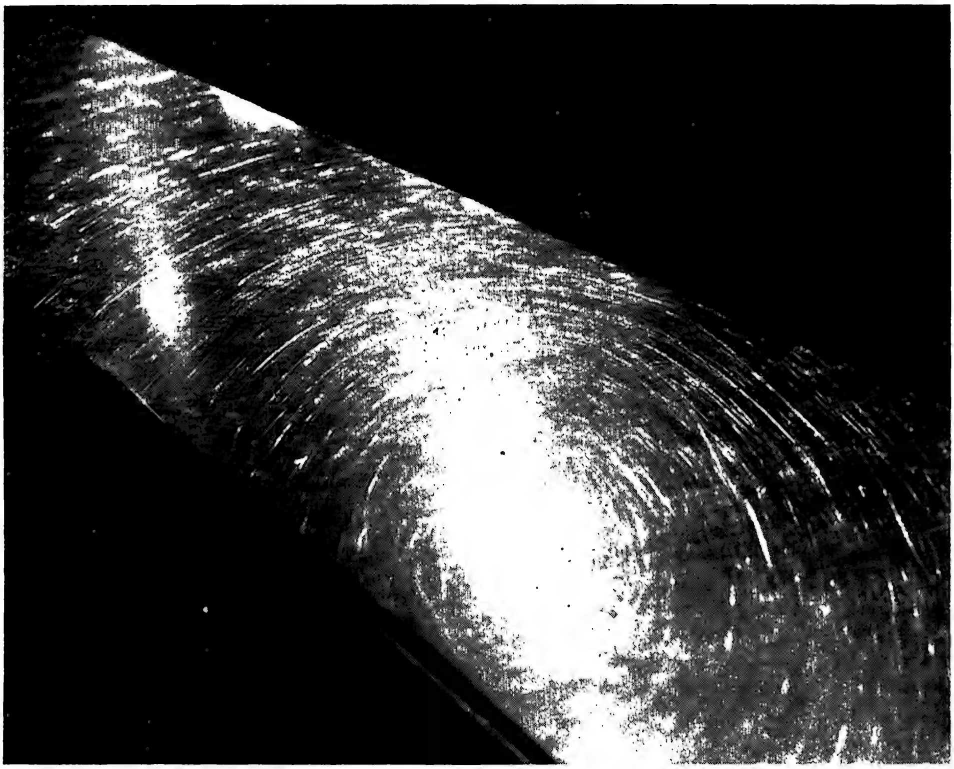
[2] Presented to the American Mathematical Society (Southwestern Section), December 1, 1917.

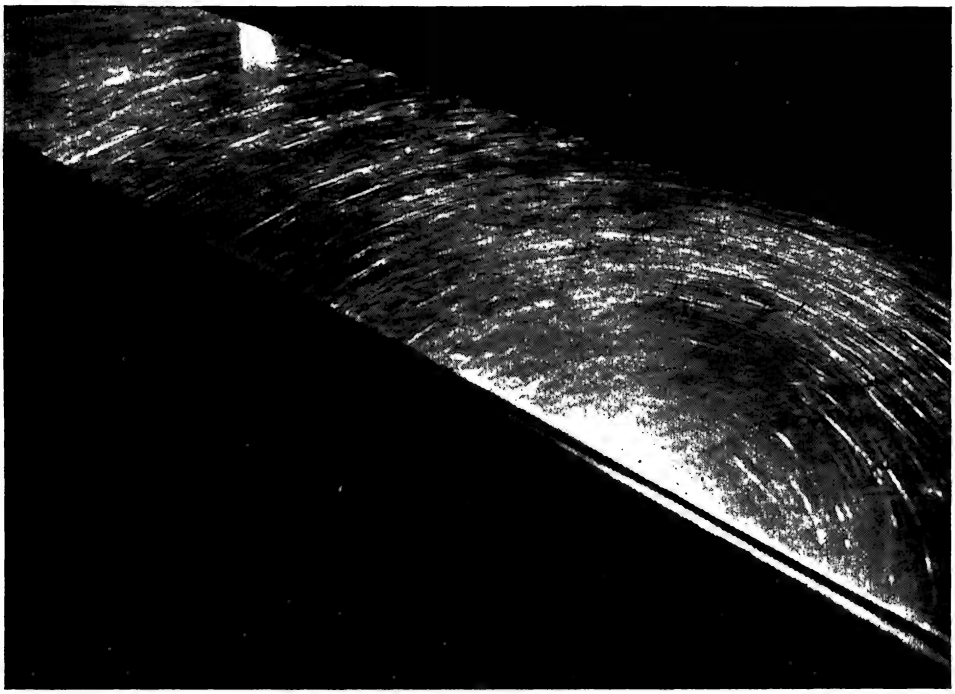
single lamp. Notwithstanding the apparently lawless nature of the manner of formation of these scratches, an observer anywhere near the counter and regardless of the direction of the illuminating electric light, will observe what appears to be a one-parameter family of illuminated ellipses which are approximately concentric and similar (see the accompanying pictures on the opposite page).

The Explanation.—A prolate spheroid, *i. e.*, an ellipsoid obtained by revolving an ellipse around its major axis, has the property that each of its points is a brilliant point with respect to its foci. In other words the focal radii to any point of the surface make equal angles with the normal to the surface at that point, and the normal lies between and in the plane of the focal radii. Consequently any reflecting surface or curve[1] which is tangent to such an ellipsoid of which a point source of light and an observer's eye are the foci, will appear to have at its point of contact with the ellipsoid a luminous point, *i. e.*, a brilliant point. The electric light and the eye of the observer are the foci of a one-parameter family of confocal ellipsoids of revolution. These ellipsoids intersect the plane of the brass-covered counter in a one-parameter family of ellipses (which are neither concentric nor similar in general, but are approximately so for the smaller curves of the family). For different positions of the observer's eye (and of the lamp, which, however, is fixed) there are, of course, different families of ellipses on the counter. Those scratches on the counter which are tangent to the members of this one-parameter family of ellipses, will have brilliant (or luminous) points at their points of contact with these curves. Owing to the fact that a scratch may have some curvature and some width and also because the source of light is not a point, it follows that not merely a point but that a small arc of the scratch becomes illuminated. These small illuminated portions of scratches (though short and disconnected) are well distributed, and even though few may lie along any individual ellipse, they do, in the aggregate, give the general impression of a one-parameter family of illuminated ellipses, *i. e.*, they make visible, so to speak, the geometric ellipses described above in much the same way that iron filings distributed in a magnetic field make visible the lines of force of that field.[2]

[1] We will regard as a curve the exterior surface of a wire of small cross section, or the gutter-like surface of a scratch.

[2] This brilliant point phenomenon is different from any of those described by the author in the *Transactions of the American Mathematical Society*, Vol. 9, No. 2, pp. 245–279; *Bulletin of the American Mathematical Society*, Vol. XXII, No. 5, p. 218; THE AMERICAN MATHEMATICAL MONTHLY, Vol. XX, No. 10, pp. 299–303, and Vol. XXI, No. 3, pp. 69–77.





"CONCERNING A METHOD FOR FINDING A PARTICULAR INTEGRAL." NOTE ON PROFESSOR COBLE'S ARTICLE.[1]

By HENRY P. MANNING, Brown University.

The method of undetermined coefficients for obtaining the particular integral in certain cases of linear differential equations is much easier and shorter than the other methods given in our text-books, except perhaps some of the mysterious symbolic methods, and it is at least as short as these. Professor Coble expresses the opinion[2] that Cohen was the first to publish it, and this may be true so far as English and American texts are concerned. Neither Johnson, nor Murray, nor Forsyth mention it. The method, however, is to be found in the second edition of the third volume of Jordan's *Cours d'Analyse*[3] which appeared some years earlier. Cohen explains the theory a little differently, but Jordan's explanation, though brief, is very clear and much simpler than Cohen's. It is so simple, in fact, that with a little elaboration it can be taught to undergraduate academic and engineering students, even when not in their books, and it can be applied not only to forms of second member that occur most frequently when the equations have constant coefficients, but also to cases of the so-called homogeneous or Cauchy equations, and to some other types that occur. It is analogous to the method of differentiation employed for certain equations that are not linear. From a given linear differential equation is obtained an equation of higher order with the second member zero, and, when the complementary function of this equation can be obtained, that part which does not belong to the complementary function of the given equation is determined by substitution so as to satisfy it as a particular integral. In practice we usually know the form of the result and have only to substitute and determine the constant coefficients.[4]

RECENT PUBLICATIONS.

REVIEW.

Analytic Geometry. By EDWIN S. CRAWLEY and HENRY B. EVANS. Philadelphia, E. S. Crawley, 1918. 12mo. 14 + 239 pages. Price, $1.60.

This book begins with seven preliminary pages given to review formulas from Algebra and Trigonometry and a set of tables, one of the three place natural values of the trigonometric functions of angles measured in radians and degrees for each degree from 0^0 to 90^0; another of three place common logarithms of numbers from 10 to 99; another of two place Napierian logarithms of numbers from 0 to 10.9; and two tables for Napierian anti-logarithms, one for positive and the

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[1] In this MONTHLY, January, 1919, pages 12–15.

[2] In a footnote on page 13.

[3] C. Jordan, *Cours d'Analyse de l'Ecole Polytechnique*, Tome 3, 2d ed., Paris, 1896; arts. 132–133, pp. 160–163.

[4] C. E. Love explains this method briefly (*Calculus*, New York, 1916, pp. 313–315).

where a, b, c, etc., are determined from the equation by giving x and y the values obtained by experiment.

This process, now commonly found in text books on analytic geometry, has a definite application in experimental sciences and ought to be given if the time is available. It furnishes also a convenient setting for a discussion of continuity.

The last chapter of the book (31 pages) is a successful adaptation of the conceptions and equations of analytic geometry of three dimensions to the needs of the student in his first course in calculus without attempting to develop fully the subject of solid analytic geometry.

The authors have not seen fit to introduce any formal treatment of limits or continuity, although they define a tangent as the limiting position of a secant and suggest the notion of continuity in the chapter on "Empirical equations." It may be a debatable question how much of this should be given in a first course. Some teachers would be satisfied with the loose notions of elementary geometry while others would demand the precise ϵ, δ definitions, but it does seem that theories so involved in analytic geometry should be formally mentioned, and an attempt made to give to the student conceptions as accurate as his ability to understand permits.

On pages 53–56 the authors discuss the "Perpendicular distance to a given point from a given line." The discussion is based upon the unnecessary Hesse normal form of the equation of the straight line. Professor Maxime Bôcher, in his *Plane Analytic Geometry* (New York, 1915), first obtains the formula on pages 38 and 39; and on page 43 he sets up the Hesse normal form in fine print. In this MONTHLY, December, 1917, page 476, Mr. R. M. Mathews finds the distance formula without making use of the Hesse normal form. Similarly, in this MONTHLY for April, 1918, page 181, Mr. H. T. Burgess comments favorably upon the derivation of Mr. Mathews, gives still another derivation, and states that "Hesse's normal form in this connection may well be relegated to Professor Miller's collection of 'Obsoletes.'"[1]

The book is well supplied with carefully selected exercises graded into "Normal exercises" and "General exercises." On the whole the book is a very teachable class room text which should enable the student to grasp the subject matter and use it successfully in the many well selected problems of the text and in other elementary problems of analytic geometry.

Strange to say, the book has no index. This seems inexcusable in a modern text.

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NOTES.

Simplified navigation for ships and circraft. A text book based upon the Saint Hilaire method. By C. L. POOR. New York, Century, 1918. 12mo. 18 + 126 pp. + 1 plate ["The line of position computer"]. Price $1.50.

[1] See also Maxime Bôcher, "Concerning Direction Cosines and Hesse's Normal Form." AMERICAN MATHEMATICAL MONTHLY, September, 1918, pages 308–10.

Contents—Chapter I: Finding one's position at sea and in the air, pp. 3–22; II: Determination of the "D. R." altitude and azimuth, 23–71; III: Corrections to be applied to the measured altitude, 72–88; IV: Simultaneous altitudes, 89–98; V: Notes and practical applications, with special reference to aërial navigation, 99–112; Tables: 113–126.

The *Journal of the Engineers' Club of Philadelphia* for June and July, 1918, contains an article by Carl G. Barth entitled "The income tax: an engineer's analysis with suggestions." The author finds a mathematical formula which agrees closely with the tax imposed by the present law and which would "smooth out" the inequalities in the gradation of this schedule. The formula is compared with Pareto's law of incomes in its estimate of the total national income of the United States; and certain criticisms of the details of the law are made, *e. g.*, that of exemption, on the mathematical basis afforded by the formula.

Four-place logarithmic and trigonometric tables together with interest tables—edited by L. C. KARPINSKI.[1] Ann Arbor, Mich., G. Wahr, 1918. 8vo. 30 pp. Stiff paper, price 30 cents.

Contents—Explanation of tables, pp. 5–7; Constants with their logarithms, 9; Logarithms of numbers, 100–999, 10–11; Logarithms of numbers, 1000–2009, 12–13; Logarithms of sines and cosines, 10′ intervals, 14–15; Logarithms of tangents and cotangents, 10′ intervals, 16–17; Logarithms of sines and tangents by minutes, 0° to 14° (Logarithms of cosines and cotangents by minutes 76° to 90°), 18–19; Numerical values of sines and cosines, 10′ intervals, 20–21; Numerical values of tangents and cotangents, 10′ intervals, 22–23; Squares, cubes, square-roots, cube-roots, and reciprocals, 1 to 100, 24; Degrees and minutes expressed in radians, Values of e^x and e^{-x}, 25; Accumulation of 1 at the end of n years, r^n, 26; The present value of 1 in n years, v^n, 27; The accumulation of an annuity of 1 per annum at the end of n years, 28; The present value of an annuity of 1 per annum for n years, 29; The annual sinking fund to accumulate to 1 at the end of n years, and by addition of i the annuity which 1 will purchase, 30.

The Metric System. A brief explanation, with Exercises and Tables of Equivalents; also a brief discussion of the Centigrade Thermometer and Foreign Money is the title of a pamphlet of sixteen pages (about 10 × 14 cm.), written by Professor J. W. YOUNG, and published last December for the Education Bureau of the National War Work Council of Young Men's Christian Associations by Association Press, 347 Madison Avenue, New York. Thirty-five thousand copies have been distributed already. [The scale of a decimeter on the edge of the outside cover is curiously inaccurate—about a millimeter too long.]

The same press is already at work upon a very short course in algebra and geometry, which is to give just enough of these subjects to enable a man to take up the study of trigonometry. This pamphlet was prepared by Professors YOUNG and MORGAN who are also at work on a very short trigonometry, to be published by the Association Press. It was originally intended that these publications should constitute a basis of instruction in Y. M. C. A. huts at army and navy camps all over the country. The need of something of the kind was everywhere keenly felt.

Amusements in Mathematics by H. E. DUDENEY. London, Edinburgh, and New York, T. Nelson and Sons, 1917. 8vo. 8 + 258 pp. Price 3s. 6d.

[1] These tables are very similar to those at the end of *Unified Mathematics* by L. C. Karpinski, H. Y. Benedict, and J. W. Calhoun. Boston, Heath, 1918.

Contents—Arithmetical and algebraical problems, pp. 1–27; Geometrical problems, 27–56; Points and lines problems, 56–58; Moving counter problems, 58–68; Unicursal and route problems, 68–76; Combination and group problems, 76–85; Chessboard problems, 85–109; Measuring, weighing, and packing puzzles, 109–111; "Crossing river" problems, 112–114; Problems concerning games, 114–117; "Puzzle games," 117–119; Magic square problems, 119–127; Mazes and how to thread them, 127–137; The paradox party, 137–141; Unclassified problems, 142–148; Solutions, 148–252; Index, 253–258.

This exceedingly interesting volume, packed with descriptions in small type and illustrated by hundreds of wood-cuts, would be a marvel of cheapness even in peace times.

Cours de mécanique professé à l'Ecole Polytechnique. Par L. CORNU. Tome 3. Paris, Gauthier-Villars, 1918. Royal 8vo. 2 + 669 pp.

Contents—Livre X: Résistance des matériaux, pp. 1–170; XI: Hydraulique, 171–310; XII: Thermodynamique, 311–428; XIII: Théorie des machines, 429–634; XIV: Notions d'aviation, 635–654.

Quotation from the "Avertissement": "Ce troisième et dernier Volume traite de la Mécanique appliquée. Les circonstances que nous traversons m'ont déterminé à entrer dans des développements dépassant les limites du Cours actuel de l'Ecole Polytechnique: tout porte à croire, en effet, que, la paix revenue, une transformation profonde s'opérera dans les études scientifiques, qui devront s'adapter plus étroitement aux réalités de la vie, en vue de mieux armer les Français pour la lutte économique succédant aux combats meurtriers . . . Il m'a donc paru qu'il convenait de préparer, en ce qui concerne la Mécanique, cette prochaine évolution."

Mr. William Allingham, who was connected with the marine department of the Meteorological Office of Great Britian for over forty years, died on January 24, 1919 in the sixty-eighth year of his age. He was a prolific writer. With D. W. Barker he was joint author of *Navigation: practical and theoretical* (London, 1896). He also edited the various editions of S. T. S. Lecky's *"Wrinkles" in practical navigation* beginning with the fifteenth in 1908.

On account of war conditions the publication of *Periodico di matematica per l'insegnamento secondario* and *Supplemento al Periodico di matematica* has been temporarily suspended. The last number of *Periodico* was volume 32, no. 6, October, 1917; of *Supplemento,* volume 20, no. 9, July, 1917.

The first number of a new scientific periodical, *El Progreso Cientifico,* has appeared recently at Saragossa under the direction of Professor Z. G. de Galdeano. It is to be a semi-annual review devoted to mathematics, physics, and chemistry, and containing papers dealing with fundamental questions, with criticism, and with scientific methodology, as well as with bibliographies, and matters pertaining to the teaching of science.

Dr. George Sarton, who has been connected with Harvard University and the Carnegie Institution of Washington for several years, expects to return to Belgium in June, but will be again at the Carnegie Institution in December. During his absence he will arrange for the resumption of the publication of *Isis,* the international quarterly (devoted to the history and philosophy of science) which he founded in 1913. The last part of volume 2, which was in the press in Brussels when the war broke out, will probable be issued in the autumn. The publication of volume 3 will take place soon after, perhaps in 1919, but at the latest in the early part of 1920.

Dr. Sarton has also announced in *Science* that "instead of publishing in four languages an effort will be made to use only French and English—chiefly, and perhaps exclusively, the latter. Articles written in other languages will be translated into English. The new *Isis* will only publish shorter articles. The longer and more monographic ones would be included in *Studies in the History and Method of Science* by Dr. Charles Singer of Exeter College, Oxford." The first volume of this work was issued by the Oxford University Press in 1917 and the second volume is now ready for the press. Dr. Singer is to share with Dr. Sarton the editorial responsibilities of the third and succeeding volumes of *Isis*. "Thus *Isis* and the *Studies* would be supplementary one to the other, and between them would provide suitable outlet for new work on the history and philosophy of science."

We list elsewhere a sketch of PETER LUDWIG MEJDELL SYLOW who died September 7, 1918, in the 86th year of his age. He had been professor of mathematics at the University of Christiania since 1898. Seven of his papers published before 1873 are listed in the *Royal Society Catalogue* and yet others in Poggendorff's *Biographisch-Literarisches Handwörterbuch.* Extensive biographical and bibliographical details are to be found in *Norsk Forfatter-Lexikon 1814–1880* of J. B. Halvorsen, volume 5 (Kristiania, 1901, pages 621–623). Sylow was co-editor of Abel's works and a member of the editorial committee of *Acta Mathematica.* A portrait and some biographical notes may be found in *Acta Mathematica 1882–1912. Table générale des tomes 1–35* (Upsala & Stockholm, 1913). The setting of some of his discoveries is indicated by H. F. BLICHFELDT in his *Finite Collineation Groups* (Chicago, 1917).

One of these discoveries is "Sylow's Theorem": "If p^a is the highest power of a prime p which divides the order of a group G, the sub-groups of G of order p^a form a single conjugate set and their number is congruent to unity, mod p."[1] The practical application of this theorem in determining the possible number of Sylow sub-groups of a given group requires considerable numerical computation before even tentative results can be obtained. This led to the publication by the Carnegie Institution of Washington, in 1916, of *A Sylow Factor Table of the First Twelve Thousand Numbers giving the possible number of Sylow sub-groups of a group of given order between the limits of 0 and 12,000,* by HENRY W. STAGER.

ARTICLES IN CURRENT PERIODICALS.

THE ALUMNI REGISTER, University of Pennsylvania, volume 21, no. 2, December, 1918: "Henry Brown Evans, M.E., Ph.D.," by H. P. Fry, 119–121.[2] [Mathematician and astronomer, appointed in 1918 dean of the Towne Scientific School.]

ANNALS OF MATHEMATICS, volume 20 (2), no. 1, September, 1918: "Functions of limited variation and Lebesgue integrals" by Goldie P. Horton, 1–8; "On the Teixeira construction of the unicursal cubic" by N. Altshiller, 9–12; "The functional equation $f[f(x)] = g(x)$" by G. A. Pfeiffer, 13–22; "The existence of the functions of the elliptic cylinder" by Mary F. Curtis, 23–34;

[1] Sylow, "Théorèmes sur les groupes de substitutions," *Mathematische Annalen*, vol. 5, 1872, pp. 584–594. For the particular statement of the theorem used above Stager refers to Burnside, *The theory of groups of finite order*, 2. ed., Cambridge, 1911, §§ 120 *et seq.*

[2] Unless stated to the contrary such numbers refer to pages.

"The gamma function in the integral calculus" by T. H. Gronwall, 35–76—No. 2, December: "The gamma function in the integral calculus" (concluded), 77–124; "Invariants which are functions of parameters of the transformation" by O. E. Glenn, 125–135; "A theorem on exhaustible sets connected with developments of positive real numbers" by H. Blumberg, 136–141; "Solution of the differential equation $dx^2 + dy^2 + dz^2 = ds^2$ and its application to some geometrical problems" by A. Pell, 142–148; "A general method of summation of divergent series" by L. L. Smail, 149–154.

Bulletin of the American Mathematical Society, volume 25, no. 3, December, 1918: "General aspects of the theory of summable series" by R. D. Carmichael, 97–131; "On the problem of the resistance integral" by T. Hayashi, 131–132; "Note on editions of von Staudt's Geometrie der Lage" by R. C. Archibald, 132–134; "Mathematical Periodicals" [Review of Union List of Mathematical Periodicals by D. E. Smith and Caroline E. Seely] by R. C. Archibald, 134–137.

The Journal of the Indian Mathematical Society, volume 10, no. 5, October, 1918: "A note in combinatory analysis" by R. Vythynathaswamy, 414–418; "Extension of M'Cay's theorem" by S. Narayanan, 419–422; "Euclid's Book on Divisions of Figures" [Review of R. C. Archibald's book], 423–428; "Note on Legendre's relation in elliptic functions" by K. B. Madhava, 429–430.

The Mathematics Teacher, volume 11, no. 1, September, 1918: "The New York state regents syllabus in intermediate algebra" by F. F. Decker, 1–8; "A geometric representation" (concluded) by E. D. Roe, 9–25; "The Reconstruction of the mathematical requirement" by G. W. Evans, 26–33; "A geometric illustration of limits" by C. E. Stromquist, 34–35; "Character-building content of arithmetic" by J. L. Green, 36–41—No. 2, December: "Why students fail in mathematics" by Helen A. Merrill, 45–56; "A solution of equations by standard curves" by R. C. Colwell, 57–60; "War problems in mathematics" by W. E. Breckenridge, 61–79; "Arithmetical errors made by high school pupils" by J. H. Minnick, 80–89; "Some relations connecting the sums of the coaxial minors of a circulant" by W. H. Metzler, 90–93; "Canada's challenge: In Flanders Field" by J. D. McCrae, 94; "America's Answer" by R. W. Lillard, 95.

Messenger of Mathematics, volume 48, no. 1, May, 1918: "The problem of the square pyramid" by G. N. Watson, 1–16.

The Monist, volume 28, October, 1918: "Leibnitz and Pascal" by K. I. Gerhardt with critical notes and a summary by J. M. Child and translations of Leibnitz's manuscripts alluded to by Gerhardt, 530–566; "The genesis of an electromagnetic field" by H. Bateman, 586-596; "Galileo and Newton" by P. E. B. Jourdain, 629–633.

Nature, volume 102, November 7, 1918: "Prof. Olaus Henrici, F.R.S." by M. J. M. Hill, 189–190—November 28: Review of R. C. Tolman's *The Theory of the Relativity of Motion* (Berkeley, 1917), 242–243—December 19: Review by S. Brodetsky of P. Frost's *An Elementary Treatise on Curve Tracing* (Fourth edition revised by R. J. T. Bell, London, Macmillan, 1918), 303–304.

The Nineteenth Century and After, volume 84, November, 1918: "On teaching mathematics" by Mrs. K. Lucas, 942–958.

Nyt Tidsskrift for Matematik, Copenhagen, volume 29, 1918, no. 1, June, *A*: En Sætning om Trekantens Røringscirkler" by J. Hjelmslev, 1–4; "Adgangseksamen til polyteknisk Læreanstalt, 1917" 5–8; "Landbohøjskolens Eksamensopgaver i Matematik, 1917" 8–10; "Studentexamen, 1917" 11–13; "Realskolexamen" 13–14. *B*: "Une formule exacte pour la détermination du nombre des nombres premiers au-dessous de x qui appartiennent à une classe de nombres donnée" by V. Brun, 1–8; "Über Mengen, die Elemente ihrer selbst sind" by H. Eklund, 8–28—No. 2, October, *A*: "En ny metod att diskutera den allmänna andragrads ekvationen med två variabler under förutsättning av snedvinkliga koordinataxlar" by G. Forsström, 25–39; "Ludwig Sylow, 1832-1918" 46–48. *B*: "Studier over en Afhandling af Gauss" by J. L. W. V. Jensen, 29–36; "Ueber uneigentliche Redeweisen in der Mengenlehre und über einen Aufsatz des Herrn H. Eklund" by T. Brodén, 36–43; "Skoleembedseksamen, Januar 1918," 43–50.

Proceedings of the American Academy of Arts and Sciences, volume 53, no. 5, March, 1918: "The dyadics which occur in a point space of three dimensions" by C. L. E. Moore and H. B. Phillips, 387–438—No. 8, July: "Rotations in hyperspace" by C. L. E. Moore, 649–694—No. 10, September: "Benjamin Osgood Peirce (1854–1914)" by E. H. Hall, 850–854.

Proceedings of the Edinburgh Mathematical Society, volume 36, part 1 (issued June, 1918): (1) "On the plane representation of the homaloidal surfaces which have a twisted cubic as multiple curve," 2–16, (2) "On a group of transformations connected with the 27 lines

of the non-singular cubic surface" by J. F. Tinto, 17–21; "Nicole's contribution to the foundations of the calculus of finite differences" by C. Tweedie, 22–39; "On a difference equation due to Stirling" by E. Pairman, 40–60—Part 2 (issued November): "The Brocard and Tucker circles of a quadrilateral" by F. G. W. Brown, 61–83; "The apolar locus of two tetrads of points on a conic" by W. P. Milne, 84–90; "Quaternion note on the theory of confocals" by C. G. Knott, 91–93; "An approximate value for the length of an arc of a suspended rope" by E. M. Horsburgh, 94–95; "Rolling loads: a new graphical method" by R. F. Muirhead, 96-102; "A formula for the solution of algebraic or transcendental equations" by E. T. Whittaker, 103-106; "On determinants whose elements are determinants" by E. T. Whittaker, 107–115.

Proceedings of the National Academy of Sciences of the U. S. of A., volume 4, 1918, no. 8, August: "Arithmetical Theory of certain Hurwitzian continued fractions" by D. N. Lehmer, 214–218; "On closed curves described by a spherical pendulum" by A. Emch, 218–221—No. 9, September: "On the α-holomorphisms of a group" by G. A. Miller, 293–294—No. 10, October: "Invariants and canonical forms" by E. J. Wilczynski, 300–305—No. 11, November: "On certain projective generalizations of metric theorems, and the curves of Darboux and Segre" by G. M. Green—No. 12, December: "On Jacobi's Extension of the continued fraction algorithm" by D. N. Lehmer, 360–364—Volume 5, no. 1, January, 1919: "A theorem on power series with an application to conformal mapping" by T. H. Gronwall, 22–24.

Revista de Matematicas, Buenos Aires, volume 2, no. 6, January–February, 1918: "La ecuación de las trayectorias ortogonales de las superficies equipotenciales newtonianas correspondientes a dos o más centros alineados y la construcción de las mismas" by B. I. Baidaff, 161–164; "Cuestiones de matemáticas elementales; relacionadas con la teoria de los grupos y con los principios del cálculo diferencial" (continued) by J. Duclout, 164–171; "Ejemplos de integración inmediata" (continued) by E. Rebuelto, 172–175; notas, bibliografia, problemas, ejercicios, tabla de las materias, 176–192.

Science, volume 48 (2), November 29, 1918: "Maxime Bôcher" [Minute on his life and services placed upon the records of the faculty of arts and sciences, Harvard University, at the meeting of October 22, 1918], 534–535—December 6: "Means for the scientific development of mathematics teachers" by G. A. Miller, 553–560; "A Greek tract on indivisible lines" by F. Cajori, 577–578—December 20: Reviews by C. L. Poor of Arrhenius's *The Destinies of the Stars* and of Hastings's *Modern Navigation,* 621–622—December 27: "International organization of Science" by G. A. Miller, 649–650—Volume 49, January 10, 1919: "Foundations of mechanics" by P. J. Fox, 44—February 14: "Edward Charles Pickering" by H. N. Russell, 151-155; "The publication of Isis" by G. Sarton, 170–171. [Quotations from Miller's article of December 6: ". . . I believe that if a man would secure a thorough knowledge of certain nine mathematical books beyond a first course in elementary calculus he would be much better informed than the average candidate for the Ph.D. degree. . . . The nine mathematical books whose mastery, together with a fair amount of general mathematical reading, and a development of some of the thoughts contained in these books, would make us an ornament unto our profession could be selected with considerable latitude. As one such selection the following may be noted: Weber, *Lehrbuch der Algebra,* 3 volumes; Goursat, *Cours d'analyse mathématique,* 3 volumes; Veblen and Young, *Projective Geometry,* 2 volumes—the second by Veblen alone; Eisenhart, *Differential Geometry,* one volume. Those who do not read German might substitute for the three volumes of Weber's algebra the following: Bôcher, *Introduction to Higher Algebra*; Miller, Blichfeldt, and Dickson, *Finite Groups*; Reid, *Theory of Algebraic Numbers.*"]

The Texas Mathematics Teachers' Bulletin, volume 4, no. 1, November, 1918: "Some observations on course of study, and mathematics in particular" by J. M. Bledsoe, 6–22; "A mathematical soothsayer" by P. M. Batchelder, 23–24; "Orientation for heavy (coast) artillery" by E. J. Oglesby, 25–29; "Progressive teaching of mathematics" (from *School Science and Mathematics,* May, 1918) by G. W. Myers, 30–40; "The straight edge," 41.

The Tôhoku Mathematical Journal, volume 13, no. 4, June, 1918: "On irreducible equations admitting roots of the form $a + \rho e^{i\theta}$, a and ρ both rational" by A. Kempner, 253–265; "Note on Dr. Muir's paper on 'A Theorem including Cayley's on zero-axial skew determinants of even order'" by W. H. Metzler, 266–268; "Sur une propriété de la courbure de certaines courbes associées au triangle" by R. Goormaghtigh, 269–273; "On the null-system" by Y. Okada, 274–289; "Binary forms and duality" by K. Ogura, 290–295; "On the algebraic correspondence" by T. Kubota and S. Kakeya, 296–299; "Un Théorème sur les continus" by W. Sierpinski, 300–303; "Repeated solutions of a certain class of linear functional equations" by R. D. Carmichael, 304–

313; "Group-theory proof of two elementary theorems in number theory" by G. A. Miller, 314–315—Volume 14, nos. 1–2, August, 1918; "A determinantal theorem and Clifford's theorem on n lines" by T. Hayashi and K. Shibata, 1–10; "A construction-problem in elementary projective geometry" by T. Hayashi, 11–19; "Ueber die Schwerpunkte der konvexen geschlossenen Kurven und Flächen" by T. Kubota, 20–27; "Theory of the point-line convex (1, 1) in space, I" by K. Ogura, 28–63; "Theorems on convergent integrals" by T. Kojima, 64–79; "On the mean center of points on an algebraic curve" by K. Yanagihara, 80–89; "On the mean center of the contact points of tangent planes to an algebraic surface" by K. Shibata, 90–97; "Ueber die Konstruktionsaufgaben dritten und vierten Grades" by T. Kubota, 104–108; "Bemerkung zur Theorie der Approximation der irrationalen Zahlen durch rationale Zahlen" by M. Fujiwara, 109–115; "The stability of the parachute" by S. Brodetsky, 116–123; "A generalized Pascal theorem on a space cubic" by K. Ogura, 124–126; "On integral inequalities between two systems of orthogonal functions" by K. Ogura, 152–154; "Determination of the central forces acting on a particle whose equations of motion possess an integral quadratic in the velocities" by K. Ogura, 155–160.

Transactions of the American Mathematical Society, volume 19, no. 4, October, 1918; "Spiral minimal surfaces" by J. K. Whittemore, 315–330; "On the group of isomorphisms of a certain extension of an abelian group" by L. C. Mathewson, 331–340; "Concerning the zeros of the solutions of certain differential equations" by W. B. Fite, 341–352; "Differentiation with respect to a function of limited variation" by P. J. Daniell, 353–362; "Linear integro-differential equations with a boundary condition" by M. F. Fu, 363–407; "On scalar and vector covariants of linear algebras" by Olive C. Hazlett, 408–420.

University Bulletin, Louisiana State University, volume 10, new series, no. 8, August, 1918: "An appreciation of James W. Nicholson"[1] by S. T. Sanders, 1–31. [Subheadings are: "A truth lover and truth seeker," "The mathematician," "The teacher," "Typical mathematical product," "Nicholson's method," and "Nicholson the philosopher." Nicholson was president of the university and head of the department of mathematics from 1883 to the time of his death in March, 1917.]

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht aller Schulgattungen, volume 49, 1918, Heft 1, January: "Vermessungskunde in Trigonometrieunterricht" by H. Schuhmacher, 1–19 (see also 186)—Heft 2, February: "Eine Ergänzung der Archimedischen Kreismessung" by H. Dörrie, 41–45; "Ueber die Flächenwinkel einer dreiseitigen Ecke" by R. Sturm, 45–46; "Ueber Krümmungen verschiedener Ordnung" by R. Mehmke, 47–49; "Elementare Theorie der ebenen Sonnenuhren nebst einigen speziellen Bemerkungen zur Gnomonik der Araber"49–57; "Die neue preussische Prüfungsordnung für das Lehramt an höheren Schulen" by W. Lietzmann, 58–61—Heft 3, April: "Die Anfänge der analytischen Raumgeometrie" by H. Wieleitner, 73–79; "Elementargeometrische Behandlung der Dupinschen Zyklide" by K. Kommerell, 79–95—Doppel-Heft 4–5, June 13: "Ueber harmonische Kegelschnitte" by H. Pfaff, 113–127; "Ueber Rechenmaschinen und Rechenunterricht (ein Beitrag zu einer Reform der Methodik und Schematik auf kinematischer Grundlage)," 127–139; "Ergänzende Zusätze zu der Arbeit von Herrn Haentzschel 'Eine von Newton gestellte Aufgabe über Sehnenvierecke' (46. Jahrgang, S. 190–194, 1915)" by E. Lampe, 139–144; "Bemerkung zu den vorstehenden 'Zusätzen'" by E. Haentzschel, 144–145; "Das Stellenwertsystem bei den Maya und bei den Indern" by H. Wieleitner; "Berechnung rechtwinkliger Dreiecke bei den Akkadern um 2000 v. Chr." by W. Lietzmann 148–149—Heft 6, June 20; "Ueber die Rentabilität der auslösbaren Schatzanweisungen der 8. Kriegsanleihe und ihre Ermittlung" by P. Lötzbeyer, 161–164; "Eine für Ellipse und Hyperbel gleichlautende Achsenkonstruktion" by F. Redl, 165–170; "Berechnung der Einmal-Prämie für eine 'unterjährige' Leibrente" by K. Wolletz, 170–174; "Zur stetigen Teilung und zum Fünfeck" by W. Weber, 174–178—Heft 7, July: "Schulnomogramme" by P. Luckey, 193–203; "Ueber den Zusammenhang der Heronischen Inhaltsformel mit einigen Gleichungen der Kegelschnitte" by C. Ibrügger, 204–207; "Tangenssatz, Mollweidesche Formeln, Kosinussatz, acht Additionssätze an einer Figur" by W. Weber, 212–213; "Die Winkelhalbierenden des Sehnenvierecks" by C. Stengel, 213–215.

The Yale Alumni Weekly, volume 28, no. 13, December 13, 1918: "Mathematics for freshmen and sophomores, how to make the subject more useful to students in a reorganized curriculum" by E. W. Brown, 310–311.

[1] He was the author of "A simple solution of the Diophantine equation $U^3 = V^3 + X^3 + Y^3$" in this Monthly, September, 1915, vol. 22, pp. 224–225.

AMERICAN DOCTORAL DISSERTATIONS.

The fifth *List of American Doctoral Dissertations* recently published by the Library of Congress[1] contains the following 22 titles in mathematics:

SUZAN R. BENEDICT, *A comparative study of the early treatises introducing into Europe the Hindu art of reckoning* . . . [Concord, N. H., 1916.] 6 + 126 pp. (Univ. of Michigan, 1914.)

R. W. BURGESS, 1887–, *The uniform motion of a sphere through a viscous liquid.* [Reprinted from *Amer. Journ. Math.*, Baltimore, 1916.] Pp. 81–96. (Cornell Univ., 1914.)

H. C. GOSSARD, 1884–, *On a special elliptic ruled surface of the ninth order* . . . [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 431–445. (Johns Hopkins Univ., 1914.)

MARY A. GRIGGS, *The surface tension of mixed liquids.* New York, 1916. 25 pp. (Columbia Univ., 1917.)

J. O. HASSLER, 1884–, *Plane nets periodic of period 3 under the Laplacian transformation.* [Reprinted from *Rendiconti del Circolo Mat. di Palermo*, 1916.] 22 pp. (Univ. of Chicago, 1915.)

OLIVE C. HAZLETT, *On the classification and invariantive characterization of nilpotent algebras.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 109–138. (Univ. of Chicago, 1915.)

F. T. H'DOUBLER, *A study of certain functional equations for the θ-functions*, by E. B. Van Vleck and F. T. H'Doubler. [Reprinted from *Trans. Amer. Math. Soc.*, Lancaster, Pa., 1916.] Pp. 9–49. (Univ. of Wisconsin, 1910.)

G. JOHNSON, 1872–, *The arithmetical philosophy of Nicomachus of Gerasa.* Lancaster, Pa., 1916. 3 + 49 pp. (Univ. of Pennsylvania, 1911.)

H. R. KINGSTON, 1886–, *Metric properties of nets of plane curves.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 407–430. (Univ. of Chicago, 1914.)

J. R. KLINE, 1891–, *Double elliptic geometry in terms of point and order alone.* [Reprinted from *Annals of Math.*, Lancaster, Pa., 1916.] Pp. 31–44. (Univ. of Pennsylvania, 1916.)

F. J. MCMACKIN, 1888–, *Some theorems in the theory of summable divergent series* . . . Lancaster, Pa., 1916. 23 pp. (Columbia Univ., 1916.)

R. W. MARRIOTT, 1882–, *Determination of the order of the groups of isomorphisms of the groups of order p^4, where p is a prime.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 139–154. (Univ. of Pennsylvania, 1911.)

L. C. MATHEWSON, *Theorems on the groups of isomorphisms of certain groups.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 19–44. (Univ. of Illinois, 1914.)

BESSIE I. MILLER, *A new canonical form of the elliptic integral.* [Reprinted from *Trans. Amer. Math. Soc.*, 1916.] Pp. 259–283. (Johns Hopkins Univ., 1914.)

W. L. MISER, 1886–, *On multiform solutions of linear differential equations having elliptic function coefficients.* [Reprinted from *Trans. Amer. Math. Soc.*, 1916.] Pp. 109–130. (Univ. of Chicago, 1913.)

E. J. MOULTON, *On figures of equilibrium of a rotating compressible fluid mass; certain negative results.* [Reprinted from *Trans. Amer. Math. Soc.*, 1916.] Pp. 100–108. (Univ. of Chicago, 1913.)

A. L. NELSON, 1891–, *Plane nets with equal invariants.* [Reprinted from *Rendiconti del Circolo Mat. di Palermo*, 1916.] 25 pp. (Univ. of Chicago, 1915.)

D. M. SMITH, 1884–, *Jacobi's condition for the problem of Lagrange in the calculus of variations.* [Reprinted from *Trans. Amer. Math. Soc.*, 1916.] Pp. 459–475. (Univ. of Chicago, 1916.)

L. A. H. WARREN, 1879–, *A class of asymptotic orbits in the problem of three bodies.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 221–247. (Univ. of Chicago, 1913.)

C. WOODY, 1884–, *Measurement of some achievements in arithmetic.* New York, Teachers College, 1916. 3 + 63 pp. (Columbia Univ., 1916.)

C. H. YEATON, 1886–, *Surfaces characterized by certain special properties of their directrix congruences.* [Reprinted from *Annali di mat.*, Milano, 1916.] 33 pp. (Univ. of Chicago, 1915.)

MABEL M. YOUNG, *Dupin's cyclide as a self-dual surface.* [Reprinted from *Amer. Journ. Math.*, 1916.] Pp. 267–286. (Johns Hopkins Univ., 1914.)

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[1] *A List of American Doctoral Dissertations printed in 1916.* Washington, Government Printing Office, 1918.

PROBLEMS AND SOLUTIONS.

Edited by B. F. FINKEL and OTTO DUNKEL.

Send all communications about problems to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[Errata: *To the number of each problem for solution in the issue of February, 1919, 4 should be added*].

2757. Proposed by ERNEST P. LANE, Rice Institute, Houston, Texas.

Integrate by quadrature the differential equation

$$\frac{d^2y}{dx^2} - 3y\frac{dy}{dx} + y^3 = 0.$$

2758. Proposed by L. RICHARDSON, Vancouver, B. C., Canada.

Prove that, if r be a positive integer,

$$\int_0^{\pi/2} \frac{\sin (2r + 1)\psi}{\sin \psi} d\psi = \frac{\pi}{2}$$

and

$$\int_0^{\pi/2} \frac{\sin 2r\psi}{\sin \psi} d\psi = 2\left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{r-1}}{2r-1}\right\}.$$

2759. Proposed by J. L. RILEY, Stephenville, Texas.

Solve the simultaneous functional equations

$$\Phi(x + y) = \Phi(x) + \frac{\Phi(y)\cdot\psi(x)}{1 - \Phi(x)\Phi(y)},$$

$$\psi(x + y) = \frac{\psi(x)\psi(y)}{1 - \Phi(x)\Phi(y)}.$$

2760. Proposed by CHARLES N. SCHMALL, New York City.

In an arithmetical progression, if s_n be the sum of the first n terms, s_{2n} the sum of the first $2n$ terms, and s_{3n} the sum of the first $3n$ terms of the same series, prove that $s_{2n} - s_n = \frac{1}{3}s_{3n}$.

2761. Proposed by W. W. DENTON, Ann Arbor, Michigan.

Find the length of the side of an equilateral triangle whose vertices are at given distances a, b, c, from a given point.

SOLUTIONS OF PROBLEMS.

411 (Calculus) [June, 1916]. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

Prove that the volume bounded by the surface $f(x, y, z) = 0$ is $1/3\iint (z - x(\partial z/\partial x) - y(\partial z/\partial y))$ $dxdy$ integrated over the area determined by projecting the surface on the xy-plane.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

The above problem is not stated accurately for all cases. The integral should be taken over the entire surface,—that is we have a surface integral, where the element of surface, dS, is replaced by its projection on the xy-plane in each element of the sum. This is not the same as integrating over the area described above.

Let us assume that the above surface is a closed, finite one with no double points or singularities. Then the direction cosines of the normal at any point are proportional to $-\partial z/\partial x$, $-\partial z/\partial y$, 1. If we denote the direction cosines by α, β, γ, and remember that $\gamma\cdot dS\cdot = dxdy$, the given integral becomes $1/3\iint (\gamma z + \alpha x + \beta y)dS$ taken over the surface. By a special case of Green's Theorem (B. O. Peirce, *Short Table of Integrals*, formula 883) this becomes $1/3\iiint(1 + 1 + 1)dxdydz$, taken throughout the volume inclosed by the surface, which is clearly the volume required.

That the above formula does not give the volume bounded by $f(x, y, z) = 0$, the xy-plane and a cylinder whose elements are parallel to the z-axis, may be readily seen by applying it to the plane $z = c$, in which case it gives a result one-third as large as the correct result.

Also solved by the Proposer.

239 (Number Theory) [March, 1916]. Proposed by HAROLD T. DAVIS, Colorado Springs, Colorado.

Give a general method for determining the solution in integers of the equation

$$x^r - 10xy - (n + 1) + y = 0,$$

where n and r are positive integers.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Solving for y, $y = \dfrac{x^r - (n + 1)}{10x - 1}$, which must be an integer. Since the denominator is prime to 10 and the numerator integral, y will be an integer if $\dfrac{10^r x^r - (n + 1)\cdot 10^r}{10x - 1}$ is. Dividing algebraically the remainder is $1 - (n + 1)\cdot 10^r$. If then $\dfrac{10^r(n + 1) - 1}{10x - 1}$ is an integer, so is y. Hence the general process (perhaps not that desired) is the following: form $10^r(n + 1) - 1$ and factor it. Equate $10x - 1$ to any factor whose last digit is 9, and we have an integral solution.

261 (Number Theory) [March, 1917]. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that for any positive integer n (excluding powers of 2) positive integers $a_1, a_2, a_3, \cdots, a_k$ which are less than $n/2$ can be chosen in such a way that

$$2^k \cos (a_1\pi/n) \cos (a_2\pi/n) \cos (a_3\pi/n) \cdots \cos (a_k\pi/n) = 1.$$

SOLUTION BY C. F. GUMMER, Queen's University.

Since n is not a power of 2, it is of the form $(2k + 1)l$. The equation

$$\cos (2k + 1)x - \cos (2k + 1)\alpha = 0$$

has $2k + 1$ distinct roots in $\cos x$, when $\cos (2k + 1)\alpha \neq 1$, the roots being

$$\cos \left(\alpha + \frac{2i\pi}{2k + 1} \right), \qquad i = 0, 1, \cdots, 2k.$$

Also $\cos (2k + 1)x = 2^{2k} \cos^{2k+1} x - \cdots$, the absolute term being zero. Hence,

$$2^{2k} \prod_{i=0}^{2k} \cos \left(\alpha + \frac{2i\pi}{2k + 1} \right) = \cos (2k + 1)\alpha.$$

By taking the limit of each side when $\alpha \rightarrow 0$,

$$2^{2k} \prod_{i=0}^{2k} \cos \frac{2i\pi}{2k + 1} = 1,$$

that is,

$$\left\{ 2^k \prod_{i=0}^{k} \cos \frac{2i\pi}{2k + 1} \right\}^2 = 1,$$

or

$$2^k \prod_{i=0}^{k} \cos \frac{2i\pi}{2k + 1} = \pm 1.$$

By taking $j = 2i$, when $i \leqq k/2$ and $j = 2k + 1 - 2i$ when $i > k/2$, we get

$$2^k \prod_{j=1}^{k} \cos \frac{j\pi}{2k + 1} = + 1,$$

which takes the required form if $a_j = jl$.

2665, 2670 [Jan., Feb., Sept., and Dec., 1918]. **Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.**

A telegraph wire which weighs 1/10 of a pound per yard is stretched between poles on a level ground so that the greatest dip of the wire is 3 feet. Find approximately the distance between the poles when the tension at the lowest point of the wire is 140 pounds.

III. SOLUTION BY J. B. REYNOLDS, Lehigh University.

Choosing any small portion of the wire of length as resolving tangentially and normally to the curve of equilibrium and passing to differentials we get the fundamental equations

$$T\frac{d\varphi}{ds} = w\cos\varphi, \quad \frac{dT}{ds} = w\sin\varphi \tag{1}$$

in which T is the tension in the wire, w the weight per foot, φ the angle the tangent makes with the horizontal and s the length of the curve measured from the lowest point.

Dividing and integrating, we find

$$T = T_0\sec\varphi \tag{2}$$

T_0 being tension at lowest point. Putting this value of T in the second equation above we find

$$ws = T_0\tan\varphi. \tag{3}$$

Now since $\tan\psi = dy/dx$ and $ds/dx = \sqrt{1+(dy/dx)^2}$ we may, from (3), arrive at the equation

$$y = \frac{T_0}{2w}\{e^{wx/T_0} + e^{-wx/T_0}\} - \frac{T_0}{w}, \tag{4}$$

y being measured vertically from the lowest point of the curve, which is taken as the origin. Likewise, from these equations we may find

$$T_0 = \frac{w}{8d}(l^2 - 4d^2), \tag{5}$$

where l is the length of wire between poles and d is the greatest dip, and

$$l = h\left\{1 + \frac{8}{3}\left(\frac{d}{h}\right)^2 - \frac{224}{45}\left(\frac{d}{h}\right)^4\cdots\right\}, \tag{6}$$

in which h is the horizontal distance between poles. For this problem, $T_0 = 140$, $w = 1/30$; so, by (4),

$$y = 2100\{e^{x/4200} + e^{-x/4200}\} - 4200.$$

Expanding to x^2, we find, when $y = 3$, $x^2 = 25200$; whence $x = 159$ ft., $2x = 318$ ft., as a good approximation of the distance between poles.

If we use equation (5), we have $l^2 = 100836$; whence $l = 317$ ft.—a nearer value. Finally by using (6) we get $h = 316 +$ ft.

2679 [Feb., 1918]. **Proposed by J. W. LASLEY, JR., University of North Carolina.**

Show that the perpendicular from any point on a circle to any chord of the circle is a mean proportional to the perpendiculars from that point to the tangents at the ends of the chord.

SOLUTION BY C. E. GITHENS, Wheeling, W. Va.

Let CD be the perpendicular from the point C to any chord AB of the given circle, and let EC and CH be the perpendiculars from C to the tangents to the circle at the points A and B respectively. To prove $EC \times HC = CD^2$. The quadrilaterals $AECD$ and $BHCD$ are similar since the angles of the one are respectively equal to the angles of the other, that is, angle EAD = angle DBH, angle CDB = angle ADC = angle AEC = angle CHB, being right angles. Hence, the corresponding sides are proportional; that is, $EC : CD = CD : CH$, or $EC \times CH = CD^2$.

Also solved by C. A. BARNHART, PAUL CAPRON, L. E. MENSENKAMP, ROGER A. JOHNSON, I. J. FAJANS, SAMUEL COHEN, O. E. SIMONSEN, I. MILLENKY,

J. V. Fazio, S. Weinberger, N. P. Pandya, H. L. Olson, Elijah Swift, W. L. Lord, J. W. Lasley, T. R. Thomson, Abraham Platman, H. Gladstone, Peter Pumo, Isadore Rose, R. M. Mathews, J. L. Riley.

2680 [March, 1918]. **Proposed by C. C. Yen, Tangshan, North China.**

The diagonals of a maximum parallelogram inscribed in an ellipse are conjugate diameters of the ellipse. (From Joseph Edward's *Elementary Treatise on Differential Calculus.*)

Solution by L. E. Mensenkamp, Freeport, Illinois.

Let the equation of the ellipse be $x^2/a^2 + y^2/b^2 = 1$. Now, it is easily shown that a quadrilateral inscribed in an ellipse is a parallelogram when, and only when, the opposite vertices are symmetrical with respect to the origin. In other words, the diagonals of an inscribed parallelogram pass through the center of the ellipse. Let us denote one vertex of the parallelogram (which for convenience we may assume to be in the first quadrant) by $P_1 = (x_1, y_1)$. Let an adjacent vertex be $P_2 = (x_2, y_2)$. Then, the vertex opposite P_1 will be $P_3 = (-x_1, -y_1)$, and that opposite P_2 will be $P_4 = (-x_2, -y_2)$.

We may call P_2P_3 the base of the parallelogram. Its equation may be written

$$(y_1 + y_2)x - (x_1 + x_2)y - (x_1 + x_2)y_1 + (y_1 + y_2)x_1 = 0.$$

The equation in this form enables us to apply the usual formula for the distance from a point to a line to obtain the distance from P_1 to the line P_2P_3, which is the altitude of the parallelogram. This expression for the altitude, after some reduction, becomes

$$D = \frac{2(x_1y_2 - x_2y_1)}{\sqrt{(y_1 + y_2)^2 + (x_1 + x_2)^2}}. \tag{1}$$

Multiplying D by the length of the base P_2P_3, we find the area of the parallelogram to be $A = 2(x_1y_2 - x_2y_1)$. Making use of the fact that these points lie on the ellipse, and assuming for the moment that P_2 lies above the X-axis, the area becomes

$$A = \frac{2b}{b}(x_1\sqrt{a^2 - x_2^2} - x_2\sqrt{a^2 - x_1^2}).$$

The condition for a maximum is that $\partial A/\partial x_1 = 0$, and $\partial A/\partial x_2 = 0$. Both of these conditions lead to the same equation, namely,

$$x_1x_2 = -\sqrt{a^2 - x_1^2}\sqrt{a^2 - x_2^2}. \tag{2}$$

Therefore,

$$\frac{y_1y_2}{x_1x_2} = \frac{y_1y_2}{-\sqrt{a^2 - x_1^2}\sqrt{a^2 - x_2^2}} = -\frac{b^2}{a^2}.$$

The last member of this equation follows from the elimination of y_1 and y_2 by means of the equation for the ellipse. Since y_1/x_1 and y_2/x_2 represent the slopes of the diagonals of the parallelogram, and since their product equals $-b^2/a^2$, it is seen (Bôcher, *Plane Analytic Geometry*, p. 153) that the diagonals are conjugate diameters of the ellipse.

If P_2 is assumed to be below the X-axis, we must remember to use

$$y_2 = -\frac{b}{a}\sqrt{a^2 - x_2^2},$$

but the final result is the same. It is evident that the equation (2) gives a maximum, and not a minimum, parallelogram; for, assuming P_2 temporarily fixed while P_1 varies, we see that the area is zero when P_1 coincides with either P_2 or P_4.

Also solved by J. B. Reynolds.

2681 [March, 1918]. **Proposed by Philip Franklin, College of the City of New York.**

Given n letters of one kind and $n - 1$ letters of another kind, in how many ways can they be arranged so that, moving along the arrangement from one end to the other, the number of letters of the first kind passed over is greater than the number of the second kind at any instant?

SOLUTION BY C. F. GUMMER, Queen's University.

Let $f(p, q)$ be the number of ways of arranging p letters, some A and some B, so that, on going from left to right, we pass *always* more A's than B's, and *finally* q more A's than B's. Clearly $f(p, q) = 0$ when $p - q$ is odd and when $q < 1$.

Such an arrangement will end with an A in $f(p-1, q-1)$ cases, $(p > 1)$; and it will end with a B in $f(p-1, q+1)$ cases $(q \neq 0)$. Therefore,

$$f(p, q) = f(p-1, q-1) + f(p-1, q+1)$$

under the above restrictions.

To remove the restriction $q \neq 0$, let us define

$$g(p, q) = f(p, q), \quad (q \geqq 0);$$

$$g(p, q) = -f(p, -q), \quad (q < 0).$$

Then

$$g(p, q) = g(p-1, q-1) + g(p-1, q+1)$$

provided $p > 1$.

By successive application,

$$g(p, q) = g(p-2, q-2) + 2g(p-2, q) + g(p-2, q+2) = g(p-3, q-3) + 3g(p-3, q-1) + 3g(p-3, q+1) + g(p-3, q+3) = \cdots = g(1, q-p+1) + \binom{p-1}{1} g(1, q-p+3) + \binom{p-1}{2} g(1, q-p+5) + \cdots.$$

Now $g(1, q) = 0$, except that $g(1, 1) = 1$ and $g(1, -1) = -1$.

Hence, if $p - q$ is even,

$$g(p, q) = -\binom{p-1}{(p-q-2)/2} + \binom{p-1}{(p-q)/2} = \frac{2q}{p+q}\binom{p-1}{(p-q)/2}.$$

For the particular problem, $p = 2n - 1$, $q = 1$, and

$$f(p, q) = g(p, q) = \frac{1}{n}\binom{2n-2}{n-1} = \frac{\lfloor 2n-2}{\lfloor n \, \lfloor n-1}.$$

2683 [March, 1918]. Proposed by J. R. HITT, Mississippi College.

The height of a frustum of a cone is h, the radii of the upper and lower circular bases are a and b, respectively. Deduce the formula for finding the center of gravity of the frustum.

SOLUTION BY H. C. GOSSARD, U. S. Naval Academy.

Since the center of gravity is, obviously, on the axis of the frustum, let y be the required distance of the center of gravity above the lower base. Complete the cone and let x be the altitude of the upper cone. From similar triangles, $x : x + h = a : b$; whence, by division, $x : h = a : b - a$, or $x = ah/(b - a)$. The altitude of the entire cone is $x + h = bh/(b - a)$.

The volume of the whole cone is $\frac{\pi}{3}\frac{hb^3}{(b-a)} = V_1$ and the volume of the upper cone is $\frac{\pi}{3}\frac{ha^3}{(b-a)} = V_2$.

Hence, the volume of the frustum is $\frac{\pi}{3}\frac{hb^3}{(b-a)} - \frac{\pi}{3}\frac{ha^3}{(b-a)} = \frac{\pi}{3}h(a^2 + b^2 + ab) = V_3$.

Remembering that the center of gravity of a cone is on its axis of symmetry and 1/4 of the distance from the center of gravity of the base to the vertex measured from the base, we have, on taking moments about any diameter of the base, $V_3 y = \frac{1}{4}bh/b - aV_1 - \frac{1}{4}ah/b - aV_2$. Substituting the values of V_1, V_2, and V_3 and solving for $\bar{y}$ and reducing to simplest form, we have

$$y = \frac{h}{4}\left(\frac{b^2 + 2ab + 3a^2}{b^2 + ab + a^2}\right).$$

Also solved by H. L. OLSON, ALBERT N. NAUER, JOSEPH B. REYNOLDS, C. E. FLANAGAN, G. PAASWELL, and HERBERT N. CARLETON.

2684 [March, 1918]. Proposed by B. J. BROWN, Kansas City, Missouri.

Find the locus of the center of a conic passing through four fixed points.

I. SOLUTION BY H. D. THOMPSON, Princeton University.

This is an exercise given in books on coördinate geometry.

Take the x-axis through two of the four points, and the y-axis, oblique, through the other two. Let $1/l$ and $1/l'$ be the abscissas of the two points on the x-axis and $1/m$ and $1/m'$, the ordinates of the two points on the y-axis. Then $lx + my - 1 = 0$ and $l'x + m'y - 1 = 0$ form another pair of lines, containing the four points in pairs. All conics through the four points are given by $\lambda xy + (lx + my - 1)(l'x + m'y - 1) = 0$, when λ is the parameter of the system.

The center of a representative conic is given by $2ll'x + (lm' + l'm)y - (l + l') + \lambda y = 0$ and $(lm' + l'm)x + 2mm'y - (m + m') + \lambda x = 0$. Eliminating λ, the locus of the center is $2ll'x^2 - 2mm'y^2 - (l + l')x + (m + m')y = 0$, a conic through the origin. The center of the locus is $\{1/4(1/l' + 1/l), 1/4(1/m' + 1/m)\}$. As any pair of opposite sides of the complete quadrilateral with the original four points as vertices can be taken as the axes, the locus of the centers will pass through the three points of intersection of opposite sides of the complete quadrilateral.

The locus is an hyperbola when $ll'mm'$ is positive, that is, when the original four points may be taken as the vertices of a convex polygon; and it is in this case only that the original conic may be a parabola (two).

II. SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The equation of a conic passing through the four given points $\pm\alpha_1$, $\pm\beta_1$, $\pm\gamma_1$, trilinear coördinates being used, is of the form

$$l_1\alpha^2 + m_1\beta^2 + n_1\gamma^2 = 0 \tag{1}$$

with the condition

$$l_1\alpha_1^2 + m_1\beta_1^2 + n_1\gamma_1^2 = 0. \tag{2}$$

The coördinates of the center of (1) are given by

$$\frac{l_1\alpha}{a} = \frac{m_1\beta}{b} = \frac{n_1\gamma}{c}, \tag{3}$$

a, b, c, being the sides of the fundamental triangle.

Eliminating l_1, m_1, n_1 from (3) and (2), we have

$$\frac{a\alpha_1^2}{\alpha} + \frac{b\beta_1^2}{\beta} + \frac{c\gamma_1^2}{c} = 0, \tag{4}$$

the required locus.

This is the nine-point conic of the quadrilateral whose vertices are the four given points.

Also solved by PAUL CAPRON and ELIJAH SWIFT.

2685 [March, 1918]. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

A particle is describing an ellipse of eccentricity $\sqrt{2/3}$ as a central orbit about a focus when the attracting force suddenly becomes repulsive without changing its magnitude and the particle begins to describe an equilateral hyperbola; find where the change occurred and the angle that the major axis of the new orbit makes with that of the old orbit.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The focal equation of the ellipse is

$$p_1^2 = \frac{a_1^2(1 - e_1^2)r_1}{2a_1 - r_1} \tag{1}$$

and of the hyperbola,

$$p_2^2 = \frac{a_2^2(e_2^2 - 1)r_2}{2a_2 + r_2} \tag{2}$$

an r being a radius vector; e, eccentricity; a, semi-major axis; p, perpendicular from a focus upon the corresponding tangent. In the ellipse, $e_1^2 = 2/3$, and in the hyperbola, $e_2^2 = 2$.

For the central force in (1),

$$F_1 = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{a_1(1-e_1^2)} \cdot \frac{1}{r_1^2}, \tag{3}$$

h being the usual constant in the theory of central forces.

By condition, we may write

$$-\frac{h^2}{a_1(1-e_1^2)} \cdot \frac{1}{r_2^2} = \frac{h^2}{p_2^3}\frac{dp_2}{dr_2}. \tag{4}$$

Integrating,

$$\frac{1}{a_1(1-e_1^2)} \cdot \frac{1}{r_2} = -\frac{1}{2p_2^2} + C \tag{5}$$

Let $r_2 = c = r_1$ at the instant of change in the nature of the force, when also

$$p_2^2 = p_1^2 = \frac{1}{a_1^2(1-e_1^2)}\,\frac{2a_1 - c}{c} \tag{6}$$

by (1); then

$$C = \frac{4a_1 - c}{2a_1^2(1-e_1^2)c} \tag{7}$$

and (5) becomes

$$\frac{1}{p_2^2} = \frac{(4a_1 - c)r_2 - 2a_1c}{a_1^2c(1-e_1^2)r_2} \tag{8}$$

or,

$$p_2^2 = \frac{a_1^2c(1-e_1^2)}{4a_1 - c}r_2 \div \left\{ r_2 - \frac{2a_1c}{4a_1 - c} \right\}$$

which is plainly the hyperbola in (2).

Now, (9) and (2) are identical if

$$a_2^2(e_2^2 - 1) = \frac{a_1^2c(1-e_1^2)}{4a_1 - c} \tag{10}$$

and

$$2a_2 = -\frac{2a_1c}{4a_1 - c}. \tag{11}$$

Eliminating a_2, and solving for c, $c = a_1$ (12), showing that the nature of the force changes at an extremity of the minor axis of the ellipse.

Let α = the angular coördinate of this point, the position of the center of force being the pole, γ = the required angle of the problem, and l_1, l_2, the latera recta of the curves; then the focal polar equations of the curves are

$$\frac{l_1}{r} = 1 - e_1 \cos\theta \quad (13), \qquad \frac{l_2}{r} = 1 - e_2\cos(\theta - \gamma) \quad (14).$$

The equations of the tangent to these at the common point α are

$$\frac{l_1}{r} = \cos(\theta - \alpha) - e_1\cos\theta \quad (15), \qquad \frac{l_2}{r} = \cos(\theta - \alpha) - e_2\cos(\theta - \gamma) \quad (16),$$

or, in cartesian coördinates,

$$(\cos\alpha - e_1)x + y\sin\alpha = l_1 \tag{17}$$

$$(\cos\alpha - e_2\cos\gamma)x + (\sin\alpha - e_2\sin\gamma)y = l_2. \tag{18}$$

These are identical if

$$\frac{\cos\alpha - e_1}{l_1} = \frac{\cos\alpha - e_2\cos\gamma}{l_2} \quad (19), \qquad \frac{\sin\alpha}{l_1} = \frac{\sin\alpha - e_2\sin\gamma}{l_2} \quad (20).$$

But from the ellipse, $\cos\alpha = e_1$, and (19) gives

$$\cos\gamma = \frac{e_1}{e_2} = \tfrac{1}{3}\sqrt{3}. \tag{2}$$

It may be added that the condition that (13) and (14) have the kind of contact consistent with the nature of the problem is

$$l_1^2(1 - e_2^2) + l_2^2(1 - e_1^2) - 2l_1l_2 + 2l_1l_2e_1e_2 \cos \gamma = 0. \tag{22}$$

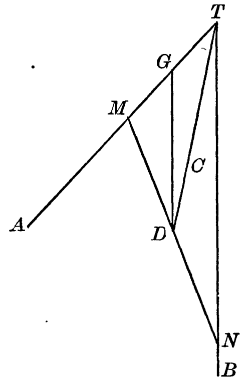
Also solved by HORACE L. OLSON.

2687 [March, 1918]. Proposed by N. P. PANDYA, Sojitra, India.

An ellipse intersects a parabola in A and B, and the tangents at A and B to the parabola meet at T. The center C of the ellipse lies within the space enclosed by the parabola and the tangents. Draw a third tangent to the parabola such that C may be the centroid of the triangle formed by the three tangents.

SOLUTION BY HORACE L. OLSON, Heidelberg University, Tiffin, Ohio.

This problem can be solved, if at all, without reference either to the ellipse or to the parabola. For this purpose let us alter the problem to read as follows: "Given two straight lines, TA and TB, intersecting at T, and a point C; draw a third line which shall form, with TA and TB, a triangle whose centroid shall be the point C." The problem, as thus stated, has a unique solution. If the third straight line is tangent to the parabola mentioned above, it is the solution of the original problem; if not, there is no solution. Draw the line TC, and extend it beyond C to D, so that $CD = \frac{1}{2}TC$. D will then be the mid-point of the third side of the required triangle. Through D, draw the line GD parallel to TB.



Lay off $GM = TG$. The line MDN, intersecting TB at N, will then be the required line; for, since the line GD is parallel to the side TN of the triangle TMN and bisects the side TM, it must also bisect the side MN. Since this demonstration is reversible, MN is the only line-segment included between the lines TA and TB, and bisected at D. Hence, if MN is tangent to the given parabola, it is the solution of the original problem; if not, there is no solution.

Also solved analytically by WILLIAM HOOVER.

2691 [April, 1918]. Proposed by ROGER A. JOHNSON, Hamline University.

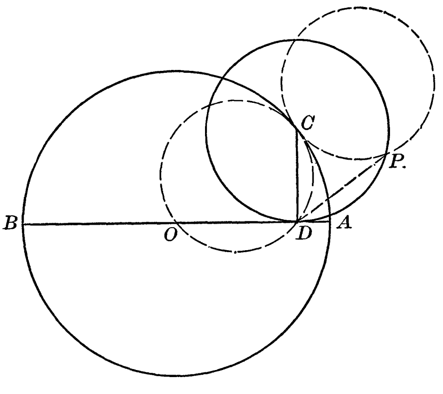
Show by purely geometric methods, without the use of the calculus, that the envelope of the circles whose centers are on a fixed circle and which touch a fixed diameter of that circle is a two-arched epicycloid. Cf. problem 423 (calculus) [February and September, 1917].

SOLUTION BY THE PROPOSER.

First, let us make a few general remarks about the envelope of a system of circles. In general, the points where one of a system of curves meets the envelope of the system are the limiting positions of the points of intersection of that curve with a near-by one, as the latter comes into coincidence with the former. But two circles generally intersect in two points; the perpendicular bisector of the line connecting them is the line joining the centers of the circles. Hence,

If a system of circles has an envelope, then generally each circle of the system meets the envelope at two points; the chord connecting these points is perpendicular to the diameter to said circle which is tangent to the curve of centers.

In the problem before us, a part of the envelope is known; namely, the fixed diameter. Let AB be a diameter of the circle with center O, radius R; let CD be the perpendicular from a point C of the circumference on the diameter AB. Then the circle with center C, radius CD, is one of the system whose envelope is to be found; and our problem reduces to that of determining the locus of the point P, which is symmetrical to D with regard to the tangent to the given circle at C.



Consider the circle having OC as diameter. Since ODC is a right angle, it passes through D. Hence, the circle symmetrical to it, in other words, the circle whose diameter equals R and which touches the given circle externally at C, passes through P.

Further, arc PC = arc DC = arc CA, since the one radius is half the other, and the angle COD inscribed in the one arc is a central angle with regard to the other.

Hence, P is the locus of a point fixed on the circle PC, of radius 1/2 R, as this circle rolls without sliding around the circumference of the given circle. In other words, the locus is a two-arched epicycloid.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Illinois.

The April meeting of the Chicago Section of the American Mathematical Society will be held at the University of Chicago on Friday and Saturday, April 4, 5. Besides the usual research papers there will be a Symposium on "Geometry of Numbers" on Friday afternoon. Two formal papers will be presented; one by Professor H. F. BLICHFELDT, of Leland Stanford University, and the other by Professor L. E. DICKSON, of the University of Chicago.

A school of navigation is being conducted this winter at Queen's University, Kingston, Canada. The course is for Canadian lake and river mariners. Instruction is given for master's and instructor's certificates, and for those who cannot attend the day courses classes are held at night. The University mathematics and physics departments furnish the lectures on these scientific subjects.

Colleges and universities offering summer courses in mathematics are requested to report an outline of them to the editor of Notes and News as soon as possible. The following reports have been received:

Brown University, June 30–August 30, is to offer four courses each with a session of five hours per week: Solid geometry by Professor R. C. ARCHIBALD; Topics in geometry by Professor Archibald; Elementary calculus by Professor C. H. CURRIER; Analytic geometry and calculus (for engineers) by Professor CURRIER.

University of California, June 30–August 9. The University will conduct two Summer Sessions, one at the State Normal School in Los Angeles, the other at the University in Berkeley. The courses offered are: At Los Angeles, Professor F. P. BRACKETT: Plane trigonometry, Introduction to calculus; Professor B. A. BERNSTEIN: Plane analytic geometry, Logic of mathematics (Teacher's course). At Berkeley, Professor PUTNAM: Graphic algebra, Analytic geometry; Dr. PAULINE SPERRY: Solid geometry, Plane trigonometry; Professor F. CAJORI: Teaching of secondary mathematics, History of mathematics. Professor F. MORLEY, of Johns Hopkins University, will give a course on special fields of Geometry, together with a seminary on topics covered in the lecture course.

University of Iowa, Summer Session, first term, June 16–July 26. Professor H. L. RIETZ: Teacher's Course; Professor E. W. CHITTENDEN: Analytic Geometry, Calculus, Reading and Research; Mr. R. E. GLEASON: College Algebra; Trigonometry; Differential Equations. Second term, July 28–Aug. 30. Professor E. W. CHITTENDEN: Analytic Geometry, Calculus, Reading and Research.

Ohio State University. The following courses will be offered by Professors R. D. BOHANNAN, KUHN and C. C. MORRIS: College algebra, Plane trigonometry, Analytic geometry (10 hours); Differential calculus (5 hours); Integral Calculus (5 hours); Differential equations (3 hours); Fundamental concepts of algebra and geometry (2 hours). A course for teachers and those intending teaching as a profession.

Dr. C. E. WILDER has accepted a position at Clark University.

Dr. W. E. MILNE, formerly of Bowdoin College but more recently a first lieutenant in the ordnance reserve corps, has been appointed assistant professor of mathematics in the University of Oregon.

Professor C. N. MILLS has resumed his teaching in South Dakota State College after having served in the National Army.

Professor J. B. SMITH, formerly professor of mathematics in Hampden-Sidney College, is serving with the rank of first lieutenant as chief of the statistical division, office of the director of purchase and storage, at Washington.

Mrs. E. R. BECKWITH, assistant professor of mathematics in the College for Women, Western Reserve University, was in charge of a round table discussion

on accounting, insurance, statistics, and drafting, at the fourth vocational conference for women held at Oberlin College on February 8.

Dr. W. H. Garnett, Dean and Professor of Mathematics at Wesleyan College, Winchester, Ky., is spending the year in California to rest, after thirty-seven years of continuous service at Wesleyan.

At the University of Kentucky, Mr. W. W. Elliott has been appointed Assistant in Mathematics and Mr. H. P. Pettitt holds a fellowship in mathematics.

Professor A. L. Rhoton, of the mathematics department at Georgetown, Ky., College, has resigned to accept the position of Associate Professor of Education at Pennsylvania State College.

Mr. Will E. Edington, who has been engaged in the study of exterior ballistics in the meteorological section, Division of Science and Research, Signal Corps, at Fort Monroe, has returned to the University of Illinois as assistant in mathematics and will continue his graduate work.

Professor E. R. Hedrick, of the University of Missouri, Director of Mathematics in connection with the work of the Y. M. C. A. for the American soldiers in France, will probably be gone for a year or more. Professor J. W. Young of Dartmouth College is engaged in executive work in connection with organizing this overseas school for the soldiers. His headquarters for this work are in New York City.

G. X. P. Koenigs, professor of physical and experimental mechanics at the University of Paris, has been elected a member, in the section of mechanics, of the Academy of Sciences of the Institute of France, in place of the late Professor H. C. V. J. Léauté.

Dr. F. W. Reed has been appointed instructor in mathematics at the University of Illinois.

Mr. J. S. Mikesh has been appointed instructor in mathematics at Sheffield Scientific School, Yale University.

Mr. A. D. Campbell and Dr. C. M. Smith have been appointed instructors in mathematics at Cornell University.

Charles Wolf, the French astronomer, died July 4, 1918, in the ninety-first year of his age.

E. C. Pickering, professor of practical astronomy and director of the Harvard College observatory since 1876, died February 3, 1919, in the seventy-third year of his age.

Miss Z. FERGUSON, instructor in mathematics at the University of Missouri for the S. A. T. C. term, returned to her previous position in St. Joseph, Mo., in January.

Dr. E. A. KIRCHER, for a time Benjamin Peirce instructor at Harvard, but more recently battery commander, battalion adjutant and acting battalion commander on the Western Front, has accepted a position in connection with the National City Bank, New York City.

The Royal Society Copley Medal for 1918 was awarded to Professor H. A. LORENTZ, of the University of Leiden, for his distinguished researches in mathematical physics.

Mr. P. A. FRALEIGH has returned from military service at the Aberdeen Proving Ground to resume his duties as instructor in mathematics at Cornell University.

Dr. PAUL CARUS, editor and manager of the *Open Court* and *The Monist* at La Salle, Illinois, died on February 11, 1919, in the sixty-seventh year of his age. He was a very prolific writer. Among his books are: *The Foundation of Mathematics*, 1908, and *The Principle of Relativity in the Light of the Philosophy of Science*, 1913. He wrote also some chapters in *Magic Squares and Cubes* by W. S. Andrews, 1908.

In the list (published in *The Times*, London, January 9) of promotions and appointments in connection with the Civil Division of the Order of the British Empire, for services in connection with the war, occur the names of Commander T. A. FERRIER, Mathematical Instrument Office, Calcutta, India; Officer A. W. CODD, chief cartographer, Hydrographic Department, Admiralty; and Officer J. W. FOORD-KELCEY, professor of mathematics and mechanics, Royal Military Academy.

In December, 1916, the Indian Mathematical Society held its "first conference" at Madras. The proceedings were opened by His Excellency the Governor of Madras and the conference lasted for three days. It was announced that the second conference would meet at Bombay, January 11–13, 1919. Among the five members of the Society who were recent victims of Spanish influenza, was a frequent contributor to its *Journal*, Mr. R. J. POCOCK, director of the Nizamiah Observatory, Begumpet, who died in November, 1918.—Apart from India's mathematical activity in connection with this Society it will be recalled that A. R. Forsyth's *Lectures Introductory to the Theory of Functions of Two Complex Variables* were delivered at the University of Calcutta in 1913, and that C. E. Cullis's work on *Matrices and Determinoids*, of which two large volumes have already appeared, was the outcome of "Readership Lectures" at the University of Calcutta where Mr. CULLIS is Hardinge professor of mathematics.

The "Committee on the MAXIME BÔCHER MEMORIAL" appointed by the American Mathematical Society, with Professor T. S. FISKE as chairman, consists of 80 members at present. It is hoped that a fund of two or three thousand dollars may be procured. "Contributors are invited to express their views and wishes in regard to the use which should be made of the fund. One suggestion is that the income of the fund be used once in four years to send to the International Congress of Mathematicians one or more delegates chosen by the Council to represent the American Mathematical Society. Another suggestion is that the income of the fund be used from time to time to assist a promising student in the last year of his study for the doctorate, perhaps to send him to the University of Paris, and that the theses of all students thus assisted be suitably inscribed. Still another suggestion is that the income of the fund be used to assist in the publication of meritorious scientific works for which otherwise it might be difficult or impossible to secure a publisher. . . . Subscriptions should be addressed to the treasurer of the Committee, Professor J. H. Tanner, Cornell University, Ithaca, New York."

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FERMAT NUMBERS $F_n = 2^{2^n} + 1$.[1]

By R. D. CARMICHAEL, University of Illinois.

INTRODUCTION AND HISTORICAL REMARKS.

1. One of the most fascinating of the small separated regions of mathematics is that pertaining to the Fermat numbers[2]

$$F_n = 2^{2^n} + 1.$$

Fermat erred in the belief that every F_n is a prime; though he admitted that he had no proof. He challenged other mathematicians to furnish a demonstration of this beautiful proposition, adding that such a proof would probably give the key to penetrate all the mystery of prime numbers and would release his energies so that afterwards nothing could keep him back in these matters.

A most surprising geometric connection of the numbers F_n was brought to light when Gauss proved that a regular polygon of m sides can be constructed by ruler and compasses if m is a product of a power of 2 and distinct odd primes each of the form F_n and stated that the construction is impossible when m is not such a product.

2. For $n = 0, 1, 2, 3, 4$, the numbers F_n are 3, 5, 17, 257, 65537, respectively. It is easy to verify directly that these are primes. No other prime F_n is known as such, while for $n = 5, 6, 7, 8, 9, 11, 12, 18, 23, 36, 38, 73$ it has been shown that F_n is composite. A table of all the known factors of F_n is to be found in this MONTHLY, Vol. 21, 1914, p. 249, except for the required addition that F_{73} has the prime factor $2^{75}.5 + 1$. (See also § 16 of the present paper.) The factors actually listed in this table are all primes. The known factorization is complete for $n = 5, 6$, but is not known to be complete for greater values of n.

Euler was the first to prove the falsity of Fermat's conjecture that every F_n is prime; and this he did by pointing out that 641 is a factor of F_5. The American calculator, Zera Colburn, in his autobiography, records that while on exhibition in London, at the age of eight, he found "by the mere operation of his mind" that $F_5 = 641 \cdot 6700147$.

In view of the known facts about the factors of F_n, Fermat's question, whether $(2k)^{2^m} + 1$ is always a prime except when divisible by an F_n, is without further particular interest.

3. The principal problem so far studied in connection with the numbers F_n is that of their factorization. The most elementary and obvious method for finding the factors of a given number is to test it for divisibility by primes less than its square root; but if the given number is a large prime or has only large prime

[1] Presented to the American Mathematical Society (Southwesten section), November 30, 1918.

[2] A complete history of these numbers (with full references) is given in Dickson's *History of the Theory of Numbers*, Vol. I, 1919, Chap. XV. To this the reader is directed for further references to the literature.

factors it is obvious that the labor thus involved is prohibitive. One of the earliest used improvements on this method consists in determining certain properties of the prime factors of the numbers in consideration and then testing with only those primes which have this property. Thus Euler showed that the factors of F_n are of the form $2^{n+1} \cdot k + 1$ (Lucas later proved that k here is even; see below). At a time when it was still unknown whether F_6 is prime or composite Lucas remarked that if it is prime the demonstration of this fact by aid of Euler's theorem would require a calculator the enormous period of three thousand years of painstaking toil. Lucas himself exhibited a new method (reproduced below) by which the question of the primality of F_6 could be determined by a single calculator in thirty hours. This method consists not in seeking the factors of F_6 but, strange to say, in the inverse process of ascertaining whether it is itself a factor of a number in a certain sequence.

4. Probably by far the largest calculation yet performed in connection with the theory of numbers is that by means of which Morehead and Western[1] established the composite character of F_8. The authors have given (*loc. cit.*) an account of their method.

It is of interest to observe the enormous bigness of the number F_{73}, the largest F_n concerning which we know whether it is prime or composite. (It has the prime factor $2^{75}.5 + 1$.) In his *Mathematical Recreations*, 5th edition, p. 40, W. W. R. Ball remarks that if this number F_{73} "were printed in full with the type and number of pages used in this book, many more volumes would be required than are contained in all the public libraries of the world." To put it differently and much more strongly we may say that if it were printed in full with the type and format of the *Encyclopædia Britannica*, eleventh edition, it would require more volumes than would be contained in 10,000,000,000,000 full sets of twenty-nine volumes each. Or if printed on ordinary 400-page octavo volumes it would make a library of more than two million volumes for each man, woman and child in the world.

5. In the present paper I have gathered together all the essential facts known about the numbers F_n. For the more important of these I have given (in §§ 6–11) as simple demonstrations as I could, employing methods to be found in the literature and encumbering them as little as possible to secure the end in view. I have listed (in § 13) the outstanding conjectures and have also added (in §§ 14–17) a few minor results which appear to be novel.

Fundamental Properties of the Numbers F_n.

6. As far back as 1730 it was observed that no F_n has a proper factor less than 100, a fact easily verified directly by the aid of congruences. More recently a considerable number of negative results of this character have been given, as, for instance, that no F_n has a factor less than 10^6 other than factors known at present; likewise there is no undiscovered factor of an F_n less than 10^8 and of the form $2^a \pm 2^x + 1$. There has also been considerable examination as to the possi-

[1] *Bull. Amer. Math. Soc.*, (2), 16 (1909), 1–6.

bility of factors of the forms

$$2^t \cdot 3 + 1, \qquad 2^t \cdot 5 + 1, \qquad 2^t \cdot 7 + 1.$$

In particular, a prime of the form $2^{2t} \cdot 3 + 1$ cannot be a factor of a Fermat number F_n.[1] The smallest of the as yet known factors of each F_n is actually the least factor of each such number.

Again it was noticed early that no two F_n have a common factor. For, if F_n and F_m, $m > n$, have a common prime factor p, then

$$2^{2^m} \equiv (2^{2^n})^{2^{m-n}} \equiv (-1)^{2^{m-n}} \equiv 1 \bmod p,$$

so that $2^{2^m} + 1$ is not divisible by p.

7. The first theorem about the numbers F_n actually demonstrated is that of Euler to the effect that all factors of F_n are of the form $2^{n+1} \cdot k + 1$. We give a proof of Lucas' extension of this theorem, namely,

Every factor of a given number F_n $(n \geqq 2)$ is of the form $2^{n+2} \cdot k + 1$.

It is sufficient to prove the theorem for a prime factor p of F_n. Since

$$2^{2^n} \equiv -1 \bmod p$$

it is evident that 2^{n+1} is the exponent to which 2 belongs modulo p, so that $p - 1$ is divisible by 2^{n+1} (the result due to Euler). Now, p is of the form $8t + 1$, since $n \geqq 2$. Hence 2 is a quadratic residue modulo p. Therefore, we have

$$2^{\frac{1}{2}(p-1)} \equiv 1 \bmod p,$$

so that $\frac{1}{2}(p - 1)$ is divisible by 2^{n+1}, the exponent to which 2 belongs modulo p. From this the theorem follows at once.

By means of this theorem it is easy to establish the following proposition: *If $f = 2^{n+2} \cdot q + 1$ is a factor of F_n and $q \not> 9 \cdot 2^{n+2}$, then f is a prime.*

For if f is not prime we must have

$$f = (2^{n+2} \cdot x + 1)(2^{n+2} \cdot y + 1)$$

where x and y are integers greater than 2; whence we have $q > 9 \cdot 2^{n+2}$. [In this theorem 9 may be replaced by 15 if f is not a square number.]

For the numbers F_5, F_{12}, F_{23}, F_{36}, F_{38}, F_{73}, the least factors to be tried (in accordance with the foregoing theorem of Lucas and the fact that no two F_n have a common factor) are respectively

$$2^7 \cdot 5 + 1, \qquad 2^{14} \cdot 7 + 1, \qquad 2^{25} \cdot 5 + 1, \qquad 2^{39} \cdot 5 + 1, \qquad 2^{41} \cdot 3 + 1, \qquad 2^{75} \cdot 5 + 1;$$

and in each case such number is a factor of the corresponding F_n. Thus of the seventeen cases in which we know at present whether the given F_n is prime or composite there are five in each of which it is easily seen to be prime while in half (six in number) of the remaining cases the least number to be tried in each instance turns out to be a factor. The verification by means of congruences is

[1] *Bull. Amer. Math. Soc.*, (2), 12: 450.

easy in the first three cases and not unduly tedious in the last three cases. In view of such verification and the two theorems above it is easy to see that each of these six factors is a prime number, the last being one of twenty-four digits.

8. Since the totient $\varphi(p)$ of p is less than $p - 1$ except when p is a prime number, the following theorem of Lucas is an immediate consequence of the Fermat theorem $a^{\phi(m)} \equiv 1 \bmod m$:

If the congruence $a^x \equiv 1 \bmod p$ is true when $x = p - 1$ but is not true when x is any proper divisor of $p - 1$, then p is a prime number.

For, $a^\delta \equiv 1 \bmod p$, when δ is the greatest common divisor of $\varphi(p)$ and $p - 1$.

By means of this result it is easy to establish Pepin's theorem concerning prime numbers F_n:

For $n > 1$, F_n is prime if and only if it divides

$$k^{\frac{1}{2}(F_n - 1)} + 1,$$

where k is any quadratic non-residue of F_n, as 3 *or* 5 *or* 10.

A test of this sort for prime numbers is remarkable in that it depends not directly on seeking factors of the number in consideration but in ascertaining whether it divides some other number.

To prove Pepin's theorem we proceed as follows: If F_n is prime we have

$$k^{\frac{1}{2}(F_n - 1)} \equiv -1 \bmod F_n$$

by the theory of quadratic residues. On the other hand, if the foregoing congruence is satisfied, we have

$$k^{F_n - 1} \equiv 1 \bmod F_n$$

while there is no proper divisor d of $F_n - 1$ for which $k^d \equiv 1 \bmod F_n$, since every such divisor is a power of 2 and a factor of $\frac{1}{2}(F_n - 1)$. Applying the theorem of Lucas at the beginning of this section we now conclude to the truth of Pepin's theorem.

To apply Pepin's test, one would take consecutively the minimum residues modulo F_n of

$$k^2, k^4, k^8, \cdots, k^{2^{2^n - 1}}.$$

9. Hurwitz gave the following generalization of Pepin's theorem:

Let $F_n(x)$ denote the irreducible algebraic factor of $x^n - 1$ (of degree $\varphi(n)$) which is not a factor of any $x^v - 1$ for which $v < n$. Then if there exists an integer q such that $F_{p-1}(q)$ is divisible by p, p is a prime.

When $p = 2^k + 1$ it is easy to show that $F_{p-1}(x) = x^{2^{k-1}} + 1$.

Carmichael employed the notation $F_n(\alpha, \beta)$ for $\beta^{\phi(n)} F_n(\alpha/\beta)$, where $F_n(x)$ has the meaning just defined. He gave the following generalization of the foregoing theorem of Hurwitz:

A necessary and sufficient condition that a given odd number p is prime is that there exist relatively prime integers $\alpha + \beta$ and $\alpha\beta$, α and β not both roots of unity, such that (the integer) $F_{p-1}(\alpha, \beta)$ is divisible by p.

A proof of this theorem is to be found in *Annals of Math.*, (2), 15 (1913): 66.

10. Let $\alpha + \beta$ and $\alpha\beta$ be two relatively prime integers, α and β not both roots of unity, and consider the quantities

$$S_n = \alpha^n + \beta^n, \qquad D_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \beta^{n-1}.$$

S_n and D_n both represent integers, since each of them is a symmetric polynomial (with integral coefficients) in the roots α and β of the equation

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

whose coefficients are integers. It is obvious that D_n is a factor of D_{nk} if n and k are both integers.

Let p be an odd prime which is not a divisor of either $(\alpha - \beta)^2$ or $\alpha\beta$. We have

$$S_p - (\alpha + \beta) \equiv \alpha^p + \beta^p - (\alpha + \beta)^p \equiv 0 \bmod p$$

and

$$D_p - (\alpha - \beta)^{p-1} \equiv \alpha^{p-1} + \cdots + \beta^{p-1} - (\alpha - \beta)^{p-1} \equiv 0 \bmod p.$$

But it is easy to verify that

$$(\alpha + \beta)D_p - S_p = 2\alpha\beta D_{p-1}.$$

Therefore

$$(\alpha + \beta)(\alpha - \beta)^{p-1} - (\alpha + \beta) \equiv 2\alpha\beta D_{p-1} \bmod p.$$

Now $(\alpha - \beta)^2$ is an integer; hence from Fermat's theorem it follows that

$$(\alpha - \beta)^{p-1} \equiv \pm 1 \bmod p.$$

From the last two congruences we have therefore

$$D_{p-1} \equiv 0 \bmod p \qquad \text{if } (\alpha - \beta)^{p-1} \equiv 1 \bmod p,$$

$$\alpha\beta D_{p-1} \equiv -(\alpha + \beta) \bmod p \qquad \text{if } (\alpha - \beta)^{p-1} \equiv -1 \bmod p.$$

Now it is easy to verify that

$$D_{p+1} - (\alpha + \beta)D_p + \alpha\beta D_{p-1} = 0;$$

and hence we see that

$$D_{p+1} \equiv 0 \bmod p \qquad \text{if } (\alpha - \beta)^{p-1} \equiv -1 \bmod p.$$

Therefore we have the following theorem (due to Lucas):

An odd prime p which does not divide either $(\alpha - \beta)^2$ or $\alpha\beta$ is a factor of D_{p-1} or of D_{p+1} according as $(\alpha - \beta)^{p-1}$ is congruent to $+1$ or to -1 modulo p.

That D_n and S_n are both prime to $\alpha\beta$ follows at once from the readily verified relations

$$(\alpha + \beta)^n = \alpha^n + \beta^n + \alpha\beta I_1 = S_n + \alpha\beta I_1,$$

$$D_n = \alpha^{n-1} + \beta^{n-1} + \alpha\beta I_2 = S_{n-1} + \alpha\beta I_2,$$

in which I_1 and I_2 denote integers. But it is easy to show directly that

$$D_m S_n - D_n S_m = 2\alpha^n \beta^n D_{m-n}.$$

Hence a common odd prime factor p of D_m and D_n $(m > n)$ is a factor of D_{m-n}; whence it follows without difficulty that p is a factor of D_v where v is the greatest common divisor of m and n. From this conclusion and the foregoing theorem we see that the least value k of n for which D_n is divisible by a given prime p (not dividing $(\alpha - \beta)^2$ or $\alpha\beta$) is a factor of $p - 1$ or $p + 1$ in the respective cases. Hence we have the following theorem (also due to Lucas):

If D_{q+1} $[D_{q-1}]$ is divisible by the odd number q, q being prime to $(\alpha - \beta)^2$, but D_k is not divisible by q for any factor k of $q + 1$ $[q - 1]$, then q is a prime number.

From the easily verified relation

$$S_n^2 - (\alpha - \beta)^2 D_n^2 = 4\alpha^n \beta^n$$

and the fact that D_n and S_n are prime to $\alpha\beta$ it follows that D_n and S_n have no common odd prime factor. But

$$D_{2^k} = S_{2^{k-1}} D_{2^{k-1}} = S_{2^{k-1}} S_{2^{k-2}} \cdots S_{2^2} S_2 S_1.$$

Hence no two numbers in the sequence

$$S_{2^0},\ S_{2^1},\ S_{2^2},\ S_{2^3},\ \cdots$$

have a common odd prime factor.

11. For the application of these results to Fermat numbers F_n let us take $\alpha + \beta = 2$, $\alpha\beta = -1$. Then $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = 8$, a quadratic residue of a prime number F_n, since 2 is a quadratic residue of all primes of the form $8t + 1$. All the divisors of $F_n - 1$ are powers of 2. Moreover

$$D_{2^k} = S_{2^{k-1}}\, S_{2^{k-2}} \cdots S_2\, S_1.$$

Hence we are concerned primarily with the numbers $G_1 = 6$, G_2, G_3, $\cdots$, where

$$G_s = S_{2^s}.$$

It is easy to verify directly the recurrence relation

$$G_{s+1} = G_s^2 - 2, \qquad s > 0.$$

Moreover no two G's have a common odd prime factor, as we see from the statement at the close of § 10.

From the first theorem of the preceding section it follows that $D_{F_n - 1}$ is divisible by F_n if F_n is prime. Hence F_n is composite if it divides no one of the numbers G_1, G_2, $\cdots$ $G_{2^n - 1}$.

Suppose that F_n divides G_t, $t < 2^n$, but no G with subscript less than t. Then any prime factor p of F_n divides G_t but no G with subscript less than t, since the G's are relatively prime in pairs. Hence p is a factor of $D_{2^{t+1}}$ but of no

D_s where $s < 2^{t+1}$. Hence p is of the form $2^{t+1}\cdot q + 1$. Hence it follows that F_n is prime if $2^{n-1} \leqq t \leqq 2^n - 1$; for, if F_n is not prime, we should have

$$F_n = (2^{t+1}\cdot q_1 + 1)(2^{t+1}\cdot q_2 + 1) > 2^{2(t+1)} + 1;$$

or $2(t + 1) < 2^n$.

These results yield us the following remarkable theorem of Lucas:

Consider the sequence $G_1 = 6$, $G_2 = 34$, $G_3 = 1154$, G_4, $\cdots$, G_{2^n-1}, *each term of which is 2 less than the square of the preceding. Then F_n is composite if it is a factor of no number of this sequence; if the first number of the sequence which F_n divides is G_t, then every prime factor of F_n is of the form $2^{t+1}\cdot q + 1$, and F_n is a prime of this form if $t \geqq 2^{n-1}$.*

Other Known Properties of the Numbers F_n.

12. A few other properties of the numbers F_n should be mentioned.[1]

J. Hermes indicated a test for composite F_n different from those given in the foregoing sections.

H. Scheffler noted that

$$F_n F_{n+1} \cdots F_{a-1} = 1 + 2^{2^n} + 2^{2\cdot 2^n} + 2^{3\cdot 2^n} + \cdots + 2^{2^a - 2^n}.$$

A. Cunningham considered the period of $1/N$ to base 2, where $N = F_m F_{m-1} \cdots F_{m-r}$. He noted that every $F_n > 5$ can be represented by four quadratic forms of determinants $\pm G_n$, $\pm 2G_n$, where $G_n = F_0 F_1 \cdots F_{n-1}$.

Gosset gave the complex prime factors $a \pm b\sqrt{-1}$ of the known real prime factors of composite F_n for $n = 5, 6, 9, 11, 12, 18, 23, 36, 38$.

Cunningham gave several algebraic properties of the numbers F_n (listed by Dickson at references 53, 55 and 56).

It is believed that all the essential known properties of the numbers F_n are stated or referred to explicitly in the foregoing portions of this paper.

Conjectures and Outstanding Problems Mentioned in the Literature.

13. Lucas[2] has listed five important problems in the theory of prime numbers, arranged in the "probable order of difficulty": (1) To find a prime greater than a given prime; (2) To find a function which yields only prime numbers; (3) To find the prime which follows a given prime; (4) To find the number of primes below a given limit; (5) To calculate directly the prime number of given rank. The first two of these would be simultaneously solved if one could prove the following conjecture:

All the numbers of the sequence

$$2^2 + 1, \quad 2^{2^2} + 1, \quad 2^{2^{2^2}} + 1, \quad \cdots$$

are primes. (The prime $2^8 + 1$ does not belong to this sequence.)

[1] See Dickson, *loc. cit.*, for references.

[2] *Théorie des nombres*, I, p. 354.

A proof of the foregoing conjecture would also carry with it the establishment of a theorem stated by Eisenstein, namely, that there is an infinitude of primes F_n. No proof of this theorem has yet been given.

Lipschitz[1] separated all integers into classes, the primes of one class being Fermat numbers, and gave a new interest to the question of the infinitude of primes F_n.

MINOR ADDITIONAL RESULTS.

14. Let p be a prime proper factor of F_n and write

$$p = 2^t \cdot k + 1$$

where k may or may not contain the factor 2 but is not a power of 2. A proper prime factor of F_n we shall call a Fermat prime factor.

Let us consider first the case in which $t = 2^\alpha$. (It is clear that 2^α may, if desired, be taken not less than the highest power of 2 not exceeding $n + 2$.) From the congruences

$$2^{2^n} \equiv -1, \qquad 2^{2^\alpha} \cdot k \equiv -1 \bmod p$$

it is clear that we have

$$k^{2^{n-\alpha}} + 1 \equiv 0 \bmod p. \tag{1}$$

From this we have readily the following theorem:

Every Fermat prime factor p has the property that it is a factor of a number of the form

$$a^{2^s} + 1$$

in which a is a factor of $p - 1$ belonging modulo p to the exponent 2^{s+1} and is not a power of 2.

The exponent s need not be taken greater than $n - \alpha$ where p is a factor of F_n and $2^\alpha \leqq n - 2$.

For the case of the prime factor p, $p = 2^{2^\alpha} \cdot k + 1$, of F_n it is obvious that $\alpha \leqq n - 1$. Consider the possibility that $\alpha = n - 1$. Then from (1) it follows that $k^2 + 1$ is divisible by p and hence that $k > 2^{2^{n-1}}$; whence we have a contradiction. Therefore, $\alpha \leqq n - 2$.

We have $k^4 + 1$ divisible by p when $\alpha = n - 2$; in view of (1) it is clear that it cannot be divisible by p when $\alpha < n - 2$. Hence the Fermat prime factors p of the form

$$p = 2^{2^{n-2}} \cdot k + 1$$

are all divisors of $k^4 + 1$ and no other Fermat prime factor $2^{2^\alpha} \cdot k + 1$ is a divisor of $k^4 + 1$.

In general, the Fermat prime factors of the form

$$2^{2^{n-v}} \cdot k + 1$$

[1] *Journ. für Math.*, 105, 152–156.

are divisors of $k^{2^v} + 1$ and no other Fermat prime factor $2^{2^a} \cdot k + 1$ is a divisor of $k^{2^v} + 1$.

15. In the Fermat prime factor p of F_n,

$$p = 2^t \cdot k + 1,$$

let t be an odd number. Then integers r and s exist such that

$$r \cdot 2^n - s \cdot t = 1.$$

From the congruences

$$2^{2^n} \equiv -1, \qquad 2^t \cdot k \equiv -1 \bmod p \tag{2}$$

we have

$$2^{r \cdot 2^n} \equiv (-1)^r, \qquad 2^{st} \cdot k^s \equiv (-1)^s \bmod p.$$

Hence we have

$$k^s \equiv (-1)^{r+s} \cdot 2 \bmod p.$$

It is clear that s is odd. Hence we have

$$k^s \equiv (-1)^{r+1} \cdot 2 \bmod p. \tag{3}$$

Similarly, integers r_1 and s_1 exist such that

$$s_1 t - r_1 2^n = 1,$$

and we have readily

$$2k^s \equiv (-1)^{r_1+1} \bmod p. \tag{4}$$

If we multiply (3) through by 2 and compare the result with (4) we see that

$$k^{s-s_1} \equiv (-1)^{r+r_1} \cdot 4 \qquad \text{or} \qquad 4k^{s_1-s} \equiv (-1)^{r+r_1} \bmod p \tag{5}$$

according as $s > s_1$ or $s < s_1$.

Again, from (2) we have immediately that

$$k^{2^n} + 1 \equiv 0 \bmod p \tag{6}$$

so that $2^{2^n} + 1$ and $k^{2^n} + 1$ have the common factor p.

From the two equations

$$r2^n - st = 1, \qquad s_1 t - r_1 2^n = 1 \tag{7}$$

we see that

$$2^n(r + r_1) = t(s + s_1). \tag{8}$$

If $r, s;\ r_1, s_1$ are the least positive integers satisfying (7) it is clear that s and s_1 are both less than 2^n. Hence from (8) we see that

$$s + s_1 = 2^n, \qquad r + r_1 = t. \tag{9}$$

Then equations (5) are respectively

$$k^{s-s_1} + 4 \equiv 0, \qquad 4k^{s_1-s} + 1 \equiv 0 \bmod p. \tag{10}$$

16. By means of (1) and the known factorizations of F_n one may readily obtain one numerical factor each of several binomial expressions. These are

given in the following table which contains in its first column a number F_n, in its second column a prime factor of F_n and in its third column a binomial number having the same factor, the table being complete as to all known prime factors of composite F_n:

F_5	$2^7\cdot 5 + 1$	$40^8 + 1$
F_5	$2^7\cdot 52347 + 1$	$418776^8 + 1$
F_6	$2^8\cdot 3^2\cdot 7\cdot 17 + 1$	$1071^8 + 1$
F_6	$2^8\cdot 5\cdot 52562829149 + 1$	$262814145745^8 + 1$
F_9	$2^{16}\cdot 37 + 1$	$37^{32} + 1$
F_{11}	$2^{13}\cdot 3\cdot 13 + 1$	$1248^{256} + 1$
F_{11}	$2^{13}\cdot 7\cdot 17 + 1$	$3808^{256} + 1$
F_{12}	$2^{14}\cdot 7 + 1$	$448^{512} + 1$
F_{12}	$2^{16}\cdot 397 + 1$	$397^{256} + 1$
F_{12}	$2^{16}\cdot 7\cdot 139 + 1$	$973^{256} + 1$
F_{18}	$2^{20}\cdot 13 + 1$	$208^{2^{14}} + 1$
F_{23}	$2^{25}\cdot 5 + 1$	$2560^{2^{19}} + 1$
F_{36}	$2^{39}\cdot 5 + 1$	$640^{2^{31}} + 1$
F_{38}	$2^{41}\cdot 3 + 1$	$1536^{2^{33}} + 1$
F_{73}	$2^{75}\cdot 5 + 1$	$10240^{2^{67}} + 1$

In addition to the information furnished by these fifteen Fermat prime factors it is known that F_0, F_1, F_2, F_3, F_4 are prime and that F_7 and F_8 are composite, as we have said above.

17. In a similar way (3), (4) and (10) yield factors of other simple binomial expressions. We shall merely illustrate this by means of two examples involving (3). Taking the first factor $2^7\cdot 5 + 1$ of F_5 we have $2\cdot 32 - 9\cdot 7 = 1$. Hence $s = 9$, $r = 2$. Therefore

$$5^9 + 2 \equiv 0 \bmod 2^7\cdot 5 + 1.$$

Again, taking $n = 9$ we may write the given factor in the form $2^{15}\cdot 74 + 1$. Then we have $8\cdot 512 - 273\cdot 15 = 1$; whence

$$74^{273} + 2 \equiv 0 \bmod 2^{16}\cdot 37 + 1.$$

SOME GEOMETRICAL RELATIONS OF THE PLANE, SPHERE, AND TETRAHEDRON.

By PHILIP FRANKLIN, College of the City of New York.

Professor Chrystal has remarked that mathematical works should be read backwards as well as forwards, *i. e.*, advanced notions of mathematics often throw light on elementary problems. While this "back tracking" is frequently brought out in analysis, it is seldom emphasized in connection with plane and solid geometry.

Many theorems of plane geometry have solid analogues which are but seldom developed. Especially is this true in the case of constructions which are slighted in solid geometry, owing to their theoretic nature. (The only way of actually drawing them being by the very indirect methods of descriptive geometry.)

For example, the problem of the "tangencies"—the construction of a *circle* in a plane from *three* elements, these being *points*, tangent *lines* or tangent *circles* —with its ten cases easily leads to the determination of a *sphere* from *four* elements, these being *points*, tangent *planes* or tangent *spheres*. There are fifteen cases in all, which are reduced to six by the following two principles, analogous to those used for the plane problem.

PRINCIPLE I.—If the given elements are only spheres and planes, and if the radii of the spheres be changed by that of one of the spheres, and if the planes be translated normal to themselves by a like amount, obviously the sphere determined by the new elements will be concentric with that determined by the original set. This principle serves to replace one of the tangent spheres by a point.

PRINCIPLE II.—If a sphere is tangent to two other spheres (sphere and plane) and goes through a given point, it goes through the inverse of this point, the center of inversion is a center of similitude of the two spheres (sphere and plane) and the radius of inversion is such as to make one of the spheres the inverse of the other (or of the plane). This also replaces one of the given spheres by a point.

The constructions follow; the proofs will be easily supplied by the reader familiar with the plane problem.

1. *Point, Point, Point, Point.*—The center of the required sphere is the intersection of the perpendicular bisecting planes of the joins of the points two and two. (Circum-sphere of a tetrahedron.) One solution.

2. *Plane, Plane, Plane, Plane.*—The center of the required sphere is the intersection of the planes bisecting the dihedral angles formed by the given planes. (In-sphere of a tetrahedron.) One solution.

3. *Point, Point, Point, Plane.*—Construct the perpendicular bisecting planes of the joins of the points intersecting in a line. Draw a sphere with center on this line to touch the given plane, and with the intersection of this line and the given plane as a "center" multiply this sphere to pass through the given points. Two solutions.

4. *Plane, Plane, Plane, Point.*—Draw the bisecting planes of the dihedral angles of the given planes, and draw a sphere with center any point on their line of intersection to touch the planes, then with the point of intersection of the given planes as a "center," multiply this sphere to pass through the given point. Two solutions.

5. *Point, Point, Point, Sphere.*—Draw any sphere through the three points to intersect the given sphere in a circle, and from the line of intersection of the plane of this circle with the plane of the three points draw a tangent plane to the given sphere, locating the point of contact of the required sphere. Then use (1). Two solutions.

6. *Plane, Plane, Plane, Sphere.*—Reduce by Principle I to (4). Four solutions.

7. *Plane, Plane, Point, Point.*—Locate the locus of the center as the intersection of the bisecting plane of the given dihedral angle and the perpendicular bisector of the join of the two points, draw a sphere with center on this line to touch the given planes, and with the intersection of the line of centers with the given planes as "center" multiply it to pass through the given points. Two solutions.

8. *Point, Point, Plane, Sphere.*—Reduce by Principle II to (3). Four solutions.

9. *Plane, Plane, Point, Sphere.*—Reduce by Principle II to (7). Four solutions.

10. *Point, Point, Sphere, Sphere.*—Reduce by Principle II to (5). Four solutions.

11. *Plane, Plane, Sphere, Sphere.*—Reduce by Principle I to (9). Eight solutions.

12. *Plane, Point, Sphere, Sphere.*—Reduce by Principle II to (3). Eight solutions.

13. *Point, Sphere, Sphere, Sphere.*—Reduce by Principle II to (5). Eight solutions.

14. *Plane, Sphere, Sphere, Sphere.*—Reduce by Principle I to (12). Eight solutions.

15. *Sphere, Sphere, Sphere, Sphere.*—Reduce by Principle I to (13). Sixteen solutions.

This last case can also be solved by a direct construction after Gergonne's construction for a circle tangent to three circles,[1] and, like it, in degenerate forms includes many remaining cases of the problem.

With center the intersection of the four radical planes of the spheres, and a radius equal to the tangent from this point to one of the given spheres, describe the sphere orthogonal to the given four spheres. Draw the planes of intersection of this sphere and each of the given spheres intersecting an "axial plane of similitude" of the spheres in four lines. Tangent planes from these lines to the corresponding spheres determine eight points of contact. The two spheres through these four and four are tangent to the given spheres. The eight "axial planes of similitude" determine the sixteen solutions.

The properties of plane triangles are often generalized for spherical triangles. These results, however, are unlike the ordinary ones of geometry, since they often involve trigonometric functions, replace straight lines by great circles, and are really a representation in Euclidean three-dimensional space of a non-euclidean plane.

The tetrahedron is the more natural spacial analogue of the plane triangle, but its properties are seldom brought out. The necessary and sufficient condi-

[1] "Recherche du cercle qui en touche trois autres dans un plan," *Annales de Mathêmatiques* vol. 7, 1817.—Editor.

tions for the co-planarity of six points on the different edges or for the concurrency of six planes through the different edges is easily obtained by applying Ceva's or Menelaus' theorem to three different faces, the condition being three equations in each case. The generalized "Ceva's theorem" might be used to prove the existence of the "axial planes of similitude" of four spheres mentioned above.

The existence of a median point (center of mean position or center of mass of the four vertices), *i. e.*, the intersection of the lines joining the centroids of the faces with the opposite vertices, and of the in- and circum-centers is usually proved, but the analogy is carried no further.

If an extension of an ortho-center be attempted, whether we define an altitude as a perpendicular from a vertex to the opposite face or as the join of a vertex with the ortho-center of the opposite face, we find that in general such altitudes do not intersect. The necessary and sufficient condition for either of these sets of "altitudes" being concurrent is the coincidence of the foot of one perpendicular with the ortho-center of the face to which it is perpendicular, in which case both sets of altitudes coincide, and the tetrahedron has its opposite edges perpendicular. Such a tetrahedron has many interesting properties.

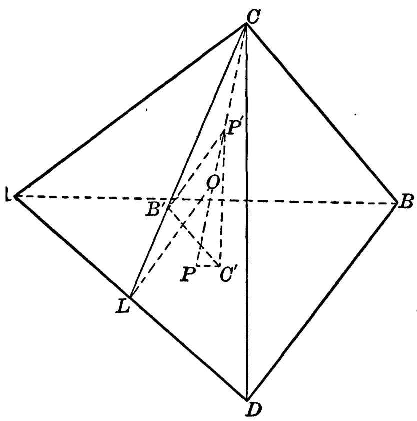

FIG. 1.

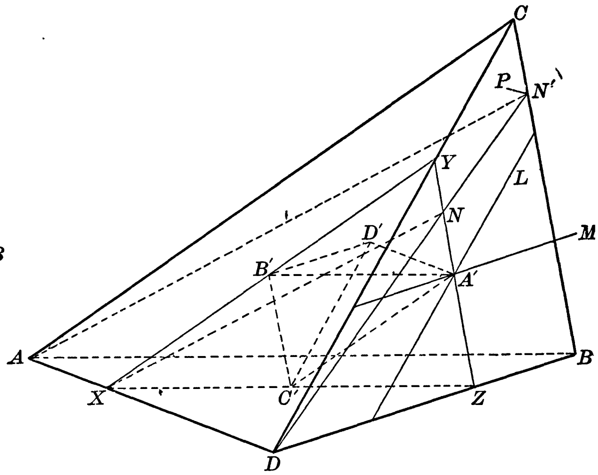

FIG. 2.

For example (Fig. 1). Figs. 1 and 2 are orthogonal projections of the spacial configurations they represent), if $ABCD$ is such a tetrahedron, the four centroids of the faces (as B', C'), the four feet of the altitudes (as P), and the four points dividing the segments of altitudes beyond the ortho-center in the ratio of $1:2$ (as P', such that $CP' = 2P'O$) all lie on a sphere, which might be called the "twelve-point" sphere after the "nine-point" circle.

Proof. $AO \perp$ Plane DBC

$DO \perp$ Plane ABC

$\therefore$ Plane $AOD \perp$ Planes DBC and ABC or to CB their intersection

$\therefore LO \perp CB$

But $CB' : B'L = CP' : P'O = BC' : C'L = 2 : 1$.

$\therefore LO \parallel B'P'$ and $CB \parallel B'C'$.

$\therefore B'P' \perp B'C'$.

But $PP' \perp PC'$;

$\therefore$ Sphere on diameter $C'P$ passes through B' (and similarly through remaining centroids of faces) and also through P and P'. By a similar proof, the sphere through the centroids may be shown to pass through the other points corresponding to P and P'.

Such a tetrahedron also possesses an "Euler" line. Since $A'B'C'D'$ is similarly situated with regard to $ABCD$ with respect to the median point M, T the

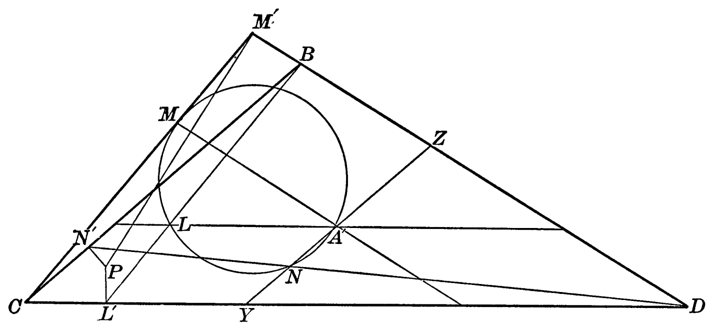

FIG. 3.

center of the twelve-point sphere, K the circum-center and M are collinear, and $KM : MT = 3 : 1$. Also since $OP' : OC = 1 : 3$, etc., the twelve-point sphere and the circum-sphere are similarly situated with regard to 0, and therefore O, T and K are collinear and $OT : OK = 1 : 3$. Also from the similar situation of $A'B'C'D'$ and $ABCD$ O, M and Q the intersection of the perpendiculars at the centroids of the faces are collinear and $MQ : MO = 1 : 3$. Therefore

O = the ortho-center,
T = the center of the twelve-point sphere,
M = the median point,
Q = the intersection of the perpendiculars at the centroids of the faces and
K = the center of the circum-sphere

are on line in the above order, and the segments $OT : TM : MQ : QK = 2 : 1 : 1 : 2$.

Conversely, solid constructions may give rise to plane theorems. If the centroids of the faces of any tetrahedron $ABCD$ (Fig. 2) $A'B'C'D'$ be located and a sphere passed through them, this sphere cuts plane $A'B'C'$ in the nine-point circle of triangle XYZ and therefore passes through N the foot of the altitude of this triangle on YZ. Similar points may be located corresponding to faces $A'B'D'$ and $A'C'D'$. But DN cuts BC in N', the foot of the A-altitude of triangle ABC,

and N' is also the foot of a perpendicular on BC from P, the foot of a perpendicular from A to BDC.

Since for different tetrahedra P becomes any point in plane BCD, the following theorem is obtained (Fig. 3): If from a point (P) perpendiculars be drawn to the sides of a triangle (BCD) produced if necessary, and their feet ($L'M'N'$) be joined to the opposite vertices, the three intersections of these lines with the parallels to the sides through the median point (LMN) and the median point (A') lie on a circle whose center divides the join of the circum-center of the original triangle and the center of mean position of the three vertices of the triangle and the arbitrary point externally in the ratio 1 : 4. The second part of the theorem follows from the fact that $B'C'D'$ and BCD (Fig. 2) are similarly situated with respect to the center of mass of the four vertices of the tetrahedron, and the orthogonal projection of the circum-center of $A'B'C'D'$ on plane BCD coincides with that of the circum-center of triangle $B'C'D'$.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

NEW QUESTIONS.

37. Criticise the following as fundamental definitions of elementary geometry:

A plane surface is the limit approached by a finite portion of the surface of a sphere as the radius increases without limit.

A straight line is the limit approached by a finite portion of the circumference of a circle as the radius increases without limit.

38. As the several courses in secondary and collegiate mathematics are now taught there is a noticeable difference of treatment with respect to the relative emphasis placed on logical accuracy and on development of technique. In elementary algebra, technique predominates, while in plane and solid geometry and advanced algebra logic is more strongly stressed. Trigonometry, analytic geometry, and the calculus are less easily classified; but there is at least a general tendency to emphasize logic in analytic geometry and technique in the calculus. It is suggested that a general appraisal of the reasons for this difference in treatment, its value, and possible alterations, would be helpful.

DISCUSSIONS.

THE EXISTENCE OF CUSPS ON THE EVOLUTE AT POINTS OF MAXIMUM AND MINIMUM CURVATURE ON THE BASE CURVE.

By G. H. LIGHT, University of Colorado.

The purpose of this paper is to show that whenever a curve $F(x, y) = 0$ has a point of maximum or minimum curvature the evolute has at the correspond-

ing point a cusp at which the curvature is infinite.[1] The assumption is made that at the point on the curve T under consideration $F_y \neq 0$.

Hence the value of dy/dx may be written

$$dy/dx = - F_x/F_y,$$

from which one easily obtains

$$d^2y/dx^2 = - B/F_y^3,$$

where

$$B = F_x^2F_{yy} - 2F_xF_yF_{xy} + F_y^2F_{xx}.$$

By definition, the curvature has the following value;

$$k = - \frac{B}{(F_x^2 + F_y^2)^{3/2}}.$$

Hence at a critical value its derivative must vanish; *i. e.*, $dk/dx = 0$. Performing the necessary work, it is found that

$$\frac{dk}{dx} = - \frac{(C - 3D)}{F_y(F_x^2 + F_y^2)^{5/2}},$$

where C and D have the values:

$$C = (F_x^2 + F_y^2)(F_y^3F_{xxx} - 3F_y^2F_xF_{xxy} + 3F_yF_x^2F_{xyy} - F_x^3F_{yyy}),$$

$$\begin{aligned} D &= (F_xF_yF_{xx} + F_y^2F_{xy} - F_x^2F_{xy} - F_xF_yF_{yy})(F_y^2F_{xx} - 2F_xF_yF_{xy} + F_x^2F_{yy}) \\ &= F_y^4F_{xx}F_{xy} - 2F_y^3F_xF_{xy}^2 - F_y^3F_xF_{xx}F_{yy} + F_y^3F_xF_{xx}^2 - 3F_y^2F_x^2F_{xx}F_{xy} \\ &\quad + 3F_y^2F_x^2F_{xy}F_{yy} + F_yF_x^3F_{xx}F_{yy} + 2F_yF_x^3F_{xy}^2 - F_yF_x^3F_{yy}^2 - F_x^4F_{xy}F_{yy}. \end{aligned}$$

From this results

THEOREM 1. At a point of either maximum or minimum curvature the relation $C = 3D$ must hold.

Having found the condition which holds at maximum and minimum points, the nature of the corresponding point on the evolute will be investigated. The equation of the evolute will be written in parametric form, x being regarded as the parameter:

$$\alpha = x - \frac{F_xF_y^2 + F_x^3}{B}, \qquad \beta = y - \frac{F_x^2F_y + F_y^3}{B}.$$

The curvature, K_1, of the evolute is given by the formula

$$K_1 = \frac{d^2\beta}{d\alpha^2} \Big/ \left[1 + \left(\frac{d\beta}{d\alpha} \right)^2 \right]^{3/2}$$

and is readily computed when it is remembered that

$$d\beta/d\alpha = F_y/F_x$$

[1] Cf. Goursat-Hedrick, *A Course in Mathematical Analysis*, vol. I, p. 438.

and therefore

$$\frac{d^2\beta}{d\alpha^2} = \frac{d\left(\dfrac{d\beta}{d\alpha}\right)}{dx} \cdot \frac{1}{\dfrac{d\alpha}{dx}}.$$

After some computation, it is found that

$$\frac{d\alpha}{dx} = \frac{F_x(C - 3D)}{F_y \cdot B^2}. \tag{1}$$

Likewise

$$\frac{d\beta}{dx} = \frac{C - 3D}{B^2} \tag{2}$$

and

$$\frac{d^2\beta}{d\alpha^2} = \frac{B^3}{F_x^3} \cdot \frac{1}{C - 3D}.$$

Therefore

$$K_1 = \frac{B^3}{(F_x^2 + F_y^2)^{3/2}} \cdot \frac{1}{C - 3D}. \tag{3}$$

From (1), (2), (3) it is seen that when $d\alpha/dx$ and $d\beta/dx$ vanish, K_1 becomes infinite and a cusp occurs on the evolute.

THEOREM 2. When the curvature of T is either a maximum or a minimum (at a point where neither F_x nor F_y is infinite) the relation $C = 3D$ must be satisfied. Moreover, the evolute of the normals has a cusp at the same time and its curvature is infinite. Furthermore, since at a point of maximum curvature the radius of curvature is a minimum, it is seen that the corresponding point on the evolute is nearer T than at any other time and that the cusp points *towards* T. In like manner, at a point of minimum curvature the cusp points *away* from T.

As a simple example consider the ellipse.

Here

$$F(x, y) \equiv b^2x^2 + a^2y^2 - a^2b^2 = 0$$

and

$$\begin{aligned} F_x &= 2b^2x, & F_y &= 2a^2y, & F_{xy} &= 0, \\ F_{xx} &= 2b^2, & F_{yy} &= 2a^2, & F_{xxy} &= 0, \\ F_{xxx} &= 0, & F_{yyy} &= 0, & F_{xyy} &= 0, \end{aligned}$$

from which

$$C = 0, \qquad D = -\,64a^6b^6c^2xy$$

and

$$C - 3D = 192a^6b^6c^2xy.$$

This vanishes only when $x = 0$ or $y = 0$, and verifies the fact that maximum and minimum curvature occurs only at the vertices of the ellipse.

From the well-known form of the evolute of the ellipse,

$$(a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2 - b^2)^{2/3},$$

the curvature, K_1, is found to be

$$K_1 = -\frac{a^{2/3}b^{2/3}(\alpha - \beta)}{3\beta^{2/3}\alpha^{2/3}(b^{4/3}\alpha^{2/3} + a^{4/3}\beta^{2/3})^{3/2}},$$

and this becomes infinite only when $\alpha = 0$ or $\beta = 0$. Thus cusps occur on the evolute at the points $\left(\pm\frac{c^2}{a}, 0\right)$, $\left(0, \pm\frac{c^2}{b}\right)$ corresponding to points of maximum and minimum curvature on the ellipse.

RECENT PUBLICATIONS.

REVIEWS.

Exercises in Algebra (including trigonometry). By T. P. NUNN. 2 volumes. London and New York, Longmans, 1913–1914. Vol. I, 11 + 421 pp. Vol. II, 11 + 551 pp. Price 4 + $6\frac{1}{2}$ shillings.

The Teaching of Algebra (including trigonometry). By T. P. NUNN. London and New York, Longmans, 1914. 12mo, 14 + 616 pp. Price $7\frac{1}{2}$ shillings.

Among the contributions that have recently been made to the solution of the problem of teaching secondary and collegiate mathematics, Dr. Nunn's work is one of the most significant. His plan is to build up a course which shall combine the most vital and essential parts of algebra, trigonometry, analytic geometry, and elementary calculus, and present them in a way which shall meet the needs both of the student who is studying mathematics merely for its cultural, disciplinary, or informational value, and of the student who plans to go on to further study of the subject. Many teachers have serious doubts as to the wisdom of giving the same work to these two different classes of students; but Dr. Nunn's course seems to the reviewer to furnish a successful demonstration that the thing can be done.

The two volumes of *Exercises* are designed for the student, and the third volume, on *Teaching,* indicates to the teacher the reasons for the order in which topics are taken up, and gives suggestions as to methods of presentation. The first volume of the *Exercises* contains little detailed explanation for the student, this being left for the teacher to give verbally on lines indicated clearly in the *Teaching.* The second volume of the *Exercises,* however, contains more discussion of the topics treated, with the object of developing in the student the ability to read and understand mathematical writing.

The work in elementary algebra is centered about two topics, the graph and the formula. By making skillful and persistent use of both, the author gives a vitality and meaning to the fundamental laws of operation which the student can rarely gain through the mere study of the axioms and formal laws. To give one illustration, the distributive law of multiplication is not first *stated,* and then

illustrated by geometrical figures, as is done in most texts, but rather it is *developed* in the mind of the student as a convenient short cut in area problems that require the value of $ac \pm bc$. Thus the usual order of procedure is here reversed, factoring being considered *before* multiplication as a formal algebraic process. Furthermore, the formal laws are first used with signless numbers, and not until the second section of the course (27th exercise, page 155) are directed numbers introduced. Much is made of approximation formulas as illustrations of the usefulness of such identities as $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

Direct proportion is introduced early, as a particularly important relation between variables, and the opportunity is taken to define the trigonometric functions of acute angles. This is surely an admirable idea, and offers a suggestion that might well be considered by American writers of texts on elementary algebra. After the introduction of directed numbers, arithmetic series are studied, and the summation applied to the problem of finding certain areas in a manner that prepares the ground very skillfully for the elementary work in integral calculus which is to follow not much later. After an elementary study of the quadratic equation and the principles of transformation of coördinates by translation, the remarkable innovation is introduced of taking up "area functions" (Exercise 47, p. 250). The method employed is the development of what the author calls "Wallis's Law," and may be illustrated as follows: in order to obtain the area under the curve $y = kx^2$, from $x = 0$ to $x = x$, the student is led to discover by induction that the ratio of the required area to that of the rectangle of base x and height kx^2 is approximately

$$\frac{0^2 + 1^2 + 2^2 + \cdots + m^2}{(m+1)m^2} = \frac{2m+1}{6m} = \frac{1}{3} + \frac{1}{6m},$$

where m is the number of subdivisions of the base. The conclusion is, that we may with no appreciable error take $\frac{1}{3}kx^3$ as the area under the curve. Similar work is carried through for other areas. At about this same period the subject of gradients and simple differentiation formulas are also introduced. It certainly demands great skill on the part of the teacher to present this work to boys of 15 and 16 years, and it would be interesting to know if the experiment has been tried in any American secondary schools.

The subject of logarithms is introduced by means of "growth problems," that is, problems depending upon the value of a^x for certain integral or fractional values of x. The historical method of Napier in constructing his system of logarithms is then employed, a procedure that seems rather questionable, but which is certainly worthy of careful experimental comparison with the usual method of approach.

Part II of the *Exercises* carries the student much farther into the field of what we know as collegiate mathematics, including functions of two variables, the trigonometry of the sphere, complex numbers, periodic functions (this section containing also a very good geometric introduction to the hyperbolic functions), and limits, which includes linear differential equations, Taylor's Theorem, and

even a good presentation of Weierstrass's derivativeless continuous function, of Peano's space-filling curve, and of Moore's "crinkly curve."

Constant use is made of the applications of mathematics. Thus, in Part I the introductory topics, the graph and the formula, are employed in connection with problems from mensuration, commercial arithmetic, physics, engineering, botany, and astronomy. Navigation is begun as a simple illustration of the use of the trigonometric functions (problems in parallel sailing, pages 118, 119) and is continued and elaborated in Part II, pages 105–142. In the latter place we find Sanson's, Lambert's, and Mercator's projections explained and applied in numerous problems; further, great circle sailing, the construction and use of gnomonic projections, and the elementary principles of practical astronomy (pages 143–169) are considered. The increasing importance of navigation in our national life suggests the advisability of enriching our mathematics courses, both in secondary school and in college, with much of this material. The subject of annuities and life insurance occupies pages 63–78 of Part II, compound interest having been considered briefly in Part I as an application of geometric progressions. The usually under-valued subject of statistics is given a very satisfactory treatment at the end of Part II (pages 431–514).

The mere mention of these topics is sufficient to show that the author has laid out an ambitious program for the secondary school course; but the original and stimulating manner in which all of the topics are presented tends to remove one's doubts as to the feasibility of carrying the young student into so much of the work which we have usually placed much later in the mathematics course. At any rate, we have in Dr. Nunn's work a valuable contribution to a most important problem, and one that can not fail to be suggestive and stimulating to any teacher of mathematics. The three volumes should find a place in every high school and college library.

R. B. McClenon.

Interpolation Tables, or Multiplication Tables of Decimal Fractions, giving the product to the nearest unit of all numbers from 1 to 100 by 0.01 to 0.99 and from 1 to 1,000 by 0.001 to 0.999. By Henry B. Hedrick. Carnegie Institution of Washington. Washington, 1918. 9 + 139 pp. Folio. Price $5.00.

Until about the beginning of the eighteenth century the multiplication of one number by another must have been considered a serious undertaking. In the old arithmetics very simple examples of the processes are given with as painful detail as is now given to an example in the extraction of the cube root. Napier's "bones" were hailed as a wonderful invention and yet they are of little use to one who knows his multiplication table as far as the nines. As late as 1841 an immense and complicated table was published by Bretschneider giving the product of all numbers up to 100,000 by 2, 3, 4, 5, 6, 7, 8 and 9. Nevertheless, as early as 1610 a table giving the product of any two numbers less than 1,000 was published by Herwart von Hohenburg in an immense folio volume of over a thousand pages.

In 1864 Crelle published a table which is still an indispensable part of every computer's library.[1] It gives in about five hundred pages the product of any two numbers less than 1,000 and may easily be used to multiply any two numbers together. It is also conveniently used as a division table.

The tables prepared by Dr. H. B. Hedrick and published by the Carnegie Institution of Washington are intended to cover the same range of numbers as Crelle's tables using 139 pages instead of 500. The abridgment is obtained by giving only three significant figures in the product, which for many purposes is sufficient. The arrangement of the table is also new in that while one factor is listed at the top of the page the other is found in the body of the table, the product being listed at the side of the page. Only a few numbers are necessary to list in the body of the table, since if only the first three significant figures are desired the products of .293, for example, by 279, 280 and 281 are identical. Only the smallest, 279, is listed. This effects a considerable saving of space.

The reviewer is not convinced that the smallness of the table is a sufficient advantage over Crelle's table to offset the manifest disadvantages. In order to test the matter he made a list of one hundred products of three digits by three digits. He first found the products "by hand" using his own method of cross-multiplication by which the product is obtained from the left. (See *Science*, July 11, 1902.) The products were then found from Dr. Hedrick's table, then from Crelle's table, and finally by means of the Monroe Calculating Machine. The results were as follows:

By hand	28 minutes.
By Hedrick's table	32 minutes.
By Crelle's table	29 minutes.
By Monroe machine	26 minutes.

The reviewer has not had any practice with Crelle's table for many years, and was so unused to it that he had to read the directions for using it. He has had, quite recently, considerable practice in using the calculating machine, so that probably he would be a little faster than the average engineer in using it. The comparison with Crelle's table is interesting as bearing out what the reviewer has always maintained, that is, that if one has to turn over pages it is as easy and convenient to turn over five hundred pages as one hundred, and that there is no appreciable saving in time in using a four place table of logarithms instead of a six place table, and when interpolations are necessary there is a distinct loss of time and accuracy in using the smaller table.

D. N. LEHMER.

A Treatise on Differential Equations. By A. R. FORSYTH. Fourth edition. London, Macmillan, 1914. 8vo, 584 pp. Price 17 shillings.

Solutions of the Examples in a Treatise on Differential Equations. By A. R. FORSYTH. London, Macmillan, 1918. 8vo, 249 pp. Price 10 shillings.

Early in the eighties Macmillan and Company consulted Forsyth with regard to the preparation of a new edition of Boole's *Treatise on Differential Equations*, a work which had already

[1] Other multiplication tables are described on page 327 of J. W. L. Glaisher's article, "Table, Mathematical," in the eleventh edition of the *Encyclopædia Britannica*—EDITOR.

gone through four editions. The result was that Forsyth gave them, in 1885, a work of his own with the same title. The second edition appeared in 1888; the third, much enlarged, in 1903, and the last, eleven years later, still further increased in size. The Italian translation, by Arbicone, of the second English edition was published in 1901[1] and the German edition by Maser (with an appendix containing solutions of the problems) in 1889; the second German edition came out in 1912,[2] but the solutions given were only of the problems of the second English edition.

In preparing his volume of solutions Forsyth worked out each example himself and found that, "with possibly three exceptions, all the equations have proved soluble." He remarks further: "The first German edition . . . translated by the late Dr. H. Maser contained the solutions of most of the examples in my own first and second editions—not a few of them were supplied to him by me. But my later editions contained a vast added array of equations, not solved in the second German edition. To make the volume complete, all the questions (except where the results were taken from original memoirs) were worked out anew from the beginning."

"I have had in mind principally the needs of teachers and only subsidiarily the case of students who are in a position to follow courses of instruction."

Text-book on Navigation and Nautical Astronomy. By J. GILL. Revised and enlarged by W. V. Merrifield. New edition. London, Longmans, 1918. 8vo, 8 + 438 + 12 pp. Price 20 shillings.

Contents: Part I, pp. 1–262: Projections; definitions and instruments; logarithms; solution of plane triangles; solution of spherical triangles; the sailings; tides; reduction of soundings; time; elements from the 'Nautical Almanac'; sidereal and solar time; corrections of altitudes; latitude by meridian altitudes; latitude by ex-meridian altitude; compass error by amplitudes; compass error by azimuths; longitude by chronometer; Sumner's method of projection; the chart; use of Napier's diagram; great circle sailing; latitude by double altitudes; finding error of chronometer; calculation of altitudes; longitude by linear distances. Part II, pp. 263–342: Construction of charts; laws of storms; magnetism and deviation of the compass; syllabus; miscellaneous problems. Appendix, pp. 343–402. Answers, 403–438. Tide Tables 1–12.

Quotations from the preface to the new edition: "The new features in this edition are—(1) Rearrangement, by which the proofs of the various problems are inserted at the beginning of the separate chapters instead of being collected in one chapter at the end of the book. . . . (2) All the astronomical examples are fresh, the necessary rules are given for working them, and, where possible, an example is worked direct from the figure, without using special formulæ. . . . (3) The necessary elements from the 'Nautical Almanac' are embodied in the book in the form found in the 'N. A.' . . . (4) The chapters on trigonometry have been entirely rewritten. . . . (5) New chapters have been introduced on the projections by which any figure may be drawn to scale, and on the correction of altitudes. (6) The chapters on tides and tidal soundings have been rewritten to suit the recent tide tables, extracts from which are found at the end of the book."

A Vest-pocket Handbook of Mathematics for Engineers. By L. A. WATERBURY, with special sections by G. A. GOODENOUGH and H. H. HIGBIE. Third edition, enlarged (total iss e twelve thousand). New York, Wiley, 1918. 16mo, 13 + 278 pp. Flexible cover, gilt. Price $1.50.

The first edition of this *Handbook* was in 1908; tables were added in 1909. To the second edition in 1915 sections on hydraulics and reinforced concrete, and a table of conversion factors were added. The principal additions to the present edition are the sections on heat engineering by Professor Goodenough, and on electrical engineering by Professor Higbie, with the corresponding related tables. The contents of the work are as follows: Algebra (exponents and logarithms; proportion; progressions; quadratic, cubic, higher equations; graphical solutions of

[1] *Trattato sulle equazioni differenziale.* Prima versione dall'inglese di Alfredo Arbicone. Livorno, 1901. 12 + 337 pp.

[2] *Lehrbuch der Differentialgleichungen.* Von A. R. FORSYTH (mit den Auflösungen der Aufgaben von Hermann Maser). Zweite Auflage, nach der dritten des englischen Originals besorgt und mit einem Anhang von Zusätzen versehen von W. Jacobsthal. Braunschweig, Vieweg, 1912. 22 + 921 pp. Reviewed by R. D. CARMICHAEL in *Bull. Amer. Math. Soc.*, Vol. 19, pp. 256–259. Feb., 1913.

equations), pp. 1–7; Trigonometry (plane and spherical triangles; hyperbolic functions), 8–13; Analytic Geometry (transformation of coördinates; straight line; circle; parabola; ellipse; hyperbola; cycloid; miscellaneous curves; solids), 14–22; Differential and integral calculus, 23–38; Theoretical mechanics, 39–53; Mechanics of materials, 54–91; Hydraulics, 92–101; Heat engineering, 102–123; Electrical engineering formulæ, 124–155. Tables (logarithms of numbers, sines, cosines, tangents and cotangents; natural sines, cosines, tangents and cotangents; conversion factors; properties of saturated steam (Goodenough); pressure-entropy table for steam (Goodenough), 156–269.

La Vie universitaire à Paris. Ouvrage publiè sous les auspices du Conseil de l'Université de Paris. Avec 92 photographies hors texte. Paris, Colin, 1918. Royal 8vo, 231 pp. Price 12 francs.

The two chapters of this sumptuous work which are of especial interest to the mathematician are chapter 4: "La faculté des sciences" by Maurice Caullery, 50–70; and chapter 5: "L'Ecole Normale Supérieure" by E. Lavisse, 71–74. Plate 15 presents Pirou's well-known portrait of Hermite and a much less familiar portrait of Poincaré by Manuel. There are also interesting views of the Sorbonne.

Business arithmetic. By C. W. SUTTON and N. J. LENNES. Boston, Allyn and Bacon, 1918. 8vo, 6 + 466 + 8 pp. Price $1.40.

Brief business arithmetic. By C. W. SUTTON and N. J. LENNES. Boston, Allyn and Bacon, 1918. 8vo, 5 + 297 + 6 pp. Price $1.16.

The titles of the last twenty-one chapters of the larger work are as follows: Promissory notes, partial payments, banks and bank discount, cash balance in bank, savings accounts, stocks and shares, bonds, the stock exchange, graphic representation, domestic exchange, foreign exchange, equation of payments and accounts, property insurance, life insurance, direct taxation, income tax, United States customs, partnership, storage, building and loan associations, and depreciation.

NOTES.

The United States Ordnance Department is to publish a treatise on exterior ballistics. It is being prepared for the press under the direction of Captain Dunham Jackson.—Teubner announces the publication of: C. CRANZ, *Lehrbuch der Ballistik.* Herausgegeben unter Mitwirkung von K. Becker. Band 4: Sammlung von Zahlentafeln, Diagrammen, und Lichtbildern. 2. vermehrte Auflage, Leipzig, 1918.

It is announced that the fourth, fifth, and sixth volumes of the *Handbuch der angewandten Mathematik* edited by H. E. Timerding (Leipzig, Teubner) are to be respectively entitled: *Die graphischen Methoden der technischen Mechanik*, *Mathematische Statistik und Versicherungsrechnung*, and *Grundzüge der Astronomie*. The first three volumes, published in 1914–15, were *Praktische Analysis* by H. von SANDEN, *Darstellende Geometrie* by J. HJELMSLEV, and *Grundzüge der Geodäsie* by M. NÄBAUER.

Three bulletins have been issued by the Naval Consulting Board of the United States and the Engineering Council's War Committee of Technical Societies. In *Problems for Aëroplane Improvement*, bulletin No. 3, August, 1918, there is a "Working Bibliography," pages 19–29. This bibliography includes portions of one published by *Aviation and Aëronautical Engineering*, May 1, 1917, entitled "Two hundred books on aëronautics" by E. N. Fales. Paul Brockett's com-

prehensive volume, *Bibliography of Aëronautics* published by the Smithsonian Institution in 1910, is now being brought up to date.

Macmillan's spring announcements include an entirely new edition of CAJORI'S *History of Mathematics;* a new work in the Hedrick series, KENYON and INGOLD'S *Elemements of Plane Trigonometry; A short Course in Mathematics for Freshmen* by R. E. MORITZ; and an elementary work by BERTRAND RUSSELL entitled *Introduction to Mathematical Philosophy* (published by Allen and Unwin, London)—The Cambridge University Press announces that J. H. JEANS'S *Problems of Cosmogony and Stellar Dynamics* is soon to appear.

Teubner announces the completion of F. Dingeldey's edition of the genial Fiedler's free translation of Salmon's *Conic Sections.* The first English edition was published in 1848, and the first in German in 1860. This translation has appeared in two parts since the fifth edition in 1887. Dingeldey's edition constitutes the eighth edition (1915) of the first part, and the seventh edition (1918) of the second part—a work of over 900 pages. The references to sources are very numerous.

Salmon was born at Dublin in 1819 and graduated from Trinity College there in 1841 when he immediately became a fellow and lecturer in mathematics. He was made a divinity lecturer as well in 1845. His first mathematical paper was in 1844 and his first publication in a theological subject in 1849. He was regius professor of divinity from 1866 till he became provost in 1888. He died in 1904. From Fiedler's "Zum Gedächtnis George Salmons" (pp. vi–xv of the first part) the following sentences are quoted: "He was a master in chess and was once chosen as an opponent of Morphy whom he met successfully. I have already mentioned that he was a great reader along many lines including much of a lighter character. . . . So well was he acquainted with the literature for young people, he was able to contribute to the 39th volume of the *Fortnightly Review* an interesting inquiry on the theme, 'What boys read'; he held the girls' book, *Alice in Wonderland,* in high esteem. Above all things he rejoiced in humor—both the very refined and the more rugged." In John Ossory's sketch in the *Dictionary of National Biography* occur the lines: "Hospitable and kindly, Salmon had many friends and interests. In youth a competent musician and a chess player of remarkable power, he cultivated both recreations until an advanced age. He was always an omnivorous reader (except in the two departments of metaphysics and poetry, for which he had no taste) and had a special affection for the older novelists, being accustomed to recommend the study of Jane Austen as a liberal education. The homely vigor and the delightful wit of the long letters which he was accustomed to write to his friends entitle him to rank as one of the best letter-writers of the last century."

Three parts of the *Encyklopädie der mathematischen Wissenschaften* were published in 1918: (1) Band III 1, Heft 6: M. Zacharias, "Elementargeometrie und elementare nichteuklidische Geometrie in synthetischer Behandlung." Zweiter Theil mit Zusätzen von W. F. Meyer; (2) Band III 2, Heft 7: C. Segre, "Mehrdimensionale Räume"; (3) Band VI 1 B, Heft 4: E. von Schweidler, "Atmosphärische Elektrizität"; A. Schmidt, "Erdmagnetismus und verwandte Erscheinungen." There are now eight complete volumes of the *Encyklopädie;* the two for "Arithmetik und Algebra," two of the four volumes for "Analysis," and four of the five volumes for "Mechanik" (the Registerband alone is lacking). At present an extra charge, amounting to 30 per cent. of old prices, is made for all parts of the *Encyklopädie.*

Teubner has recently published two additional parts of Materialien für eine

wissenschaftliche Biographie von Gauss—Heft 4: *C. F. Gauss als Zahlenrechner* von A. Galle (Pp. 1–24); Heft 5: *C. F. Gauss als Geometer* von P. Stäckel (Pp. 25–142), 1918. Earlier parts were—Heft 1: *Ueber Gauss' zahlentheoretische Arbeiten* von P. Bachmann, 1911; Heft 2: *C. F. Gauss, Fragmente zur Theorie des arithmetisch-geometrischen Mittels aus den Jahren 1797–1799,* and Heft 3: *Ueber Gauss' Arbeiten zur Funktionentheorie,* von L. Schlesinger, 1912.

The first part of the tenth volume of Gauss' collected works appeared in 1917. It is a volume of nearly 600 pages and is entitled "Nachträge zur reinen Mathematik. Nachbildung und Abdruck des Tagebuchs." The second part of volume 10 (Aufsätze über Gauss' wissenschaftliche Tätigkeit auf den Gebieten der reinen Mathematik—Zahlentheorie, Funktionentheorie, Algebra, Geometrie, Variationsrechnung) and the first part of volume 11 (Nachträge zur Physik, Astronomie und Chronologie) are in the press. The second part of volume 11 which is in course of preparation is to contain "Aufsätze über Gauss' wissenschaftliche Tätigkeit auf den Gebieten der angewandten Mathematik." The twelfth and last volume is to be biographical.

The following quotation is a bit of personal testimony to the value of mathematical study, from the pen of one who has become world renowned for the clarity and beauty of his literary style:

"Geometry was to teach me the logical progression of thought; it was to tell me how the difficulties are broken up into sections which, elucidated consecutively, together form a lever capable of moving the block that resists direct efforts; lastly, it showed me how order is engendered, order, the base of clarity. If it has ever fallen to my lot to write a page or two which the reader has run over without excessive fatigue, I owe it, in great part, to geometry, that wonderful teacher of the art of directing one's thought. True, it does not bestow imagination, a delicate flower blossoming none knows how and unable to thrive on every soil; but it arranges what is confused, thins out the dense, calms the tumultuous, filters the muddy, and gives lucidity, a product superior to all the tropes of rhetoric.

"Yes, as a toiler with the pen, I owe much to it. Wherefore my thoughts readily turn back to those bright hours of my novitiate, when, retiring to a corner of the garden in recreation-time, with a bit of paper on my knees and a stump of pencil in my fingers, I used to practise deducing this or that property correctly from an assemblage of straight lines. The others amused themselves all around me; I found my delight in the frustum of a pyramid." From *The Life of the Fly with which are interspersed some chapters of autobiography* by J. Henri Fabre, translated by A. T. de Mattos. New York, Dodd, Mead and Co., 1913, Chapter 12: "Mathematical memories: Newton's binomial theorem."

ARTICLES IN CURRENT PERIODICALS.

American Journal of Mathematics, volume 40, no. 4, October, 1918: "Theta modular groups determined by point sets" by A. B. Coble, 317–340; "On the asymptotic solution of the non-homogeneous linear differential equation of the nth order—a particular solution" by W. V. N. Garretson, 341–350; "A collineation group isomorphic with the group of the double tangents of the plane quartic" by C. C. Bramble, 351–365; "Proof of Pohlke's theorem and its generalizations by affinity" by A. Emch, 366–374; "Arithmetical theory of certain Hurwitzian continued fractions" by D. N. Lehmer, 375–390.

Bulletin of the American Mathematical Society, volume 25, no. 4, January, 1919: "On the evaluation of the elliptic transcendents η_2 and η_2'" by Harris Hancock, 150–157; "On plane algebraic curves with a given system of foci" by Arnold Emch, 157–161; "Quadratic systems of circles in non-Euclidean geometry" by D. M. Y. Sommerville, 161–173; "Continuous sets that have no continuous sets of condensation" by R. L. Moore, 174–176; "Derivativeless continuous functions" by M. B. Porter, 176–180; Review by R. D. Carmichael of Picard's *Les*

der Erddrehung "by W. Brunner, 254–262; "Max Simon zum Gedächtnis" (mit Bild im Text) by W. Lorey, 268–271. [A more elaborate "Nachruf" by the same author is in *Leopoldina*, 1918, no. 2. Max Simon was born in 1844 and died January 15, 1918. He was Oberlehrer at the Lyceum in Strasbourg for over 40 years[1] and "Honorarprofessor" at the university since 1903. His most successful book was probably his *Didaktik und Methodik des mathematischen Unterrichts.* The long list of his writings in elementary mathematics and mathematical history includes the mathematical articles A–L, as well as the biographies of all mathematicians, in the *Konversations-lexikon* of Brockhaus. He was also the author of the mathematical articles A–R in the fifth edition of "Meyer"].—Doppel Heft 10–11, November, 1918: "Ueber isoperimetrische Probleme in der Schule und in der Forschung" by W. Lorey, 281–293; "Nochmals die Hessesche Normalform" by K. Doehlemann, 293–299; "Ueber eine algebraische Behandlungsweise des regulären Siebzehnecks" by H. Wolff, 299–303; "Die Siebenzehn-Teilung des Kreises in elementargeometrischer Herleitung" by R. Lohnstein, 303–312; "Zur Ortsbestimmung eines Fesselballons" by A. Witting, 312–317; "Der räumliche pythagoreische Lehrsatz" by H. Wieleitner, 321–322. [Wieleitner's note deals with the history of the theorem: If a tetrahedron has each of its face angles at one vertex a right angle, the sum of the squares of the areas of the faces about the vertex is equal to the square of the area of the face opposite the vertex. Wieleitner states that this theorem was first given in *Johann Faulhabers Ulmensis Miracula Arithmetica,* Augspurg, M.DC.XXII, chapter 45, pp. 73–75. Faulhaber refers to the theorem as "ein Newe Geometrische Invention welche auss der Zahl 666 (Apocal. in 13. Cap.) Calculirt und Demonstrirt." It is pointed out that the theorem in its generality was known to Descartes who discussed it in his *Cogitationes privatæ* (written 1619–21, but first printed in 1908) where it is stated in the form: In tetraedro rectangulo, basis potentia æqualis est potentijs trium facierum simul. It is known that Faulhaber was in personal touch with Descartes about the time that these *Cogitationes* were written.]

AMERICAN DOCTORAL DISSERTATIONS.

J. W. Campbell, 1889– , *Periodic solutions of the problem of three bodies in three dimensions.* Pp. 43–84. [Reprinted from *The Proceedings of the London Mathematical Society,* 1917. (Chicago, 1915.)

A. M. Harding, *On certain loci projectively connected with a given plane curve.* 38 pp. [Reprinted from *Giornale di matematiche di Battaglini,* 1916.] (Chicago, 1916.)

W. L. Hart, 1892– , *Differential equations and implicit functions in infinitely many variables.* Pp. 125–160. [Reprinted from *Transactions of the American Mathematical Society,* 1917.] (Chicago, 1916.)

Mary G. Haseman, *On Knots, with a census of the amphicheirals with twelve crossings.* [Reprinted from the *Transactions of the Royal Society of Edinburgh,* volume 52, 1917]. Edinburgh, Neill and Co., 1918. Pp. 235–255 + 1 plate. 4to. (Bryn Mawr, 1916).

F. M. Morrison, 1871– , *On the relation between some important notions of projective and metrical differential geometry.* Pp. 199–221. [Reprinted from the *American Journal of Mathematics,* 1917.] (Chicago, 1913.)

Mary E. Wells, *On inequalities of certain types in general linear integral equation theory.* Pp. 163–184. [Reprinted from the *American Journal of Mathematics,* 1917.] (Chicago, 1915.)

MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The Maryland-Virginia-District of Columbia Section of the Mathematical Association of America met at Johns Hopkins University, Baltimore, Maryland, January 18, 1919. Among those in attendance were the following members: Oscar S. Adams, U. S. Coast and Geodetic Survey; H. G. Avers, U. S. Coast and Geodetic Survey; Clara L. Bacon, Goucher College; G. R. Clements, U. S.

[1] *Cf.* S. Wirz, *Der mathematische Unterricht an den höheren Knabenschulen sowie die Ausbildung der Lehramtskandidaten in Elsass-Lothringen.* (IMUK Abhandlung), Leipzig, Teubner, 1911.

Naval Academy; A. Cohen, Johns Hopkins University; Alexander Dillingham, U. S. Naval Academy; H. C. Gossard, U. S. Naval Academy; W. M. Hamilton, U. S. Nautical Almanac Office; Frank Morley, Johns Hopkins University; F. D. Murnaghan, Johns Hopkins University; R. E. Root, U. S. Naval Academy; C. A. Shook, U. S. Naval Academy; Clara E. Smith, Wellesley College (Goucher, 1918–1919); H. Ivah Thomsen, Johns Hopkins University; G. F. Winslow, Jr., U. S. Coast and Geodetic Survey; L. S. Hulburt, Johns Hopkins University.

The program consisted of a single session in the afternoon, and contained the following papers:

1. Content of a course in analytic geometry. By Dr. G. R. Clements, U. S. Naval Academy. Discussion led by Professor Clara L. Bacon, Goucher College.

2. Lambert's mapping. By Professor Frank Morley, Johns Hopkins University.

3. The teaching of the subject of limits. By Professor Clara E. Smith, Wellesley College.

4. The polyconic projection and the quadrillage associated therewith. By Mr. O. S. Adams, U. S. Coast and Geodetic Survey.

After the program the members present were guests of Johns Hopkins University at the Hopkins Club, where a very enjoyable supper was served.

R. E. Root, *Secretary.*

UNDERGRADUATE MATHEMATICS CLUBS.

Edited by U. G. Mitchell, University of Kansas, Lawrence.

CLUB ACTIVITIES.

The Mathematics Club of Brown University, Providence, R. I. [*1918*, 33–34].

For 1918–19 the club is to be guided by

Chairman: Professor Nathaniel F. Davis;

Committee on Program: Professor Roland G. D. Richardson, Winona M. Perry Gr., Frances W. Wright '19, Harley F. Carey '20, Robert B. Lindsay '20;

Committee on Arrangements: Professor Theodore H. Brown, Rachel T. Easterbrooks '20, Clarence R. Adams Gr., Everett L. Sweet '21.

The following meetings have been held.

November 30, 1918: "The method of solving certain problems in the Ahmes papyrus" by Arnold B. Chace, Chancellor of the University.

February 14, 1919: "Archimedes" by Frances W. Wright '19; "Two methods of locating the German super-gun"[1] by Daniel B. Whitford '20; "The cattle problem of Archimedes" by Frances M. Merriam '20; "The logarithmic spiral" by Robert B. Lindsay '20.

[1] Report on H. F. MacNeish's article in *School Science and Mathematics*, October, 1918 (Vol. XVIII, pp. 626–628).

THE MATHEMATICS CLUB OF HUNTER COLLEGE, New York City [*1918*, 187–188].

So far as the editor knows this club is the first one to make use of special printed stationery for its correspondence. According to the letter head the officers for the year 1918–19 are as follows: President, Anita Rosenthal '19; vice-president, Kathryn McSorley '19; secretary, Louise Biehl '19; treasurer, Miriam Werner, instructor in mathematics.

The spirit of war service dominated the work of the Club throughout the year 1918. Most of the members were volunteer workers aiding the Legal and Local Draft Boards throughout the city and promoting the Hunter College drives for Liberty Loans, Red Cross, Y. W. C. A. and other organizations. A number took courses in the War Service Training School for Women and about a dozen worked as farmerettes.

The programs for the year were as follows.

March: "The need of evaluating one's self so as to render the most effective service" by Professor Emma M. Requa; "Mr. and Mrs. Ratio One to Two demonstrating practical mathematics" by Marion Graham '18 and Ellen Raymond '18; social meeting.

April: "Commercial education of the present day" by Mr. Alfred Sommerfield of Washington Irving High School, New York.

May: "High School mathematics as an aid to the interpretation and appreciation of life about us" by Professor Charles B. Upton of Teachers' College, Columbia University.

June: General business meeting; election of officers.

September: The social meeting was postponed on account of war activities.

October: Social meeting.

November: "The common place book" by Charles B. Walsh of Ethical Culture School, New York.

THE MATHEMATICAL AND ASTRONOMICAL CLUB OF SWARTHMORE COLLEGE, Swarthmore, Pa. [*1918*, 135].

The officers of the club for the year 1917–18 were as follows: First semester—president, Robert S. Blau '18; vice-president, Ethelwyn Bower '18; secretary, Gladys Pell '20. Second semester—president, John Trimmer '18; vice-president, Dorothy A. Johnson '18; secretary, Charlotte Moore '20.

The programs for the first six meetings of the year 1917–18 were given in an earlier number of the MONTHLY. The programs for the remaining meetings of the year were as follows:

December 18, 1917; "The depression range finder and its use in the present war" by John Trimmer '18; "The Barr and Strand range finder" by Ewing T. Corson, '18; "The use of a range finder for objects invisible from the place where the gun is located" by Robert S. Blau '18.

February 5, 1918: "The astrology of casting a horoscope" by Opal M. Robinson '18; "The history of mathematics" by Albert N. Nelson '18; "De Moivre's theorem and its applications" by Lena Clark '20; "Graphic methods of solving equations" by Professor John H. Pitman.

February 19: "Rifling a cannon" by Robert S. Blau '18; "The discovery of Neptune" by Boyd J. Brown '21; "Scales of notation" by Elizabeth Trorer '19; "Intra-Mercurial planets" by Professor John A. Miller.

March 5: "Methods used in passing from the properties of rectilinear figures to circular figures" by Ethelwyn Bower '18; "Properties of magic squares" by Albert N. Nelson '18; "Principles and applications of the planimeter" by Harry Yardley '19; "The use of the planimeter in finding the displacement of a ship" by John Trimmer '18.

March 19: "The solar eclipse of 1905" by Professor Miller; "The problem of mounting long focus cameras for photography" by Margaret E. Powell '19; "Theories of the formation of the corona" by Caroline H. Smedley Gr.

April 2: "The discriminant and its relation to higher plane curves" by Helen Deputy '18; "Structure of powder grains" by Frank Fetter '20; "Problems of gunnery" by John Trimmer '18; "The Gregorian calendar" by Reverend Walter A. Matos.

April 19: "The history of algebra," an illustrated lecture by Professor Louis C. Karpinski of the University of Michigan.

THE MATHEMATICAL AND PHYSICAL SOCIETY OF THE UNIVERSITY OF TORONTO, Toronto, Ontario [*1918*, 229–231].

Regular meetings of this club are held on alternate Thursdays at 4:15 p. m. The officers for the year 1918–19 are as follows: Honorary president, Professor Eli F. Burton; president, William W. Shaver '19; vice-president, Percy Lowe '20; secretary-treasurer, William S. Vaughan '20; corresponding secretary, Ila B. Giles '19; representatives of classes: Lily M. Floody '19, Arthur J. Sonley '20, Nora E. Gray '21, Dorothy G. Gavin '22.

The following programs are announced for the current year:

November 21, 1918: "The darkest year in British history" by Professor Alfred Baker.

December 5: "Mathematical puzzles" by Professor John Satterly; "The moon" by Franklin B. Keachie '19.

December 17: Social evening at 8 p. m.

January 9, 1919: "New Zealand" by Professor John C. Fields; "My first impressions of the course" by Mattie Levi '21.

January 23: Graduates' meeting. "Periodic precipitations" by Alice W. Foster Gr.; "The ancient scientists" by Mabel C. Child '18.

January 28: Skating party.

February 6: Open meeting at 8 p. m.

February 20: Debate between First and Second Year Students.

March 6: "The most important experiment in fourth year physics" by Mary I. Mackay '19; "Mathematical recreation" by Everett O. Hall '19.

March 20: "Radium: its discovery and use" by Raymond C. Dearle Gr.; annual elections.

THE UNDERGRADUATE MATHEMATICS CLUB OF THE UNIVERSITY OF WASHINGTON, Seattle, Washington.

The information given below concerning this club was furnished by Assistant Professor Eric T. Bell of the University of Washington.

The club was in existence for some ten years and had a membership of about twenty-five. Meetings were held twice a month with an average attendance of about fifteen. The outbreak of the war put an end to the club's activities, since all but two of its members were taken in the draft or in other service. The club is soon to be reorganized.

The club was managed entirely by the students without any interference or direction whatever from the faculty and the results amply justified this course. Members of the faculty were invited to attend the meetings of the club and occasionally did so. On rare occasions, when the students had not had time to prepare papers of their own, they asked a member of the faculty to give an account of some branch of mathematics not in the undergraduate courses; for example, mathematical crystallography, groups, higher arithmetic, applications of mathematics to biology, etc. Meetings were scheduled to last for an hour and a half to two hours but sometimes continued for as long as three hours.

Detailed information concerning programs is lacking since the records kept by the secretaries disappeared when the last secretary left the university at the outbreak of the war. Some of the programs given, however, were as follows.

"Ten British mathematicians." A review of Alexander Macfarlane's book (1916) of that title—"Non-euclidean geometry." Presented and discussed at several meetings by members who had studied the subject. Bonola's (1912) and Manning's (1901) books were used as sources—"Geometry of four dimensions." A presentation during several meetings of Manning's book (1914) and a sketch of the circle-representation as given in Weber-Wellstein (1905)—"Hilbert's proof of the transcendence of π"—"Continued fractions and 'Pell's equation'"—"Formal implication."

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

2762. Proposed by N. P. PANDYA, Amreli, India.

$ABCD$ is a cyclic quadrilateral inscribed in an ellipse. $AB = 2BC$ and $CD = 2DA$. Find the eccentricity of the ellipse in terms of the sides of the quadrilateral.

2763. Proposed by C. N. SCHMALL, New York City.

Show that the equation, $ky - 2k^{1/3}a^{2/3}x + x^2 = 0$, where k is a variable parameter, represents a family of parabolas passing through a fixed point, and all having the same areas, comprised between the curve and the x-axis.

Show, also, that the envelope of the family is the rectangular hyperbola whose equation is $xy = 2^5a^2/3^3$.

2740. Proposed by E. H. CLARKE, Hiram College.

If the coefficients of $(a - b)^k$ (where k is a positive integer) be multiplied term by term by the nth powers (n being zero or a positive integer), of the terms of any arithmetic progression with common difference $d \neq 0$, the sum of the products will vanish if $n < k$; will be $(-d)^k(k!)$ if $n = k$; and, if $n = k + 1$, will be the product of this last result and the sum of the terms of the arithmetic progression.

2765. Proposed by A. M. HARDING, University of Arkansas.

ABC is an equilateral triangle. A point D is taken in BC such that BD is $\frac{1}{3}$ of BC and E is taken in CA such that CE is $\frac{1}{3}$ of CA. If the lines AD and BE intersect at O, show that OC is perpendicular to AD.

2766. Proposed by N. P. PANDYA, Amreli, India.

Is it possible to find a harmonic series whose terms are positive integers such that the product of the first, second, seventh, and eighth terms is equal to the product of the third, fourth, fifth, and sixth terms?

2767. Proposed by W. W. JOHNSON, U. S. Naval Academy.

Let the complex quantities p, q, and r satisfy the relation $p^2 + q^2 + r^2 = 0$; prove that the corresponding vectors OP, OQ, and OR are such that if any two of them are taken as conjugate semi-diameters of an ellipse, the third lies on the minor axis and its length is the distance from the center to either focus.

2768. Proposed by PAUL CAPRON, U. S. Naval Academy.

Given the center, a focus, and a point of a conic, construct geometrically the circle of curvature at the point.

2769. Proposed by B. J. BROWN, Kansas City, Mo.

Expand in powers of x as far as x^2 the function $\dfrac{\cosh \lambda x}{\cosh \lambda} - \dfrac{x \sinh \lambda x}{\sinh \lambda}$ in which λ is a positive constant.

Prove that, if $\lambda \tanh \lambda > 2$, the function has only one maximum value for $x > 0$ and that the value of x for which the maximum occurs is less than 1. India Civil Service. 1912.

2770. Proposed by A. M. HARDING, University of Arkansas.

Solve the differential equation

$$\frac{d^3x}{dt^3} - 2(\mu t + \lambda)\frac{dx}{dt} + (1 - \mu)x = 0,$$

where λ and μ are constants.

2771. Proposed by GEORGE PAASWELL, New York City.

A circle is revolved through an angle of 90° about a vertical chord which does not pass through the center of the circle. Taking the origin at the lower extremity of the chord, the z-axis along the chord, and the x- and y-axes in the boundary planes, pass a plane through the x-axis making a given angle with the xy-plane. Determine the portion of the area of the surface above the plane and between the xz- and yz-planes.

NOTE.—This is a restatement of calculus problem 430, a solution of which appeared in this MONTHLY, February, 1913. As the solution there given is not according to the interpretation intended by the Proposer, we are reprinting the problem in this slightly revised form—EDITORS.

2772. Proposed by HARRY LANGMAN, New York City.

Given $1 = (-\frac{1}{2} + x)^r = (-\frac{1}{2} - x)^r$, where r is an integer. Prove that r is a multiple of 3. In general, if

$$1 = \left(\cos\frac{2\pi}{m} + x\right)^r = \left(\cos\frac{2\pi}{m} - x\right)^r,$$

where r and m are integers, prove that r is a multiple of m.

SOLUTIONS OF PROBLEMS.

2689 [April, 1918]. Proposed by E. V. HUNTINGTON, Harvard University.

Show that the maximum value of

$$y = \frac{\sin x \cos (x + \varphi)}{\cos (x + \theta) \sin (x + \varphi + \theta)}$$

is $y_1 = (\cos \theta - \rho)/(\cos \theta + \rho)$ where $\rho = \sqrt{\sin^2 \varphi - \sin^2 \theta}$.

This problem was suggested to the proposer by a professor of civil engineering, and has important applications in the theory of conjugate stresses.

Solution by J. B. Reynolds, Lehigh University.

$$y = \frac{\sin x \cos (x + \varphi)}{\cos (x + \theta) \sin (x + \varphi + \theta)}. \tag{1}$$

Taking the logarithm of (1), differentiating and setting the result equal to zero we get

$$\cot x - \tan (x + \varphi) + \tan (x + \theta) - \cot (x + \varphi + \theta) = 0 \tag{2}$$

or

$$\cot x - \cot (x + \varphi + \theta) = - \tan (x + \theta) + \tan (x + \varphi);$$

whence

$$\frac{\sin (\varphi + \theta)}{\sin (\varphi - \theta)} = \frac{\sin x \sin (x + \varphi + \theta)}{\cos (x + \varphi) \cos (x + \theta)}. \tag{3}$$

Or writing (2) in the form

$$\cot x + \tan (x + \theta) = \tan (x + \varphi) + \cot (x + \varphi + \theta)$$

we get

$$\frac{\cos \theta}{\sin x \cos (x + \theta)} = \frac{\cos \theta}{\cos (x + \varphi) \sin (x + \varphi + \theta)};$$

or

$$\frac{\cos (x + \varphi)}{\cos (x + \theta)} = \frac{\sin x}{\sin (x + \varphi + \theta)}. \tag{4}$$

(4) in (3) gives

$$\frac{\cos (x + \varphi)}{\sin x} = \sqrt{\frac{\sin (\varphi - \theta)}{\sin (\varphi + \theta)}};$$

whence

$$\cot x = \tan \varphi + \sec \varphi \sqrt{\frac{\sin (\varphi - \theta)}{\sin (\varphi + \theta)}}. \tag{5}$$

By (4) and (1), we may write

$$y_1 = \frac{\sin^2 x}{\sin^2 (x + \varphi + \theta)}$$

so that by (5)

$$\frac{1}{\sqrt{y_1}} = \cos (\varphi + \theta) + \tan \varphi \sin (\varphi + \theta) + \sec \varphi \sqrt{\sin (\varphi + \theta) \sin (\varphi - \theta)} = \frac{\cos \theta + \rho}{\cos \varphi}.$$

$$\therefore \quad y_1 = \frac{\cos^2 \varphi}{(\cos \theta + \rho)^2} = \frac{\cos^2 \theta - \rho^2}{(\cos \theta + \rho)^2} \equiv \frac{\cos \theta - \rho}{\cos \theta + \rho}.$$

To prove that we have a maximum we find:

$$\frac{dy}{dx} = \frac{\cos \theta}{\cos (x + \theta) \sin (x + \varphi + \theta)} \left\{ \frac{\cos (x + \varphi)}{\cos (x + \theta)} - \frac{\sin x}{\sin (x + \varphi + \theta)} \right\}$$

$$= \frac{\cos \theta}{\cos^2 (x + \theta) \sin^2 (x + \varphi + \theta)} \{\sin \varphi \cos (2x + \varphi + \theta) + \sin \theta\},$$

which shows that we have either a maximum or a minimum when the brace equals zero, since it is the only variable factor of odd power. Choosing the least value of x that will make the brace zero we have that $\cos (2x + \varphi + \theta)$ decreases as x increases so that the sign of dy/dx changes from positive to negative as x passes through this critical value.

2690 [April, 1918]. Proposed by E. V. HUNTINGTON, Harvard University.

Find the maximum value of

$$y = \frac{\sin x \cos (x + \varphi)}{\cos (x + \theta) \sin (x + \beta + \theta)}. \tag{1}$$

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Taking the logarithm of both sides, differentiating, and setting the result equal to zero we get

$$\cot x - \tan (x + \varphi) + \tan (x + \theta) - \cot (x + \beta + \theta) = 0 \tag{2}$$

and from the two forms of this

$$\cot x - \cot (x + \beta + \theta) = \tan (x + \varphi) - \tan (x + \theta)$$

and

$$\cot x + \tan (x + \theta) = \cot (x + \beta + \theta) + \tan (x + \varphi)$$

we get

$$\frac{\sin (\beta + \theta)}{\sin (\varphi - \theta)} = \frac{\sin x \sin (x + \beta + \theta)}{\cos (x + \varphi) \cos (x + \theta)} \tag{3}$$

and

$$\frac{\cos (x + \varphi)}{\cos (x + \theta)} = \frac{\cos (\beta + \theta - \varphi)}{\cos \theta} \cdot \frac{\sin x}{\sin (x + \beta + \theta)}. \tag{4}$$

By (3) and (4), we have

$$\sqrt{\frac{\cos (\beta + \theta - \varphi) \sin (\varphi - \theta)}{\cos \theta \sin (\beta + \theta)}} = \frac{\cos (x + \varphi)}{\sin x} = \cot x \cos \varphi - \sin \varphi.$$

Hence,

$$\cot x = \tan \varphi + \sec \varphi \sqrt{\frac{\cos (\beta + \theta - \varphi) \sin (\varphi - \theta)}{\cos \theta \sin (\beta + \theta)}}. \tag{5}$$

(4) and (1) give, where y_1 is the maximum value of y,

$$y_1 = \frac{\cos (\beta + \theta - \varphi)}{\cos \theta} \cdot \frac{\sin^2 x}{\sin^2 (x + \beta + \theta)};$$

whence

$$\frac{1}{\sqrt{y_1}} = \sqrt{\frac{\cos \theta}{\cos (\beta + \theta - \varphi)}} \{\cos (\beta + \theta) + \sin (\beta + \theta) \cot x\}$$

and eliminating $\cot x$ by (5), we have

$$y_1 = \cos^2 \varphi / \{\sqrt{\cos \theta \cos (\beta + \theta - \varphi)} + \sqrt{\sin (\varphi - \theta) \sin (\beta + \theta)}\}^2.$$

As in 2689 we may prove that y_1 is a maximum as found for the case $\beta = \varphi$ from which we may infer from the manner that β enters into the expression that it is a maximum for all values of β.

2692 [April, 1918]. Proposed by J. L. RILEY, Stephenville, Texas.

A cube is cut at random by a plane, what is the chance that the section is a hexagon?

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

Let the faces of the cube be $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$. Let the plane be

$$lx + my + nz = p,$$

where $n = \cos \theta > 0$, $l = \sin \theta \cos \phi$, $m = \sin \theta \sin \phi$. The normal from the origin is equally likely to have any direction, and, when the direction is fixed, equally likely to have any length. Since $\sin \theta d\theta d\phi$ is an element of the solid angle traced by the normal, and dp an element of its length, the relative probability of the plane lying within given limits is $\iiint \sin \theta dp d\theta d\phi$. On account of the symmetry of the cube, it may be assumed that the normal through O to the plane meets the surface of the cube within the triangle (0, 0, 1), (1, 0, 1), and (1, 1, 1); so that $n > l > m > 0$, that is, $0 < \phi < \pi/4$, $0 < \theta < \tan^{-1} \sec \phi$.

The distances from the plane, with regard to sign, of the vertices (1, 1, 1), (1, − 1, 1), (− 1, 1, 1), (1, 1, − 1), (− 1, − 1, 1), (1, − 1, − 1), (− 1, 1, − 1), (− 1, − 1, − 1) are $l + m + n - p$, $l - m + n - p$, etc.; and these are in descending order, except the two middle ones, which partake of the property only when $l + m > n$, that is when $\theta > \tan^{-1}(\cos\phi + \sin\phi)^{-1}$.

The plane will cut the cube when $l + m + n > p > - l - m - n$, and will cut it in a hexagon when $l + m - n > p > - l - m + n$. Hence, the required chance is

$$\frac{\displaystyle\int_0^{\pi/4}\int_{\tan^{-1}(\cos\phi+\sin\phi)^{-1}}^{\tan^{-1}\sec\phi}\int_{\cos\theta-\sin\theta(\cos\phi+\sin\phi)}^{-\cos\theta+\sin\theta(\cos\phi+\sin\phi)}\sin\theta\, dp\, d\theta\, d\phi}{\displaystyle\int_0^{\pi/4}\int_0^{\tan^{-1}\sec\phi}\int_{-\cos\theta-\sin\theta(\cos\phi+\sin\phi)}^{\cos\theta+\sin\theta(\cos\phi+\sin\phi)}\sin\theta\, dp\, d\theta\, d\phi}.$$

The integrations for p and θ having been effected, the inverse tangents in ϕ may be removed by integration by parts, after which the substitution $u = \tan\phi$ reduces the integrands to rational functions of u. The resulting value for the probability is $2/\sqrt{3} - (4\sqrt{2}/\pi)\tan^{-1} 1/\sqrt{2}$, or $.0465\cdots$.

2693 [April, 1918]. Proposed by W. F. HARLOW, Portland, Oregon.

A cow is tethered with a rope, length l, to a peg on the opposite side of a wall, height h, the peg being at a distance a from the wall. Find the area over which the cow can graze.

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

Let the bottom of the wall be $x = z = 0$; the top $x = 0$, $z = h$; and the peg $(-a, 0, 0)$. The cow being at $(x, y, 0)$, the lengths of the two parts of the rope being u and v, and the point at which the rope crosses the wall being $(0, t, h)$, we must have, if the rope is taut,

$$u^2 = a^2 + h^2 + t^2, \qquad v^2 = x^2 + (t - y)^2 + h^2,$$

and $u + v = l$.

The solution of these equations in t, u, v is

$$t = \frac{yp \pm lr}{2(l^2 - y^2)}, \qquad u = \frac{lp \pm yr}{2(l^2 - y^2)}, \qquad v = \frac{lq \mp yr}{2(l^2 - y^2)},$$

where $p = a^2 + l^2 - x^2 - y^2$, $q = l^2 + x^2 - a^2 - y^2$, and $r = \sqrt{p^2 - 4(a^2 + h^2)(l^2 - y^2)}$.

We require the conditions that u and v may be positive and t real.

From the obtuse triangle $(-a, 0, 0)$, $(0, 0, h)$, (x, y, h) it is clearly necessary that

(1) $$l^2 > a^2 + h^2 + x^2 + y^2.$$

Other necessary conditions are that $p^2 \geqq 4(a^2 + h^2)(l^2 - y^2)$, and (since $l > |y|$ and $|p| > r$) that $p > 0$. From these two it follows that

(2) $$x^2 \leqq l^2 + a^2 - y^2 - 2\sqrt{a^2 + h^2}\,\sqrt{l^2 - y^2},$$

which again implies that $(\sqrt{l^2 - y^2} - \sqrt{a^2 + h^2})^2 \geqq h^2$, that is (with the help of (1)) that

(3) $$y^2 \leqq l^2 - (\sqrt{a^2 + h^2} + h)^2.$$

It remains to prove that (2) and (3) are not only necessary but sufficient. For this we need only observe that they make r real (and positive) $l > |y|$, $p > r$, and q positive. The lower signs will then give a suitable solution in t, u, and v.

The area to be computed is therefore

$$A = 2\int_0^{\sqrt{l^2-(\sqrt{a^2+h^2}+h)^2}} \sqrt{a^2 + l^2 - y^2 - 2\sqrt{a^2 + h^2}\sqrt{l^2 - y^2}}\cdot dy.$$

Writing $\sqrt{l^2 - y^2} = u$, $\sqrt{a^2 + h^2} = c$, we get

$$A = 2\int_{c+h}^{l} \frac{u\sqrt{(u - c)^2 - h^2}}{\sqrt{l^2 - u^2}}\, du.$$

To reduce to standard elliptic integrals, we may make

$$u = \frac{\lambda v + \mu}{v + 1},$$

where $(\lambda - c)(\mu - c) = h^2$, $\lambda\mu = l^2$, $\lambda \geqq \mu$. If l is assumed $\geqq c + h$, as is natural, λ and μ are real, unequal, and greater than c. The result of the transformation is

$$A = 2(\lambda - \mu) \int_{\sqrt{\frac{\mu - c}{\lambda - c}}}^{\sqrt{\frac{\mu}{\lambda}}} \frac{\sqrt{(\lambda - c)v^2 - (\mu - c)}}{\sqrt{\mu - \lambda v^2}} \cdot \frac{\lambda v + \mu}{(v + 1)^3} dv.$$

The final result will be simplified if we transform again, putting

$$v = \frac{\sqrt{\mu(\mu - c)}}{\sqrt{\mu(\lambda - c) - (\lambda - \mu)cw^2}},$$

which gives

$$A = 2c(\mu - c)\int_0^1 \frac{(3\lambda c + 3\mu c - 2\lambda\mu - 4c^2)\mu^2 + (\lambda\mu + 5\mu c - 3\mu^2 - 3\lambda c)\mu c w^2 + (\lambda - \mu)\mu c^2 w^4}{(\mu - cw^2)^3\sqrt{1 - w^2}\sqrt{\mu(\lambda - c) - (\lambda - \mu)cw^2}} \cdot w^2 dw$$
$$+ 2c(\mu - c)\sqrt{\mu(\mu - c)}\int_0^1 \frac{(\lambda + \mu - 4c)\mu - (\lambda - 3\mu)cw^2}{(\mu - cw^2)^3\sqrt{1 - w^2}} \cdot w^2 dw.$$

These integrals are respectively elliptic and circular. On making the necessary reductions, we find, after much calculation,

$$A = c(\mu - c)\sqrt{\frac{\mu}{\lambda - c}}\,F\left(\sqrt{\frac{(\lambda - \mu)c}{\mu(\lambda - c)}}, \frac{\pi}{2}\right) - c\sqrt{\mu(\lambda - c)}\,E\left(\sqrt{\frac{(\lambda - \mu)c}{\mu(\lambda - c)}}, \frac{\pi}{2}\right)$$
$$- \frac{c(\mu - c)(\lambda + \mu - c)}{\sqrt{\mu(\lambda - c)}}\,\Pi\left(-\frac{c}{\mu}, \sqrt{\frac{(\lambda - \mu)c}{\mu(\lambda - c)}}, \frac{\pi}{2}\right) + \frac{\pi}{2} \cdot c(\lambda + \mu - c).$$

2697 [April, 1918]. Proposed by H. S. UHLER, Yale University.

Show how to reduce the left-hand members of the following identities to their respective right members:

$$\sin^2 (x + \tfrac{1}{2}y) - \sin (x + \tfrac{3}{2}y) \sin (x - \tfrac{1}{2}y) = \sin^2 y, \tag{1}$$

$$\sin (x + y) \sin (x + \tfrac{1}{2}y) - \sin x \sin (x + \tfrac{3}{2}y) = \sin \tfrac{1}{2}y \sin y, \tag{2}$$

$$\sin x \sin (x + \tfrac{1}{2}y) - \sin (x - \tfrac{1}{2}y) \sin (x + y) = \sin \tfrac{1}{2}y \sin y. \tag{3}$$

SOLUTION BY POLYCARP HANSEN, St. John's University, Collegeville, Minn.

The terms of the left-hand members can be expressed as follows:

(1) $$\sin^2 (x + \tfrac{1}{2}y) - \sin (x + \tfrac{3}{2}y) \sin (x - \tfrac{1}{2}y) = \frac{1 - \cos (2x + y)}{2} + \frac{1}{2}[\cos (2x + y) - \cos 2y] = \frac{1 - \cos 2y}{2} = \sin^2 y.$$

(2) $$\sin (x + y) \sin (x + \tfrac{1}{2}y) - \sin x \sin (x + \tfrac{3}{2}y) = -\tfrac{1}{2}[\cos (2x + \tfrac{3}{2}y) - \cos \tfrac{1}{2}y] + \tfrac{1}{2}[\cos (2x + \tfrac{3}{2}y) - \cos (-\tfrac{3}{2}y)] = \tfrac{1}{2}[\cos \tfrac{1}{2}y - \cos (-\tfrac{3}{2}y)] = \sin \tfrac{1}{2}y \sin y.$$

(3) $$\sin x \sin (x + \tfrac{1}{2}y) - \sin (x - \tfrac{1}{2}y) \sin (x + y) = -\tfrac{1}{2}[\cos (2x + \tfrac{1}{2}y) - \cos (-\tfrac{1}{2}y)] + \tfrac{1}{2}[\cos (2x + \tfrac{1}{2}y) - \cos (-\tfrac{3}{2}y)] = \sin \tfrac{1}{2}y \sin y.$$

Also solved by R. B. WILDERMUTH, JEROME J. JULIAN, KATHERINE S. ARNOLD, R. M. MATHEWS, H. L. OLSON, H. E. CARLSON, A. T. DINEEN, R. C. COLWELL, and ROGER A. JOHNSON.

2698 [April, 1918]. Proposed by WARREN WEAVER, Throop College of Technology, Pasadena, California.

An urn contains N balls, numbered from 1 to N. Of these n are drawn out and are arranged linearly according to the numbers on each. A certain ball is observed to be the kth in this line. What is the most probable number written on this ball?

I. Solution by Harry M. Roeser, Bureau of Standards, Washington, D. C.

The total number of selections of n balls from N balls is

$$U = \frac{\lfloor N}{\lfloor n \lfloor N - n}.$$

The least number that can be in the kth place is k. In the urn there are $(N - k)$ numbers greater than k and $(k - 1)$ numbers less than k. The numbers less than k can be arranged in groups of $(k - 1)$ each in 1 way. After this is done the numbers greater than k can be arranged in groups of $(n - k)$ numbers each in $\dfrac{\lfloor N - k}{\lfloor n - k \lfloor N - n}$ ways. Therefore the total number of sequences with k in the kth place is

$$u_0 = \frac{1 \cdot \lfloor N - k}{\lfloor n - k \lfloor N - n}$$

and by definition the probability that k will be in the kth place is u_0/U.

The next lowest number that can be in the kth place is $(k + 1)$. The k numbers less than $(k + 1)$ can be arranged in groups of $(k - 1)$ numbers each in

$$\frac{\lfloor k}{\lfloor k - 1} = k$$

ways. After this is done the $N - k - 1$ numbers greater than $(k + 1)$ can be arranged in groups of $(n - k)$ numbers each in $\dfrac{\lfloor N - k - 1}{\lfloor n - k \lfloor N - n - 1}$ ways. Therefore, the total number of sequences with $(k + 1)$ in the kth place is

$$u_1 = \frac{\lfloor N - k - 1}{\lfloor n - k \lfloor N - n - 1} \cdot k$$

and the probability that $(k + 1)$ will be in the kth place is u_1/U.

By similar reasoning it readily follows that the respective probabilities that k, $(k + 1)$, $(k + 2)$, $\cdots$ $(k + i)$, $\cdots$, are in the kth place are given by the successive terms of the series,

$$\frac{1}{U}[u_0 + u_1 + u_2 + \cdots u_i + \cdots] = \frac{\lfloor N - n \lfloor n}{\lfloor N} \left[\frac{\lfloor N - k}{\lfloor n - k \lfloor N - n} + \frac{\lfloor N - k - 1}{\lfloor n - k \lfloor N - n - 1} \cdot k \right.$$
$$\left. + \frac{\lfloor N - k - 2}{\lfloor n - k \lfloor N - n - 2} \cdot \frac{(k + 1) \cdot k}{\lfloor 2} + \cdots \frac{\lfloor N - k - i}{\lfloor n - k \lfloor N - n - i} \cdot \frac{(k + i - 1) \cdots k}{\lfloor i} + \cdots \right]$$

and the number most probably in the kth place is given by adding to k the value of i corresponding to the largest term of this series.

In the above series the ratio of the term corresponding to $(i + 1)$ to the term corresponding to i is

$$m = \frac{N - n - i}{N - k - i} \cdot \frac{k + i}{i + 1}.$$

Setting $m \gtreqless 1$ we find by reduction of the inequalities that

$$i \lesseqgtr N \frac{(k - 1)}{n - 1} - k$$

and, therefore, the number most likely to be found in the kth place is given by the integral part of $(k + i + 1)$ or the integral part of $N \dfrac{k - 1}{n - 1} + 1$.

If $N \dfrac{k - 1}{n - 1}$ is an integer there will be two successive numbers $N \dfrac{k - 1}{n - 1}$ and $N \dfrac{k - 1}{n - 1} + 1$ each equally likely to be found in the kth place and more likely to be found there than any other numbers in the urn.

II. Mr. C. F. Gummer solved the problem in a similar manner and added:

It may be of interest to find the expectation. This will be

$$\sum_{r=k}^{r=N-n+k} \frac{\binom{r-1}{k-1}\binom{N-r}{n-k}}{\binom{N}{n}} r = \frac{k\,\lfloor n}{\lfloor N} f(N, n, k),$$

where r is the number on the kth ball, and

$$f(N, n, k) = \sum_{r=k}^{r=N-n+k} \binom{N-n}{r-k} \frac{\lfloor r}{\lfloor k} \frac{\lfloor N-r}{\lfloor n-k}.$$

From the relation

$$\binom{N-n}{r-k} = \binom{N-n-1}{r-k-1} + \binom{N-n-1}{r-k},$$

it follows that

$$f(N, n, k) = (n-k+1)f(N, n+1, k) + (k+1)f(N, n+1, k+1). \tag{1}$$

Now $f(N, N, k) = 1$, being independent of k. Therefore, by (1), $f(N, N-1, k) = N+1$, also independent of k, and finally $f(N, n, k) = (N+1)N\cdots(n+2)$. Hence, the expectation for the kth ball is $k(N+1)/(n+1)$.

2699 [May, 1918]. Proposed by the late ROGER E. MOORE, University of Wisconsin.

Show that if $a_k^{(r)}$ denotes the kth term of an arithmetic progression of order r, and c_k denotes the kth binomial coefficient in the expansion of $(a-b)^n$, n being a positive integer,

$$s \equiv \sum_{k=1}^{n+1} c_k a_k^{(r)} = 0, \qquad \text{if} \qquad n > r.$$

Solution by Elbert H. Clarke, Hiram College.

Let d_0 be the first term in the arithmetic progression and let $d_1, \cdots, d_r$ denote the initial difference of each order. Using the usual abbreviated notation for binomial coefficients, we write

$$a_k^{(r)} = \binom{k-1}{0} d_0 + \binom{k-1}{1} d_1 + \binom{k-1}{2} d_2 + \cdots + \binom{k-1}{r} d_r,$$

$$c_k = (-1)^{k-1}\binom{n}{k-1}$$

and

$$\sum_{k=1}^{n+1} c_k a_k^{(r)} = \sum_{i=0}^{r} d_i \sum_{k=1}^{n+1} (-1)^{k-1} \binom{k-1}{i}\binom{n}{k-1}.$$

Consider the inner sum. Since $\binom{k-1}{i} = 0$, for $k < i+1$,

$$\sum_{k=1}^{n+1} (-1)^{k-1} \binom{k-1}{i}\binom{n}{k-1} = \sum_{k=i+1}^{n+1} (-1)^{k-1} \binom{k-1}{i}\binom{n}{k-1},$$

and the latter expression easily becomes

$$\binom{n}{i} \sum_{k=i+1}^{n+1} (-1)^{k-1} \binom{n-i}{k-(i+1)}.$$

Now put $k - i - 1 = t$ and we have

$$\binom{n}{i} \sum_{t=0}^{n-i} (-1)^{t+i} \binom{n-i}{t}.$$

But the expression under summation is simply $(-1)^i(1-1)^{n-i}$. Hence, the coefficient of every d_i is zero. Therefore,

$$\sum_{k=1}^{n+1} c_k a_k^{(r)} = 0, \qquad n > r.$$

NOTE.—This proof is good only for $n > r$. Numerical examples can easily be given to show that the formula is not true for $n \leqq r$.

Also solved by HORACE L. OLSON.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Dr. J. N. VAN DER VRIES has resigned his position as professor of mathematics at the University of Kansas to continue his work as secretary of the central district of the Chamber of Commerce of the United States, with headquarters at Chicago.

Mr. G. H. CRESSE, previously instructor in the University of Michigan, has been appointed to an instructorship in mathematics at the U. S. Naval Academy at Annapolis; he was granted the degree of doctor of philosophy at the University of Chicago in December.

Mr. F. S. NOWLAN, of Bowdoin College, has been promoted to an assistant professorship of mathematics.

Dr. MARION B. WHITE, formerly of the Ypsilanti State Normal College, Michigan, is now professor of mathematics at Carleton College.

Dr. TOBIAS DANTZIG and Dr. G. A. PFEIFFER have been appointed instructors in mathematics at Columbia University.

Professor GRIFFITH C. EVANS is now scientific attaché to the American Embassy at Rome.

Dr. W. GROSS, of the University of Vienna, has been promoted to professor of mathematics.

Professor E. WIECHERT of Göttingen has been appointed professor of geodesy and geophysics at the University of Berlin.

Efforts are being made to establish a chair of mathematical physics at the University of Edinburgh in memory of the late Professor TAIT.

At the University of Strasbourg Professor RENÉ M. FRÉCHET, of the University of Poitiers, has been appointed professor of mathematics, and PIERRE WEISS, professor at the Polytechnikum, Zurich, professor of general physics.

CHARLES L. DOOLITTLE, professor emeritus of astronomy at the University of Pennsylvania, and director of the Flower Observatory, died on March 3, 1919, aged seventy-five years.

Mr. ROGER E. MOORE, son of Professor Ransom A. Moore of the Wisconsin College of Agriculture, and a contributor to the pages of this MONTHLY, died at Camp Taylor, Kentucky, in October, 1918.

FRANÇOIS DANIËLS of Nymwegen, Holland, professor of mathematics at the University of Fribourg, Switzerland, died on November 16, 1918, at the age of fifty-eight years. He was the author of *Essai de géométrie sphérique en coordonnées projectives* (Fribourg, 1907) and of articles in *L'Enseignement Mathématique.*

From the *Jahresbericht der Deutschen Mathematiker-Vereinigung* we learn of the deaths of the following mathematicians: Professor A. BENTELI, of the University of Bern, on November 10, 1917, in his seventieth year. Professor E. OTT, of the University of Bern, on November 17, 1917, in his seventieth year. Dr. ROBERT JENTZCH, of the University of Berlin, on March 21, 1918, fallen in battle. Professor M. B. WEINSTEIN, of Berlin, in his sixty-fifth year. Professor G. VERONESE, of the University of Padua, on July 17, 1917, in his sixty-third year.

Nature, for February 27, 1919, announces the death of ALEXANDER MICHAILOVITCH LIAPOUNOFF who held the chair of applied mathematics in the Petrograd Academy. "His later, and perhaps best-known, work dealt with the stability of the pear-shaped figure of a rotating mass of liquid, a problem of the first importance to theories of cosmogony. Poincaré had developed a method for the analytical discussion of the problem in 1901, but did not carry out the necessary calculations in detail, and so reached no definite conclusion. In 1902 Sir G. DARWIN announced that he had proved the pear-shaped figure to be stable, but this announcement was followed by a paper from Liapounoff in 1905, in which it was claimed that the pear-shaped figure was unstable. Liapounoff's work was distinguished by the combination of clear physical insight and masterly analytical skill." For somewhat more accurate statements in connection with the above see *Scientific Papers* by G. H. DARWIN, Volume 5, 1916, pp. xliii–xliv.

Professor H. E. BUCHANAN, of the University of Tennessee, has been employed in Y. M. C. A. work during the past year.

Professor JOSEPH ALLEN, of the College of the City of New York, and Professor W. H. METZLER, of the University of Syracuse, have gone to France on Army educational work.

Captain P. L. THORNE, assistant professor of mathematics at New York University, has recently returned to his university work. He served at the front in France with the Sixtieth Heavy Artillery regiment.

Captain A. L. UNDERHILL, of the University of Minnesota, has been appointed Commandant at the University of Grenoble in France, where several hundred American soldiers are taking courses while awaiting their opportunity to return home.

Professor EDWARD S. SMITH, of the department of mathematics at the University of Cincinnati, was, in addition to his regular duties, Acting Commandant of the Military Department from January to August, 1918, and Executive Secretary in charge of administrative matters in connection with the S. A. T. C. during the fall term.

The following reports of Summer Sessions, supplement those given in our last issue.

At *Oberlin College*, Professor W. D. CAIRNS will give courses in Freshman mathematics in the summer session, June 20 to August 7.

The Faculty of Applied Science of *Queen's University*, Kingston, will hold this year a special summer session for returned soldiers. The date of opening, about May 1, depends somewhat upon the time of demobilization, and the session will continue until the regular session begins in October. As a great many students enlisted early in the war, before they had finished their year, this session will serve either as a "refresher" course or will count as a year towards a degree.

University of Kansas, June 17–July 25, 1919. By Professor C. H. ASHTON: Differential Calculus (3 semester hrs. credit), and Advance analytic geometry (3 hrs.).—By Professor U. G. MITCHELL: Series (3 hrs.) and Teachers' course (3 hrs.). By Professor J. J. WHEELER: College algebra (2 hrs.) and Analytic geometry (4 hrs.). Second session, July 28–August 22. By Professor E. B. STOUFFER: Trigonometry (2 hrs.) and Theory of equations (2 hrs.).

University of Wisconsin: By Professor C. S. SLICHTER, Fourier series, 1 credit; Mechanics, 2 credits; Algebra, 2 credits. By Professor L. W. DOWLING: Equations of the third and higher degrees, 1 credit; Projective geometry, 2 credits; Analytic geometry, 2 credits. By Professor W. W. HART: The content of elementary mathematics, 1 credit; The teaching of secondary mathematics, 2 credits; Plane trigonometry and logarithms, 2 credits. By Professor H. C. WOLFF: Practical computation, 2 credits; Definite integrals, 1 credit; Calculus, 2 credits. By Dr. C. P. PAINE: Integral calculus, 4 credits. By Mr. R. W. BABCOCK: Elementary analysis, 4 credits; Solid geometry, 2 credits. A course in surveying will be given in the College of Engineering.

University of Chicago, June 16–August 29, 1919. By Professor G. A. BLISS: Differential equations (Lie theory), 4 hours; Differential calculus, 5 hours. By Professor W. D. MACMILLAN: Celestial mechanics, 4 hours. By Professor F. R. MOULTON: The solution of numerical differential equations, 4 hours. By Professor H. E. SLAUGHT: Elliptic integrals, 4 hours; Integral calculus, 5 hours. By Professor E. J. WILCZYNSKI: Metric differential geometry, 4 hours; College algebra, 5 hours. By Professor J. W. A. YOUNG: Solid analytic geometry, 4 hours; Plane trigonometry, 5 hours. By Professor A. B. COBLE (University of Illinois): Elliptic modular functions, 4 hours; Plane analytic geometry, 5 hours. By Professor T. H. HILDEBRANDT (University of Michigan): Theory of functions of a real variable, 4 hours; Limits and series, 5 hours. By Professor G. W. MYERS (School of Education): The teaching of secondary mathematics, 5 hours.

The American Mathematical Monthly

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The Mathematical Association of America

Is the Only Journal of Collegiate Grade in The Mathematical Field in this Country

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The Notes and News cover a wide range of interest and information both in this country and in foreign countries.

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There are other journals suited to the Secondary field, and there are still others of technical scientific character in the University field: but the MONTHLY is the only journal of Collegiate grade in America suited to the needs of the non-specialist in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual and institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, Ohio.

VOLUME XXVI MAY, 1919 NUMBER 5

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

HOW THE MAP-PROBLEM WAS SOLVED IN THE WAR.[1]

By KURT LAVES, University of Chicago.

Introductory Considerations which Led to the Selection of the Lambert Projection for the War Maps of Northern France.

For the pre-war maps of northern France the Bonne projection was used, with the Meridian of Paris as primary meridian. The Bonne projection (Rigobert Bonne, 1727–1795) is a modified conical projection. As an "equivalent" projection any area on the map is equal to the corresponding area on the Earth's surface. For all peaceful occupations this is an essential requirement, while for the purposes of war, in particular those of heavy artillery, the essential requirement is that the angles and distances radiating from *any point* of the territory mapped should with only inappreciable errors be obtainable from the data furnished by the map. The theory of the Bonne projection will show that the distortions in angle and distance increase with the longitude from the primary meridian. At the eastern frontier of northern France the angular errors[2] amount to 18′ and the linear errors to 1 : 379. Such errors are not negligible in artillery fire. The eventual subsequent transfer of the battlefront in an easterly direction for which the military command had to make early provision in their map-department would have necessitated the introduction of a second and perhaps of a third and fourth primary meridian under the Bonne projection, in order that these errors might be kept within specified limits. Two, and eventually three or even more, sets of maps which furnish *different* data for the territory covered by them in common are a source of confusion which may lead to disastrous results. This fact, coupled with the relatively large distortions in angle and distance at the edges of each set of maps mentioned above counted heavily against the retention of the Bonne projection. The modern fire-direction of a battery requires the determination of distance and azimuth from the rectangular coördinates of target and battery. The kilometer-grid printed on the map permits of a rapid solution of this problem. The meridians on the Bonne projection are sinusoidal curves and "convergence of meridians," *i. e.*, the angle between the meridian of a point P and that of the primary meridian, is not readily obtainable by calculation since it is a function of both latitude and longitude of P. The determination of "Bonne-North" (corresponding to Lambert-North, *i. e.*, direction of primary meridian at a point) would have proved a rather lengthy and knotty process for the average battery officer even with the help of auxiliary tables.

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[1] Read at the joint meeting of the Mathematical Association of America and of the American Mathematical Society at Chicago, December 27, 1918.

[2] Walton C. Clark, *Heavy Artillery Orientation*, p. 33; published by Coast Artillery School, Ft. Monroe, Va. $0.75.

The objections mentioned against the Bonne projection do not apply to the Lambert *conform* conical map projection for the following reasons:

1. Being conform the angles on the map are equal to those on the earth; the distortions of distances are zero along the two parallels along which the map-cone cuts the earth's surface. Between these parallels the distances on the map are smaller than those on the earth's surface, beyond them greater, the maximum linear distortion being 1:2037 for the maps of northern France which cover a belt of $5^\circ.4$ in latitude.

2. Since the distortion is not a function of the longitude from the primary meridian, the Lambert map projection allows of an unlimited extension east or west of the primary meridian. The map material is homogeneous throughout.

3. The convergence of meridians is at once obtainable from the map without the objectionable computation of the angle ψ, necessary under the Bonne projection. The longitudes on the map are μ times the corresponding longitudes on the earth where μ is a constant factor smaller than 1.

4. The principal meridian has not a determining influence on the construction of the map; it may be selected in such a way as to make the construction of the kilometer grid as convenient as circumstances will permit. The meridian of Treves served this purpose for the War maps of northern France, the tangent to the meridian at Treves being parallel to an element of the map-cone.

The Bonne Projection.

For the mapping of a country which has a considerable extent in longitude but only a moderate extent in latitude the conical projections offer the greatest advantages. The surface of a cone which touches the earth's surface along a middle parallel and whose vertex is in the prolongation of the earth's axis, will, when developed, furnish a map in which the parallel-circles of the earth are represented by concentric circles, the meridians by straight lines radiating from the vertex of the cone (center of parallel circles). While meridians and parallel-circles on the map cut each other under right angles as on the earth's surface, it is evident that the parallel circles with the exception of the middle parallel will be longer than those on the earth. The Bonne projection avoids this difficulty by sacrificing the perpendicularity of meridians and parallels except on the first meridian. On the primary meridian the degrees of latitude south and north of the middle parallel circle are measured in their true lengths. Likewise on the concentric parallel circles accurate degrees of longitude are laid down east and west of the primary meridian. The end points of equal degrees to either side are joined by smooth curves. By this device Bonne secures equivalency between areas on the earth and on the map, which for all peace purposes is obviously the feature that a map should offer. If R and L are the polar coördinates of a point on the map with the vertex of the cone for the pole and the primary meridian for axis of reference, then $R \cdot dR \cdot dL$ will be the element of area on the map. Since now $RdL = r_1 d\lambda$ and $dR = -\rho \cdot d\phi$ we have $R \cdot dR \cdot dL = -r_1 d\lambda \rho d\phi$ the corresponding area of the earth, where λ stands for longitude, ϕ for latitude, ρ for the radius of

curvature of the meridian ellipse and r_1 for the radius of the parallel circle and the earth is assumed to be an ellipsoid of revolution. This shows that the Bonne projection is one of equivalence.

From the analytics of the meridian ellipse it follows that,

$$\rho = \frac{a(1 - e^2)}{\sqrt{(1 - e^2 \sin^2 \phi)^3}}, \qquad r_1 = \frac{a \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}},$$

where $2a$ is the major axis, and e the numerical eccentricity of the meridian ellipse.

The element of area will therefore be,

$$\frac{a^2(1 - e^2) \sin \chi \, d\chi \cdot d\lambda}{(1 - e^2 \cos^2 \chi)^2},$$

where $\chi = 90° - \phi$ is introduced in order that R and χ may both be increasing functions.

In order to compute the radius R of a parallel circle of the map we start from R_m, the radius of the middle parallel,

$$R_m = r_{1m} \operatorname{cosec} \phi_m = \frac{a \operatorname{ctg} \phi_m}{\sqrt{1 - e^2 \sin^2 \phi_m}}$$

and obtain

$$R = R_m - \int_{\phi_m}^{\phi} \rho d\phi = R_m + \int_{\chi_m}^{\chi} \frac{a(1 - e^2)}{\sqrt{(1 - e^2 \cos^2 \chi)^3}} d\chi.$$

The polar angle L is derived from $RdL = r_1 d\lambda$ or

$$dL = \frac{a \cos \phi \cdot d\lambda}{R \sqrt{1 - e^2 \sin^2 \phi}};$$

since R is only a function of ϕ and not of λ, we conclude,

$$L = \frac{a \cos \phi \cdot \lambda}{R \sqrt{1 - e^2 \sin^2 \phi}}, \tag{1}$$

if we count the longitudes and angles L from the primary meridian. Knowing R and L for given values of ϕ and λ, the map is easily constructed. We now turn our attention to the distortions of the map in angle and distance. We shall see that these distortions are zero for the principal meridian and the middle parallel and grow rather rapidly as the point to be represented increases in longitude and approaches the northern or southern edge of the map. To show this we consider first the law of divergence ψ of the direction of radius R of the map from the particular meridian at a point (Fig. 1). It should be remembered that L is not the longitude of the point P on the map since VP is not a meridian in the Bonne projection. In order to obtain the "convergence of meridians" with respect to the primary meridian it is necessary to find the angle ψ which the

tangent to the meridian at P makes with R:

$$\operatorname{tg} \psi = \frac{R \dfrac{\partial L}{\partial \phi} \cdot d\phi}{dR};$$

since

$$R \cdot L = \frac{a \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}} \cdot \lambda,$$

we obtain

$$R \frac{\partial L}{\partial \phi} \cdot d\phi + LdR \doteq - \frac{a(1 - e^2)\lambda \sin \phi d\phi}{\sqrt{(1 - e^2 \sin^2 \phi)^3}} = - \rho d\phi \cdot \lambda \sin \phi = dR \cdot \lambda \sin \phi.$$

The equation for $\operatorname{tg} \psi$ becomes, therefore,

$$\operatorname{tg} \psi = \lambda \sin \phi - L = \lambda \left(\sin \phi - \frac{a \cos \phi}{R\sqrt{1 - e^2 \sin^2 \phi}} \right). \tag{2}$$

ψ will be zero for two cases, (1) $\lambda = 0$, *i. e.*, on the primary meridian, (2) on the middle parallel, since

$$\sin \phi_m = \frac{a \cos \phi_m}{R_m \sqrt{1 - e^2 \sin^2 \phi_m}}.$$

The angle ψ for points east of the prime meridian will be in the NW. quadrant

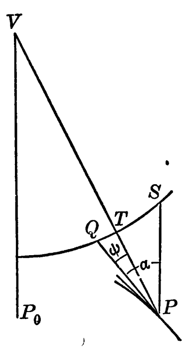


FIG. 1.

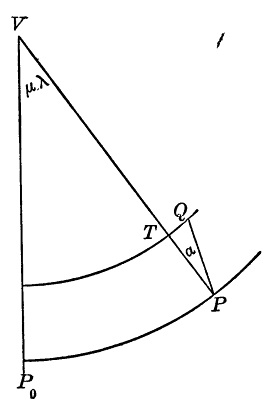


FIG. 2.

(with respect to VP) for points north of the middle parallel, and in the NE. quadrant south of this parallel.[1] If α and a are corresponding azimuths on the

[1] The convergence of meridians with respect to the primary meridian is the angle which the tangent to the meridian at P forms with the primary meridian. This convergence M is measured by $M = L + \psi$ or $= L +$ arc $\operatorname{tg}(\lambda \sin \varphi - L)$, or $\operatorname{tg}(M - L) = \lambda \sin \varphi - L$; for small values of ψ we have $M = \lambda \sin \varphi \cdots$. (In Major W. C. Clark's pamphlet *Heavy Artillery* (*Coast*) *Orientation*, p. 38, it is overlooked that M is a function of ψ.)

earth and the map, then $\alpha - a$ will be *the distortion in angle,* and we have

$$\operatorname{tg}(a-\psi)=\frac{QS-QT}{PT} \qquad \text{or} \qquad \operatorname{tg}(a-\psi)=\operatorname{tg}\alpha-\operatorname{tg}\psi,$$

or

$$\operatorname{tg}\alpha=\frac{\sin a}{\cos(a-\psi)\cos\psi}.$$

For the middle parallel ($\psi = 0$) we obtain $\alpha = a$.

The distortion $\alpha - a$ will be found by

$$\operatorname{tg}(\alpha-a)=\frac{\operatorname{tg}a[\cos a-\cos\psi\cos(a-\psi)]}{\cos(a-\psi)\cos\psi+\sin a\operatorname{tg}a}. \tag{3}$$

For $\psi = 0$ we have $\alpha = a$ as it should be. Likewise $\alpha = a$ for $tga = 0$ i. e., for the meridian.

To find the *distortion in distance* δ we have on the ellipsoid

$$\Delta\delta=\frac{a(1-e^2)\Delta\phi}{\sqrt{(1-e^2\sin^2\phi)^3}}\cdot\frac{1}{\cos\alpha},$$

and on the map

$$\Delta d=\frac{a(1-e^2)\Delta\phi}{\sqrt{1-e^2\sin^2\phi}}\cdot\frac{1}{\cos(a-\psi)}. \tag{4}$$

Hence

$$\frac{\Delta\delta-\Delta d}{\Delta d}=\frac{\cos(a-\psi)-\cos\alpha}{\cos\alpha},$$

the distortion $\Delta\delta - \Delta d$ is then likewise a function of ψ and therefore of λ and ϕ, and it increases considerably with the longitude.

The Lambert Conical Conform Projection.[1]

While in the Bonne projection the map-cone is tangential to the earth's surface at the middle parallel of the zone so that the radius of the middle map-parallel is a fundamental constant, the Lambert projection retains the properties of the cone without fixing its position from the start. Since meridians and parallel circles on the earth are orthogonal to each other a conical map projection which is to be *conform* will have to retain this orthogonal system of coördinates. Calling α the initial azimuth of a course on the surface of the earth and a the corresponding angle on the map, then for conform representation $\alpha = a$. Since $\cot\alpha = -\rho d\phi/r_1 d\lambda$, $-\rho d\phi$ being an element of the meridian arc, and $r_1 d\lambda$ that of the parallel circle, and since furthermore $\cot a = dR/RdL$, therefore $dR/RdL = \rho d\phi/r_1 d\lambda$, Fig. 2, L being here the "convergence of meridians."

We may simplify this differential equation considerably by making R only a

[1] Attention is drawn to Special Publications nos. 52 and 53 (price 25 and 10 cents respectively), of the U. S. Coast and Geodetic Survey, by Oscar S. Adams. They were issued by the Government Printing Office, Washington, in 1918, and have the following titles: *Lambert Projection Tables for the United States,* and *General Theory of the Lambert Conformal Conic Projection*—Editor.

function of ϕ and by *assuming* $dL = \mu d\lambda$, where μ is an arbitrary constant, to be determined later by special conditions.

Then we get

$$\frac{dR}{R} = \frac{-\mu\rho d\phi}{r_1} = \frac{-\mu(1-e^2)d\phi}{\cos\phi\,(1-e^2\sin^2\phi)}.$$

Integrating,

$$lg(R) = \mu lg\left[\operatorname{tg}\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{e/2}K\right],$$

K being the constant of integration. This gives

$$R = C\cdot\operatorname{tg}^{\mu}\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{\mu e/2}, \tag{5}$$

where $C = K^{\mu}$. If we put

$$\operatorname{tg}\frac{\zeta}{2} = \operatorname{tg}\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)^{e/2},$$

ζ will be determined to a sufficient degree of approximation (within a few tenths of a second) on setting

$$\operatorname{tg}\frac{\zeta}{2} = \operatorname{tg}\left(\frac{\pi}{4}-\frac{\phi}{2}\right).$$

Then R takes the final form

$$R = C\operatorname{tg}^{\mu}\zeta/2. \tag{6}$$

To determine the arbitrary constants μ and C, we assume that the map-cone cuts the earth's surface along the parallel circles whose latitudes are ϕ' and ϕ''. These are generally selected close to the northern and southern edges of the territory to be mapped. For the war maps it was desired to have the parallels of 52 grades (Central Switzerland) and 58 grades (Northern Holland) as extreme parallels. By taking $\phi' = 53$ grades and $\phi'' = 57$ grades the determination of the constants was effected for the war maps. To obtain μ, Lambert postulated that

$$\frac{R'}{R''} = \frac{r_1'}{r_1''};$$

this gives

$$\left(\frac{\operatorname{tg}\zeta'/2}{\operatorname{tg}\zeta''/2}\right)^{\mu} = \frac{\cos\phi'\sqrt{1-e^2\sin^2\phi''}}{\cos\phi''\sqrt{1-e^2\sin^2\phi'}},$$

or taking logarithms

$$\mu = \frac{lg\dfrac{\cos\phi'}{\cos\phi''}\,\dfrac{\sqrt{1-e^2\sin^2\phi''}}{\sqrt{1-e^2\sin^2\phi'}}}{lg\left(\dfrac{\operatorname{tg}\zeta'/2}{\operatorname{tg}\zeta''/2}\right)}. \tag{7}$$

In order to find C the condition is imposed that the linear distortion $k-1$ for the

Omit lines 3-6 after formula (5).

parallel of latitude ϕ' be zero; it is observed that, owing to the way μ has been found, the distortion on the parallel of ϕ'' will be zero likewise. Since

$$k = \frac{RdL}{r_1 d\lambda} = \frac{\mu R d\lambda}{r_1 d\lambda} = \mu \cdot \frac{R}{r_1},$$

we obtain

$$C = \frac{a \cos \phi'}{\mu \operatorname{tg}^{\mu} \zeta' \sqrt{1 - e^2 \sin^2 \phi'}}.$$

Since the cone is inside the earth's surface between the parallels ϕ' and ϕ'' and outside for the border area of the map, it follows that distances on the map will be smaller than corresponding distances on the earth's surface for the zone between ϕ' and ϕ'' and larger in the border-area. Computation will show that the maximum distortion amounts to 1 : 2037 or 5 : 10000; this, even for ranges of 10 km., will be a negligible quantity.

While geodetic lines on the surface of the earth are not represented by straight lines on the map but appear as slightly curved traces, the deviation for a distance of 100 km. will introduce an angular error in azimuth of less than one minute of arc.

In concluding this paper the reader's attention is once more called to Major Clark's able pamphlet which should prove a most interesting guide for students of surveying and cartography. Gretchel's *Lehrbuch der Karten-Projectionen* has been used in preparing this paper.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

32. In a discussion of the Peaucellier[1] cell by analytic methods the following equations are obtained:

$$(1)\quad (x_2 - x_1)^2 + (y_2 - y_1)^2 - b^2 = 0; \qquad (2)\quad (x_3 - x_1)^2 + (y_3 - y_1)^2 - b^2 = 0;$$

$$(3)\quad (x_2 - X)^2 + (y_2 - Y)^2 - b^2 = 0; \qquad (4)\quad (x_3 - X)^2 + (y_3 - Y)^2 - b^2 = 0;$$

$$(5)\quad x_2^2 + y_2^2 - K^2 = 0; \qquad (6)\quad x_3^2 + y_3^2 - K^2 = 0;$$

$$(7)\quad x_1^2 + y_1^2 - 2cx_1 = 0.$$

The result of eliminating $x_1, y_1, x_2, y_2, x_3, y_3$ gives an equation of the first degree, which establishes that the linkage will trace a straight line. There are various ways of effecting this elimination.

1. What element of the situation is left unused by the following procedure in the elimination?
   (*a*) From equations (1), (3), (5) eliminate x_2 and y_2 and obtain an equation

$$(8)\quad f_1(x_1, y_1) = 0.$$

---

[1] In the accompanying figure, taken from the article on "Linkages" in the December, 1915, MONTHLY, by Mr. Leavens, the coördinates of the points of the linkage are: O (0, 0); C (c, 0); P_1 (x_1, y_1); M (x_2, y_2); M' (x_3, y_3); P_2 (X, Y).

(b) From equations (2), (4), (6) eliminate x_3 and y_3 and obtain an equation

$$(9) \quad f_2(x_1, y_1) = 0.$$

(c) From equations (7), (8), (9) eliminate x_1 and y_1 and obtain the desired equation.

2. How should this procedure be supplemented to secure the result?

REPLY BY J. K. WHITTEMORE, Yale University.

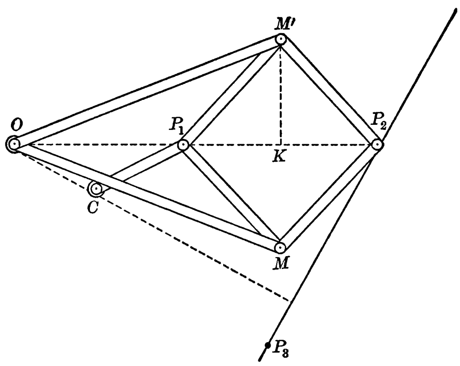
The method of elimination proposed might be expected to fail, since (8) and (9) are apparently identical, for equations (2), (4), (6) differ from (1), (3), (5) only in containing x_3, y_3 in place of x_2, y_2. But the functions f_1 and f_2 of (8) and (9) do not, as a matter of fact, exist at all. If we seek to determine f_1 we may write from (1) and (5)

$$2x_1x_2 + 2y_1y_2 = K^2 - b^2 + x_1^2 + y_1^2,$$

and from (3) and (5)

$$2Xx_2 + 2Yy_2 = K^2 - b^2 + X^2 + Y^2.$$

These equations may be formally solved for x_2, y_2 and the results substituted in (5), apparently giving f_1; but this work is illusory. Equations (2), (4), (6) give identically the same expressions for x_3, y_3, so that, unless (x_2, y_2) coincides with (x_3, y_3), the two equations above are equivalent and cannot be solved, and



$$(A) \quad \frac{x_1}{X} = \frac{y_1}{Y} = \frac{K^2 - b^2 + x_1^2 + y_1^2}{K^2 - b^2 + X^2 + Y^2}.$$

Since (x_2, y_2) and (x_3, y_3) do *not* coincide equations (A) are true. It is indeed obvious from the figure that $Yx_1 - Xy_1 = 0$. The equation of the locus of (X, Y) is obtained by eliminating x_1, y_1 from (A) and (7). The elimination is simply carried out as follows: Let the common value of the fractions in (A) be $\lambda \neq 1$. Then (7) gives

$$\lambda(X^2 + Y^2) - 2cX = 0.$$

Equating to λ the last member of (A),

$$(\lambda^2 - \lambda)(X^2 + Y^2) - (\lambda - 1)(K^2 - b^2) = 0.$$

Since $\lambda - 1 \neq 0$, this factor is cancelled from the last equation; then subtracting the preceding equation,

$$2cX = K^2 - b^2.$$

NOTE. This reply points out the necessity of using the fact that the points (x_2, y_2) and (x_3, y_3) are distinct and that the points (x_1, y_1) and (X, Y) are distinct, the latter condition being of course implied by $\lambda \neq 1$. It is likewise clear from the last part of the proof that neither (x_1, y_1) nor (X, Y) may coincide with $(0, 0)$; this may be insured by requiring that $K \neq b$. There are therefore several elements of the situation left unused by the process suggested in the question. —EDITOR.

DISCUSSIONS.

In a recent number of the MONTHLY, Professor E. J. Moulton considered the nature of a course in advanced calculus which should serve as a bridge between the work in elementary calculus and more advanced subjects. In the first discussion below, Professor Gillespie presents the claims of elementary differential equations for adequately effecting this transition. The content which he proposes is by no means that of the course in elementary differential equations as it is usually given; neither does it coincide with the usual course in advanced calculus. Further expressions of opinion on the questions raised will be welcome.

Among the topics presented in courses common to the secondary and collegiate fields, probably no other is so difficult, and at times so discouraging, to teach satisfactorily, as mathematical induction. Professor Weaver, in the second discussion of this number, gives an amusing account of an analogy from life, which may be useful in illustrating the essential nature of the method of proof involved.

In the third discussion Professor Noble considers the process of altering the strength of a mixture by continuous addition of quantities of one component. Mathematically the situation turns out to involve a neat illustration of the notion of a derivative, leading to a simple differential equation; some of our readers will find here a useful supplementary problem for the class-room.

I. Advanced Calculus or Differential Equations.

By D. C. Gillespie, Cornell University.

The second course in calculus as it is often given,—following Goursat-Hedrick, *Mathematical Analysis*, Vol. I, let us say to be explicit,—breaks suddenly with what has gone before. It does not appeal to everyone as a natural continuation of previous work. The point of view is different, the emphasis is different, and even among those who have done well in the more elementary courses there are students with whom it is not successful. Professor E. J. Moulton's suggestive paper in the December number of the MONTHLY proposes a plan which would accomplish much that it is desirable to accomplish by the second course in the calculus and at the same time make the change less abrupt.

Even in the case of the future mathematicians, Professor Moulton feels, "It is too early in the student's career to introduce the critical attitude of higher mathematics." I am not sure that this phrase is quite clear. The student is perhaps fortunate to escape altogether the attitude that is more concerned with detecting error than with acquiring knowledge. On the other hand the desire and ability to analyze and see on what assumptions theorems are founded and how the results follow from the assumptions are, at every stage of the student's development, great assets. The point is, it seems to me, that while the teacher may encourage this desire he can not introduce this ability. The scales are not going to fall from the student's eyes at the teacher's command; they wear off, and indeed slowly. The plan proposed has rather this clear advantage, that,

instead of halting the progress of the student until he has mastered the fundamental concepts of the calculus, it offers him a course useful and interesting in itself where the ideas of the calculus are continually applied.

Somewhat the same purpose may be accomplished, I think, by an elementary course in differential equations. At most collegiate institutions such a course is given each year. This course may quite naturally cover a good deal of the outline proposed for the second course in calculus.

It is true in my experience with both text books and lectures that courses in differential equations divide themselves rather sharply into two classes: the first is purely formal, where the time is spent in the monotonous and, if long continued, rather deadening work of learning schemes of integration; the second consists in courses devoted to the study of existence theorems and properties of solutions. One must go to books on differential geometry or mechanics or mathematical physics to find the calculus used to state problems and the geometrical and physical notions involved further employed to aid in the solution and in the interpretation of results. It would however be necessary for the purpose at hand to include these applications of calculus under the name elementary differential equations.

The course in differential equations would include methods of integration. It should perhaps include the power series existence theorem for ordinary differential equations of the first order. There could be added many problems from geometry, mechanics and physics which are not stated as differential equations but which the student himself must formulate, solve and interpret. Such a course is a continual review of the ideas of function, continuity, derivative, integral. Families of curves, their trajectories, envelopes and singularities would be included. The plane pendulum problem gives a natural introduction to elliptic integrals and functions. The brachistochrone or some similar problem takes one at least to the Euler equation in the calculus of variations. The existence theorem mentioned or an approximation method leads to some study of Taylor's series and infinite series generally.

The student should acquire in a course of this nature considerable knowledge and technique, as well as a better understanding of the calculus. This would allow more time for the essentially slow process of the accurate statements and proofs of advanced calculus when it follows.

II. Note on the Teaching of the Principle of Mathematical Induction.

By Warren Weaver, University of Wisconsin.

The writer well remembers when he first met a proof by mathematical induction. He felt as it he had been introduced to a scientific three-shell artist. While he could find no definite loop-hole in the argument it seemed too mysterious and unreal a process to have actually effected any definite proof. The attitude of classes meeting this principle for the first time seems to justify the conclusion that this is no unusual circumstance. The fact that some ridiculously simple

illustration from one's everyday experience often wonderfully clears up hazy notions will perhaps justify the following allegory, which has been successful in bringing forth that first, wide-eyed, understanding "oh!" in several cases.

A long—indeed apparently endlessly long—line of people is seen to be standing waiting to buy tickets at a window. One wonders if some far-famed magician is to give an exhibition of extracting square roots from apparently empty silk hats. Our hero, Mr. Kueedee, being the proprietor of a rival attraction, wishes to persuade this line of people to stop waiting, and come to his show. He looks at the people, and seeing a friend, Mr. Kayplusone, he goes over to him.

"Kayplusone," he says earnestly, "I feel that you are a typical average sort of a chap. I come to you to see if I can learn the attitude which all these people take towards the proposition of coming at once to my show."

"I have indeed been thinking that matter over," replies Kayplusone, "and I will agree to come if my friend Kay ahead of me here in line will also go. And I can save you a lot of questions by telling you that mine is a typical attitude. Kay says that he, in turn, will go if the man ahead of him will."

Kueedee ponders for but a moment, and then with sudden inspiration goes to the end of the line, and is seen to be in serious conversation with the man next to the ticket window. Suddenly this man, whose name happens to be One, grabs Kueedee by the hand and calls out, "I'll do it." He speaks hurriedly to the man behind him, and starts off. This man, likewise, speaks to the man behind him, and starts off also. So that Mr. Kueedee goes, calm in the confidence that even the infinite capacity of his house is going to be taxed, and himself puts out the sign,

STANDING ROOM ONLY.

Q. E. D.

III. The Curve of Concentration for a Liquid Mixture.

By C. A. Noble, University of California.

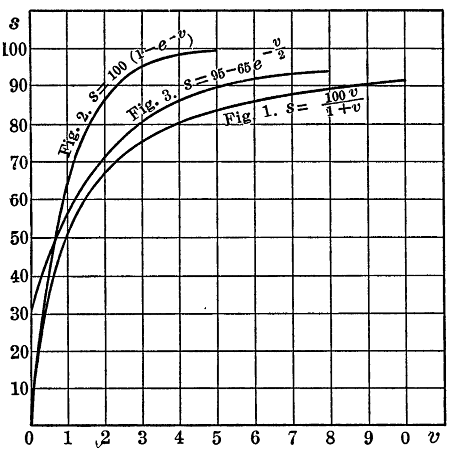
When strong alcohol is added to weak alcohol the concentration of the mixture may be representated by the arc of a hyperbola. If, for example, to a volume v_0, of percentage-strength s_0, is added a volume v, of percentage-strength s_1, the strength of the mixture is given by

$$s = \frac{s_0 v_0 + s_1 v}{v_0 + v}.$$

Figure 1 shows the curve of concentration for the case where pure alcohol is added to a unit volume of pure water. Then $s_0 = 0$, $s_1 = 100$, $v_0 = 1$, and the equation of the curve is

$$s = \frac{100v}{1 + v}.$$

One can obtain more rapidly a desired volume of required concentration if one keeps the volume of the mixture constant, by drawing off from the stirred mixture a volume equal to that of the strong alcohol which has been added. The smaller the steps taken, in this process of adding a small quantity, stirring the mixture, and then pouring off an equal quantity, the more rapidly does the mixture increase in strength. The following discussion is for the case where the process goes on by infinitesimal steps, *i. e.*, continuously.



With the notation mentioned above, let the volume $\Delta v = v/n$ be added to v_0, the mixture stirred, and then the volume Δv of the mixture poured off. The strength of the remaining mixture (volume v_0) is

$$\frac{s_0 v_0 + s_1 \Delta v}{v_0 + \Delta v}.$$

Abbreviate this to k/v_1. Again add Δv of the strong alcohol, stir the mixture, and pour from it Δv. The strength of the mixture is now

$$\frac{s_1 v_1 \Delta v + k v_0}{v_1^2}.$$

After another repetition of this process, the mixture has the strength

$$\frac{s_1 \Delta v (v_1^2 + v_0 v_1) + k v_0^2}{v_1^3}.$$

After $n - 1$ such repetitions, the mixture has the strength

$$s_n = \frac{s_1 \Delta v (v_1^{n-1} + v_0 v_1^{n-2} + \cdots + v_0^{n-2} v_1) + k v_0^{n-1}}{v_1^n}$$

$$= \frac{s_1 \Delta v}{v_1} \cdot \frac{1 - \left(\frac{v_0}{v_1}\right)^{n-1}}{1 - \left(\frac{v_0}{v_1}\right)} + \frac{k}{v_1}\left(\frac{v_0}{v_1}\right)^{n-1}.$$

If the process is to be continuous, n must become infinite, and the limit s of s_n will be the ultimate concentration. Now,

$$\lim_{n=\infty} \left(\frac{v_0}{v_1}\right)^{n-1} = \lim_{n=\infty} \left[\frac{1}{1 + \frac{v/v_0}{n}}\right]^{n-1} = e^{-v/v_0},$$

and

$$\lim_{n=\infty} \frac{1 - \left(\frac{v_0}{v_1}\right)^{n-1}}{1 - \frac{v_0}{v_1}} \cdot \Delta v = \frac{v(1 - e^{-v/v_0})}{\lim_{n=\infty} n \left[1 - \frac{v_0}{v_0 + \frac{v}{n}}\right]} = v_0(1 - e^{-v/v_0}).$$

Moreover,

$$\lim_{n=\infty} \frac{s_1}{v_1} = \frac{s_1}{v_0},$$

and

$$\lim_{n=\infty} \frac{k}{v_1} = s_0.$$

Inserting these values in the above expression for s_n, one has

$$s = \lim_{n=\infty} s_n = s_1 - (s_1 - s_0)e^{-v/v_0}.$$

This curve has a steeper slope than the hyperbola mentioned above, that is, this method of continuous dilution is the more economical of the two. It is actually employed in biological work at the University of California by Professor Long, who has devised appropriate apparatus, and at whose suggestion the equation of the curve which should represent the process was deduced.

Figure 2 is the curve which represents the pouring of pure alcohol into one unit volume of pure water. $s_1 = 100$, $s_0 = 0$, $v_0 = 1$, and the equation is

$$s = 100(1 - e^{-v}).$$

Figure 3 represents the pouring of 95 per cent. alcohol into two unit volumes of 30 per cent. alcohol. $s_1 = 95$, $s_0 = 30$, $v_0 = 2$, and the equation is

$$s = 95 - 65e^{-v/2}.$$

The differential equation corresponding to this process is simple, and may be deduced as follows:

Let s be the strength of the mixture, the volume of which is maintained at v_0 in the way above indicated. When the small volume Δv of strength s_1 is added, the strength of the mixture changes to

$$\frac{sv_0 + s_1\Delta v}{v_0 + \Delta v}.$$

The increment which this brings to s is

$$\Delta s = \frac{sv_0 + s_1\Delta v}{v_0 + \Delta v} - s = \frac{(s_1 - s)\Delta v}{v_0 + \Delta v}.$$

The difference-quotient becomes

$$\frac{\Delta s}{\Delta v} = \frac{s_1 - s}{v_0 + \Delta v}.$$

Hence, the rate of change of the strength of the mixture with respect to the volume of liquid added, is given by

$$\frac{ds}{dv} = \frac{s_1 - s}{v_0} \qquad \text{or} \qquad \frac{ds}{s_1 - s} = \frac{dv}{v_0}.$$

The value of v, the volume necessary to raise the strength from s_0 to s, is given by

$$\frac{v}{v_0} = \int_{s_0}^{s} \frac{ds}{s_1 - s} = -\log \frac{s_1 - s}{s_1 - s_0},$$

i. e., by

$$s = s_1 - (s_1 - s_0)e^{-v/v_0}$$

as before.

February, 1919.

RECENT PUBLICATIONS.

REVIEWS.

VEBLEN AND YOUNG'S PROJECTIVE GEOMETRY.

Projective Geometry. By O. VEBLEN and J. W. YOUNG. Boston: Ginn. 8vo. Vol. 1, 1910, reprinted in 1916. 10 + 344 pages; price $4.00. Vol. 2, 1918. 12 + 511 pages; price, $5.00.

To dispel at the outset any misunderstanding concerning the following comment and criticism of the reviewer it must be stated that, in his opinion, Veblen and Young's Projective Geometry is a very scholarly and profound treatise on the axiomatic foundations of projective geometry, its classification, and the main body of projective propositions, or theorems. As such it undoubtedly reflects credit upon American scientific scholarship.

All through the two volumes one is constantly impressed with the fact that the axiomatic foundations are extremely important, and that the chain of principal theorems is merely a byproduct of axiomatic, and not the main object of, scientific research in general. This impression may of course be due to the fact that there are two irreconcilable classes of scientific minds: idealists and realists, just as in philosophy.

For this reason it is as a rule useless to argue a case belonging to the philosophy, or psychology, of this domain between representatives of the two different classes. The idealist maintains that the mind is free to create anything it pleases, without reference to any sense-perception whatsoever; mathematics in its foundation may be based upon pure logic alone. It is the purely logical process which, for him, is the essential thing. The realist, on the other hand, holds fast to the famous doctrine: "Nihil est in intellectu quod non prius fuerit in sensu." According to him our entire knowledge, fundamentally, may be traced back to the impressions made upon our mind by the influx through the senses. The choice

of a set of axioms for a reasonable, not only theoretically possible, scientific doctrine, must be dictated by the desire to create a system whose possible interpretations and applications do not collide in essential parts with our empirical knowledge of the external world. For him the logical structure is erected for the purpose of housing mathematical content conveniently and elegantly. That the foundations of such a system cannot be established without intuitional direction, at least in parts, seems clear to the realist. Poincaré,[1] for example, maintains that some of Hilbert's axioms of order are not free creations of the mind, on the contrary that they bear the mark of intuitional truth, that they are intuitional propositions. According to Poincaré, who, as is well known, has meditated profoundly upon analysis situs, the axiomatic foundations of this important branch of mathematical science cannot be laid entirely without some direction by intuition.

The idealist and realist agree, of course, that after the establishment of a set of axioms the erection of the scientific system, or systems, for which the set is valid, must proceed rigorously according to the demands of formal logic.

But here is another difficulty. How are we going to prove that a system is consistent without the aid of some previously known consistent system? Or are we supposed to go on building the structure and to say that it is consistent as long as it does not collapse; *i. e.*, as long as no contradictions turn up? In this connection see the foot-note on page 9 of volume 1.

After this short digression it would seem apparent that the set of assumptions A, E, K for a real projective space, for example, in spite of the author's assertion, is not "very arbitrary," but is chosen with the purpose of creating a real projective geometry, as it is customarily understood.

Volume I is the product of the joint labor of Veblen and Young, while for Volume 2 Veblen alone holds himself responsible. According to some statements in the preface the authors realize that from the standpoint of pedagogy the contents might possibly be treated and may be arranged differently. "The second volume has been arranged so that one may pass on a first reading from the end of chapter VI, volume 1, to the beginning of volume 2. The later chapters of volume 1 may well be read in connection with the part of volume 2 from chapter V onward."

On page 95, volume 1, the following *provisional assumption of projectivity* is made:

P. *If a projectivity leaves each of three distinct points of a line invariant, it leaves every point of the line invariant.* This is, of course, von Staudt's theorem, to which more explicit reference should be made. Based upon this the fundamental theorem of projective geometry, printed on the same page, then follows easily: *If* 1, 2, 3, 4 *are any four elements of a one-dimensional primitive form, and* 1′, 2′, 3′ *are any of three elements of another or the same one-dimensional primi-*

[1] *Dernières Pensées*, 1917, pp. 92–97. See in this connection also recent discussions in the *Monist* (January, 1919) by Richard A. Arms on "the relation of logic to mathematics" and by V. F. Lenzen on "independence proofs and the theory of implication."

tive form, then for any projectivities giving $1234 \barwedge 1'2'3'4'$ and $1234 \barwedge 1'2'3'4_1'$, *we have* $4' = 4_1'$.

The foregoing provisional assumption contains the key to the whole situation, and its rigorous proof from a set of axioms of connection, order, and continuity has required the successive efforts of von Staudt, Klein, Lüroth, Zeuthen, Darboux, and others.

The discussion of the foundations is again taken up in chapter I of volume 2, with the object of obtaining a classification of various projective geometries. In this the symbols denoting the various assumptions have the following meaning:

A: Alignment,
E: Extension,
P: Projectivity,
K: A geometric number system is isomorphic with the real number system of analysis,
H: If any harmonic sequence exists, not every one contains only a finite number of points,
H_0: The diagonal points of a complete quadrangle are non-collinear.
J: A geometric number system is isomorphic with the complex number system of analysis,
C: Continuity,
R: On at least one line, if there is one, there is not more than one chain.
$\bar{R}$: On some line, l, not all points belong to the same chain.
I: Through a point P of any chain C of the line l, and any point J on l but not in C, there is not more than one chain of l which has no other point than P in common with C,
$\bar{H}$: If any harmonic sequence exists, at least one contains only a finite number of points.
Q: There is more than one net of rationality on a line.
S: Senses of ordered projective spaces. A space satisfying assumptions.
A, E is a general projective space,
A, E, P is a proper projective space,
A, E, H is a non-modular projective space,
A, E, $\bar{H}$ is a modular projective space,
A, E, S is an ordered projective space,
A, E, $\bar{H}$, Q is a rational modular projective space,
A, E, H, Q is a rational non-modular projective space,
A, E, H, C, R or A, E, K is a real projective space,
A, E, H, C, $\bar{R}$, I or A, E, J is a complex projective space.

A set of assumptions is said to be categorical, if there is essentially only one system for which the assumptions are valid, i. e., if any two such systems may be made simply isomorphic, page 6, volume 1. Thus A, E, H, C, R form a categorical set.

This classification, the establishment of series of propositions in these various geometries, and the logistic theory leading to them, characterizes the whole

treatise. It is very commendable that the authors have accomplished this task, which was not a very easy one, with marked success. Americans, and for that matter all geometers to whom the book is accessible, will be glad to refer students who desire more information on foundational methods in geometry to Veblen and Young's competent treatment of this branch of mathematics.

As far as the didactic treatment is concerned, the reviewer expresses the opinion that in a projective geometry a more central, or prominent position should be given to the fundamental theorem, and it would have been more consistent to have incorporated the first chapter of volume 2 in volume 1 before the fundamental theorem was reached. Enriques's *Vorlesungen über projektive Geometrie* offers an excellent example of such a treatment. Also Del Pezzo's *Principi di Geometria Projettiva* follows a similar plan with much success.

Although a logistic treatment of geometry should be possible without figures of any kind, Veblen and Young were wise not to exclude the mechanical auxiliary of graphic illustrations, as is done by Del Pezzo. I ask the question whether anybody without preliminary knowledge, or the analytic equivalent, could understand Del Pezzo's book at all without drawing most of the figures to which he refers? Again, who is able to write a treatise on projective geometry without either the knowledge of the analytic equivalent, or the drawing of at least some of the figures as a purely mechanical auxiliary? If an author makes use of figures in preparing a treatise, why should those figures be deleted on publication? But the point is really this: Since a logistic system starts with a set of *undefined elements*, and a set of assumptions between them, independent of intuition, the use of figures in such a system should be justified. What relation connects the points and lines of the logistic system with the physical points and lines of the figure and the sheet of paper upon which they are drawn? It seems to me that this difficulty can only be overcome by a set of *axioms of construction* admitting the possibility of actual graphic representation, isomorphic with the logistic system.

As to the content of the two volumes in general, it would be impossible to give a detailed account of it within the narrow space of this review. I shall therefore confine myself to some features of particular interest.

The fourteen chapters of the first volume deal with theorems of alignment and the principle of duality; projection, section, perspectivity, elementary configurations; projectivities of the primitive geometric forms of one, two, and three dimensions; harmonic constructions and the fundamental theorem of projective geometry; conic section; algebra of points and one-dimensional coördinate systems; coördinate systems in two and three-dimensional forms; geometric constructions, invariants; projective transformations of two-dimensional forms; families of lines.

As has been pointed out before, von Staudt's theorem, and thereby implicitly the fundamental theorem, is assumed for a proper projective geometry and receives its conclusive aspect at the beginning of the second volume. Thus the theory of conics in chapter 5, for example, is based upon this assumption P

(and, of course, A, E, H_0). An account of the historic development, or of the evolution, of the fundamental theorem, and a statement as to the motive for making one particular choice of assumptions in preference to another, would have greatly added to the clearness of the exposition. The reviewer shares with Poincaré the opinion that the beginner should be given the reason why a particular choice of axioms is made, because it seems to be evident that the choice cannot be "very arbitrary" if the geometer has in view the establishment of a certain system of geometry. Moreover the student undoubtedly shows increased interest if he is familiarized with the historic roots of the science which he undertakes to study. The authors, in their treatise at least, show little interest in the historic side of the subject. Historic and some other references given are rather meager and partly insufficient. Thus, in a foot-note on page 109 we are told that Jakob Steiner (1796–1863) was the first to define the first four one-dimensional forms of the second degree by projective pencils. Why not add when and where?, *i. e.*, in the *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander*, etc., which appeared in 1832.

In the definition of homogeneous coördinates in the plane it is important to state explicitly that the ratios of any two coördinates of a point (x_1, x_2, x_3), or a line (u_1, u_2, u_3), may be represented by cross-ratios, which are not changed by collineations. This is the reason why they are frequently called projective coördinates, as introduced into geometry by Fiedler.[1] It would be more natural to incorporate exercise 1, on page 179, into theorem 3 on the same page. Choosing the same triangle of reference for point and line coördinates, and a unit line $(\rho u_1 = 1, \rho u_2 = 1, \rho u_3 = 1)$ independent of the unit point $(\delta x_1 = 1, \delta x_2 = 1, \delta x_3 = 1)$, the condition for coincidence for a point (x_1, x_2, x_3) and a line (u_1, u_2, u_3) is $k_1u_1x_1 + k_2u_2x_2 + k_3u_3x_3 = 0$, where k_1, k_2, k_3 are three constants which do not vanish simultaneously. When the unit-line is the triangle-polar of the unit-point, then $k_1 = k_2 = k_3$ and the necessary and sufficient condition for coincidence reduces to $u_1x_1 + u_2x_2 + u_3x_3 = 0$. A similar remark may be made with respect to projective coördinates in space of three dimensions.

In Chapter IX on geometric constructions is introduced the term *geometric operation*, by which the authors give expression to the feeling that something should be done in passing from an axiomatic to a concrete system. This is precisely the point which I had in view when I wrote of the necessity of suitable axioms of construction. The statement, on pp. 236–237, that there is no familiar mechanical device for drawing lines and planes in space is not correct. Just as one can manufacture a ruler, one can manufacture a *planoid*, *i. e.*, a plane surface, by means of which planes in space may be drawn, or mechanically fixed. Moreover we have Koenig's planigraph,[2] a linkage, by which a plane may be described in analogy with the description of a straight line by Peaucellier's inversor.

On account of the importance of the subject it would have been well if the

[1] *Vierteljahrsschrift der naturforschenden Gesellschaft in Zürich*, Vol. 15, p. 152f. (1870).

[2] *Leçons de Cinématique*, pp. 295–297.

classification of collineations and their group-properties in Chapter X had received a more extensive treatment. The same is true of the discussion of pairs of conics on pp. 287–293. The parametrically linear form which is of importance in this discussion is the pencil of conics and the classification of such pencils. We also miss the statement that, in general, the discriminant of a pencil of conics $\phi + \lambda\psi = 0$, set equal to zero, is a cubic in λ, so that, in general, there are three degenerate conics in a pencil.

The last chapter on families of lines deals with the regulus, its principal properties, and the figures of lines depending on these properties. It is important to know that, from a certain point of view, the problem of projectivity is closely related with the theory of the regulus. In arts. 109 and 110 a very brief treatment of Plücker's line coördinates is given. The fact ought to be mentioned that the coördinates

$$\rho_{ik} = \begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}, \qquad \phi_{ik} = \begin{vmatrix} u_i & v_k \\ v_i & u_k \end{vmatrix}$$

are commonly known as *homogeneous radial,* and *axial coördinates* of the line, respectively. Considering their great importance in other fields, the linear complex and in particular the null system, and line-congruences deserve fuller consideration than is given to them in the book under review. The authors realized, of course, that in their treatment space did not permit of too extensive accounts of even important specific geometric systems.

The scope of the second volume, written by Veblen, is very clearly stated in the preface:

"We have in mind two principles for the classification of any theorem of geometry: (*a*) the axiomatic basis, or bases, from which it can be derived, or, in other words, the class of spaces in which it can be valid; and (*b*) the group to which it belongs in a given space. . . . Having fixed attention on any particular space, we have a set of groups of transformations to each of which belongs its geometry. For example, in the complex projective plane we find among others, (1) the group of all-continuous one-to-one reciprocal transformations (analysis situs), [why add the word reciprocal?], (2) the group of birational transformations (algebraic geometry), (3) the projective group, (4) the group of non-Euclidean geometry, (5) a sequence of groups connected with Euclidean geometry. . . . The two principles of classification, (*a*) and (*b*), give rise to a double sequence of geometries, most of which are of consequence in present day mathematics. It is the purpose of this book [volume 2] to give an elementary account of the foundations and interrelations of the more important of these geometries (with the notable exception of (2))." Veblen suggests the desirability of other books taking account of this logical structure, but dealing with particular types of figures. I take it for granted that the author does not want to intimate by this that there are no treatises on geometry in existence which are based upon a logical structure. The projective geometry of that three dimensional space which is simply isomorphic with analytic space containing as points all quadruples

(x_1, x_2, x_3, x_3) of real or complex numbers, of which not all four ever vanish simultaneously, and as planes all linear equations $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ between the coördinates of these points, with the a's as arbitrary constants from the same number system, for example, is well established according to the requirements of a logical system.

The second volume contains nine chapters on foundations, elementary theorems on order, the affine group in the plane, Euclidean plane geometry, ordinal and metric properties of conics, inversion geometry and related topics, affine and Euclidean geometry of three dimensions, non-Euclidean geometries, theorems on sense and separation.

Affinity is given a prominent place in Chapters III and VII, which treat of the affine groups in the plane and in space, respectively. As the systems of affinity in the plane as well as in space express properties invariant under a given group of transformations, they may be called geometries in the sense of Klein and properly specified by the word affine. Considering the fact that affinity is recognized as a geometry of marked importance in a number of recent investigations, the 112 pages reserved for the two chapters prove indeed that the author is aware of this fact. Chapter III is based entirely on assumptions A, E, P, H_0. In fact a large number of theorems depend only on A, E, H_0. The affine group of three dimensions, of which the whole affine plane geometry is a part, in a nonhomogeneous coördinate system in which π_∞ is the singular plane, consists of the set of all projectivities of the form

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{10},$$

$$y' = a_{21}x + a_{22}y + a_{23}z + a_{20},$$

$$z' = a_{31}x + a_{32}y + a_{33}z + a_{30},$$

where

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0.$$

The set of points not on π_∞ is called a *Euclidean space* and π_∞ is called the *plane at infinity* of this space, which is invariant in the affine group of three dimensions. The corresponding geometry is called *affine geometry,* or simply *affinity of three dimensions*.

To non-Euclidean geometries in Chapter VIII only 31 pages are given. These are approached in the most natural manner by means of *the absolute* as introduced by Cayley and clearly brought out by Klein in its connection with non-Euclidean geometries. I wonder what our friend Halsted will say if he finds that in the references to the history and exposition of parts of non-Euclidean geometry on page 362, his name is omitted?

In comparison to the size of this chapter, the one following on theorems on sense and separation with 116 pages, more than one fifth of the whole book, seems exceptionally large. But this chapter contains much of Veblen's own investigations on this branch of analysis situs.

A very commendable feature of the whole treatise is the large number of exercises, which are either original, or selected from many well-known standard works and monographs. Among them are problems of considerable importance in real projective geometry, which by some authors are included in the main subject matter.

The mechanical make up of the two books is excellent, with the exception of many figures, which, from the standpoint of a connoisseur of graphic arts, are by no means the product of expert draftsmen. Pleasing figures, however, are as much to be desired as pleasing type. In this respect the second volume shows a decided improvement over the first.

May be it is a whim of an æsthetic crank if he misses the same kind of imprint on the backs of the two volumes. Why is that of the first volume gilt, that of the second volume black? But we shall not insist further upon such trivialities, and close this review by thinking with Voltaire: Le secret d'ennuyer est celui de tout dire.

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An Elementary Treatise on Curve Tracing. By P. FROST. Fourth edition, revised by R. J. T. BELL. London, Macmillan, 1918. 8vo. 16 + 210 pp. + 17 plates. Price 12s. 6d.

The first edition of this work appeared in 1872 and reprints were issued as second and third editions in 1892 and 1911 respectively. Concerning the present edition (which is the first revision of the original work) Mr. Bell states that it does not differ from the previous one except in places where alterations were necessary to remove ambiguities or to correct mistakes in analysis and diagrams. A useful classified list of the curves discussed has been added on pages 203–208, and the typography of the new edition is much better than that of the earlier one. Except for 7 types of curves whose equations involve trigonometric functions, all curves analyzed have rational algebraic equations; these include 23 cubics, 69 quartics, 31 quintics, 18 sextics, 9 curves of the seventh degree and 2 of the eighth. As the calculus is nowhere employed in the discussion great ingenuity is displayed in the analyses, and the work as a whole is exceedingly interesting.

In his preface of 1872 (reprinted in the present edition) Frost wrote that it would be difficult to find another subject

"which, with a very limited extent of reading, combines, to the same extent, so many valuable hints of methods of calculations to be employed hereafter, with so much pleasure in its use.

For example, the subject of Graphical Calculations is coming more into use every day, and is applied with success to many difficult problems in Statics, Engineering and Crystallography; hints of this the student will find in the practical solution of divers questions and in the determination of the number of their real roots, which are obtained by graphical methods with great facility.

Again, the methods of successive approximations which are employed in Optics and Astronomy are illustrated in the process of finding asymptotes and approximations to the forms of curves at a finite distance.

The comparison of large and small quantities of different orders of magnitude contains the staple of many of the most important applications of Mathematical Analysis; the Lunar and Planetary Theories depending almost entirely upon such considerations of relative magnitude.

The habit of looking towards an infinite distance, and discussing what takes place there, will render less startling a multitude of conceptions having in them a tendency to produce a feeling of vagueness, such, for instance, as the treatment of the mechanical effect of a couple as synonymous with that of an infinitely small force acting at an infinitely great distance.

As an important point, I would mention the tentative character of the inverse problem in which the form of a curve being given, its equation is to be investigated; the kind of uncertainty which will remain on the mind on account of defective estimation of magnitudes; and the necessity of a selection of what may appear the best of many possible solutions; all this will prepare the student for disappointment which, having perhaps a wrong notion of what is meant by calling mathematics an exact science, he will feel in the conflict of theories by which it is attempted to reconcile the results of experiment in such subjects as Heat, Light, Electricity, and Molecular action generally; for an instance of this I may refer to the battle of philosophers about the direction of vibration of the ether in Plane Polarization."

Contents—Chapter I: Introductory theorems; definitions; tracing by points; symmetry, 1–8. II: Orders of small quantities; forms of parabolic curves near the origin; cusps; tangents to curves; curvature, 9–19. III: Forms of parabolic curves at an infinite distance; examples of tracing curves; trigonometrical curves; illustrations of theory of equations; rules for approximation, 20–37. IV: Forms of curves in the neighbourhood of the origin; simple tangents; direction and amount of curvature; multiple points of two branches; curvature of branches at multiple points; multiple points of higher orders, 38–57. V: Forms of branches whose tangents at the origin are the coördinate axes, 58–67. VI: Asymptotes; points of intersection at an infinite distance; asymptotes parallel to the axes, 68–87. VII: Asymptotes not parallel to the axes; asymptotes to homogeneous curves, 88–106. VIII: Curvilinear asymptotes, 107–116. IX: The analytical triangle [Newton's parallelogram and De Gua's triangle]; properties of the analytical triangle, 117–132. X: Singular points; division into compartments; special curve of the fourth degree, 133–166. XI: Systematic tracing of curves; repeating curves, 167–185. XII: Inverse process; determination of the equation of a given curve, 186–202.

Matrices and Determinoids. By C. E. CULLIS. Cambridge: at the University Press. Royal 8vo. Volume 1, 1913; 12 + 430 pp. Price 21 shillings. Volume 2, 1918; 24 + 555 pp. Price 42 shillings.

Extracts from prefaces: "The present work is an amplification of a course of lectures given for the University of Calcutta in the winter of 1909–10. Its chief feature is that it deals with rectangular matrices and determinoids as distinguished from square matrices and determinants, the determinoid of a rectangular matrix being related to it in the same way as a determinant is related to a square matrix. . . . The first volume contains the most fundamental portions of the theory, and concludes with the solution of any system of linear algebraic equations, which is treated as a special case of the solution of a matrix equation of the first degree. . . . The second volume contains those parts of the theory which naturally precede any investigation of the special properties of functional matrices, i. e., matrices whose elements are rational integral functions of a finite number of variables. It deals almost exclusively with matrices whose elements are constants, which may be arbitrary parameters, and with those transformations of such matrices which are classed as equigradent. It does not however contain all the properties of such matrices. There remain many properties which it will be more convenient to consider after a preliminary study of functional matrices. . . . The following is a list of the books which have had most influence on the work as a whole: Bôcher's *Introduction to Higher Algebra*, Heffter and Koehler's *Lehrbuch der Analytischen Geometrie*, Muth's *Elementarteiler*, Netto's *Vorlesungen über Algebra*, Veronese's *Fondamenti di geometria a piu dimensioni*, Whitehead's *Universal Algebra*. My indebtedness to these and other writers will be more easily recognized in those portions of the work, occurring chiefly in volume 3, which are interpolations in the original scheme." It is expected that the third volume will contain the completion of the theory and "applications to vector analysis and the theory of invariants. The complete exposition was in fact undertaken with a view to these last mentioned applications."

Contents—Volume 1, chapter I: Introduction of rectangular matrices and determinoids, pages 1–21; II: Affects of the elements and derived products of a matrix or determinoid, 22–54; III: Se-

quences and the affects of derived sequences, 55–85; IV: Affects of derived matrices and derived determinoids, 86–104; V: Expansions of a determinoid, 105–152; VI: Properties of a product formed by a chain of matrix factors, 153–208; VII: Determinoid of a product formed by a chain of matrix factors, 209–247; VIII: Matrices of minor determinants, 248–264; IX: Rank of a matrix and connections between the rows of a matrix, 265–298; X: Matrix equations of the first degree, 299–363; XI: Solution of any system of linear algebraic equations, 364–417; Index, 419–430—Volume 2, XII: Compound matrices, 1–36; XIII: Relations between the elements and minor determinants of a matrix, 37–106, 515–520; XIV: Some properties of square matrices, 107–164, 521–530; XV: Ranks of matrix products and matrix factors, 165–227; XVI: Equigradent transformations of a matrix whose elements are constants, 228–308; XVII: Some matrix equations of the second degree, 309–377; XVIII: The extravagances of matrices and of spacelets in homogeneous space, 378–462, 531–534; XIX: The paratomy and orthotomy of two matrices and of two spacelets of homogeneous space, 463–514; Index 535–555.

Graphical and Mechanical Computation. By J. Lipka. New York, Wiley, 1918. 9 + 264 pp. + 2 scales in pocket. Price $4.00.

Contents—I: Scales and the slide rule, 1–19. II: Network of scales; charts for equations in two and three variables, 20–43. III–V: Nomographic or alignment charts, 44–119. VI: Empirical formulas—non-periodic curves, 120–169. VII: Empirical formulas—periodic curves, 170–208. VIII: Interpolation, 209–223. IX: Approximate integration and differentiation, 224–259.

Extract from the Preface—"This book embodies a course given by the writer for a number of years in the Mathematical Laboratory of the Massachusetts Institute of Technology. It is designed as an aid in the solution of a large number of problems which the engineer, as well as the student of engineering, meets in his work. . . .

"Engineers have recognized for a long time the value of graphical charts in lessening the labor of computation. Among the charts devised none are so rapidly constructed nor so easily read as the charts of the alignment or nomographic type—a type which has been most fully developed by Professor M. d'Ocagne of Paris. Chapters III, IV, and V aim to give a systematic development of the construction of alignment charts; the methods are fully illustrated by charts for a large number of well-known engineering formulas. It is the writer's hope that the simple mathematical treatment employed in these chapters will serve to make the engineering profession more widely acquainted with this time and labor-saving device.[1]

"Many formulas in the engineering sciences are empirical, and the value of many scientific and technical investigations is enhanced by the discovery of the laws connecting the results. . . . Chapter VII considers the case where the data are periodic, as in alternating currents and voltages, sound waves, etc. and gives numerical, graphical, and mechanical methods for determining the constants in the equation.

"When empirical formulas cannot be fitted to the experimental data, these data may still be efficiently handled for purposes of further computation,—interpolation, differentiation, and integration,—by the numerical, graphical, and mechanical methods developed in the last two chapters.

"Numerous illustrative examples are worked throughout the text, and a large number of exercises for the student is given at the end of each chapter. The additional charts at the back of the book will serve as an aid in the construction of alignment charts. Bibliographical references will be found in the footnotes.

"The writer . . . owes the idea of a Mathematical Laboratory to Professor E. T. Whittaker of the University of Edinburgh."

[1] The second edition, revised and corrected, of D'Ocagne's *Calcul graphique et nomographie* (Paris, O. Doin, 1914), contains an interesting and extensive bibliography down to the year 1912 (pages 381–386). Among eighteenth century items are: (1) L. Pouchet, "Arithmétique linéaire" appendix to *Echelles graphiques des nouveaux poids, mesures*, Rouen, 1795; and (2) J. von Segner, "Methodus simplex et universalis omnes omnium æquationum radices detegendi," *Acad. Petrop. Novi Comment.*, tome 7, 1761. E. V. Huntington briefly discusses nomography on pages 178–185 of his *Handbook of mathematics for engineers* (1918) which is reprinted from L. S. Marks's *Mechanical Engineer's Handbook* (New York, 1916). For an application of nomography to the geological problem of finding faults, see "The faultless faultfinder" by W. S. Weeks and E. V. Huntington, *Engineering and Mining Journal*, August 15, 1914.

Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche. Di F. ENRIQUES. Pubblicate per cura del Dott. O. Chisini. Bologna, Zanichelli. 8vo. Vol. 1, 1915, 16 + 398 pp.; vol. 2, 1918, 713 pp. Price 12 + 30 lire.

Contents—volume 1: Libro I, Introduzione (1. Le equazioni $f(x) = 0$ e i gruppi di punti sulla retta; 2. Interpretazioni fondamentali dell' equazione $f(xy) = 0$: curve e corrispondenze; 3. Nota sul significato dell' espressione "in generale" e sui computi di costanti), 5–153. Libro II, Il principio di corrispondenza e le sue applicazioni (1. Le involuzioni e i gruppi finiti di proiettività sulla retta; 2. Teoria elementare delle curve piane; 3. Nota sulle funzioni algebriche e sulle rappresentazioni reali dell'immaginario), 155–386—Volume 2: Libro III, La teoria elementare delle curve piane basata sulla polarità (1. Polarità e curve covarianti; 2. Il problema delle intersezioni e i caratteri plueckeriana delle curve; 3. La cubica piana; 4. Appendice: realità e continuità; geometria numerativa), 5–321. Liber IV, Le singolarità delle curve algebriche (1. Le singolarità e gli sviluppi in serie di Puiseux; 2. Le singolarità rispetto alle trasformazioni quadratiche; 3. Le singolarità rispetto al calcolo differenziale; 4. Appendice: singolarità delle curve gobbe e delle superficie), 323–686. Indice alfabetico, indice dei capitoli, and errata-corrige e addizioni, 687–713.

The History of Statistics, Their Development and Progress in Many Countries, in Memoirs to commemorate the seventy-fifth anniversary of the American Statistical Association, collected and edited by J. KOREN. New York, Macmillan, 1918. 8vo. 12 + 773 pp. Price $7.50.

Contents—Book I: Historical and commemorative addresses, "The American Statistical Association, 1839–1914," by John Koren, president, pages 3–14; "Seventy-five years of progress in statistics: the outlook for the future" by S. N. D. North. Book II: History and development of official statistics in many countries: Australia by G. H. Knibbs, 53–82; Austria by R. Meyer, 83–122; Belgium by A. Julin, 123–176; Canada by E. H. Godfrey, 177–198; Denmark by A. Jensen, 199–214; France by F. Faure, 215–330; Germany by E. Würzburger, 331–362; Great Britain and Ireland by A. Baines, 363–390; Hungary by L. v. Buday, 391–414; India by A. Baines, 415–426; Netherlands by C. A. V. Stuart, 427–444; Norway by A. N. Kiaer, 445–466; Russia by A. Kaufmann, 467–534; Sweden by E. Arosenius, 535–570; United States: Federal statistics by J. Cummings, State statistics by C. F. Gettemy, 571–739.

NOTES.

Il Pitagora has not been published since the completion of "anno 23" in June, 1918—*Rendiconti del circolo matematico di Palermo* resumed publication in February, 1919; the last number which had appeared previously was volume 42, fasc. 1, January–April, 1917.

The publication of *Mathematical Questions and Solutions in Continuation of the Mathematical Columns of the Educational Times,* which had been issued in monthly parts since January, 1916, was discontinued "for the present," in December, 1918, with the issue of the last number of volume 6. The three series of this publication total 110 volumes. (For further details see this MONTHLY, volume 3, 1896, pages 159–163, and volume 22, 1916, page 100).

On Professor Poul Heegaard's appointment as professor at the University of Christiania he transferred the *Nyt Tidsskrift for Matematik, A* and *B,* of which he had long been editor, to the Mathematical Society of Copenhagen. The publication is now being continued as *Matematisk Tidsskrift, A* and *B;* the first numbers appeared in February, 1919. *A* is edited by I. L. W. JESSEN and O. A. SMITH; *B* by H. BOHR, and T. BONNESEN. *Nyt Tidsskrift, A* and *B,* published 1890–1918, constituted a continuation of *Tidsskrift for Matematik,* 1859–89.

The *Subject Index of the Modern Books added to the Library of the British Museum* 1911–1915 (London, 1918), lists many books and memoirs of mathematical interest under such headings as: Aëronautics; algebra; arithmetic; calculus; functions, mathematical; hydraulics; hydrodynamics; mathematics; mechanics; surveying; tables; and trigonometry. The writings of American authors are well represented.

In *The Carnegie Foundation for the Advancement of Teaching, Thirteenth Annual Report of the President and of the Treasurer* (New York, October, 1918), are brief biographies of CHARLES LEE CRANDALL (1850–1917) and of NATHAN FELLOWES DUPUIS (1836–1917). Professor Crandall was connected with Cornell University most of his life and among books of which he was the author are: *Notes on descriptive geometry* (1888, 1893), *The transition curve* (1893, 1899), and *Textbook on geodesy and least squares* (1907). Professor Dupuis was a Canadian and he taught at Queen's University, his alma mater, for more than forty years. He built several well-known astronomical clocks, a sidereal clock built by him being still in use in the university observatory. In addition to numerous articles for periodicals Professor Dupuis wrote a *Treatise on geometrical optics* (1868); *Geometry of the point, line and circle* (1889); *Principles of algebra* (1893); *Elements of synthetic solid geometry* (1893); *Elements of trigonometry for practical science students* (1902); *Spherical trigonometry and astronomy* (1906); *Descriptive and mechanical astronomy* (1910); and *Measurement of time* (1915).

ARTICLES IN CURRENT PERIODICALS.

ANNALI DI MATEMATICA PURA ED APPLICATA, series 3, volume 28, no. 1, November, 1918: "Ulisse Dini" by L. Bianchi, i–ii. [Dini was born at Pisa in November, 1845, and died in his native city, October 28, 1918. He was professor of mathematics in the University of Pisa for over 50 years, and director of the R. Scuola Normale Superiore in Pisa since 1901. His books which were, for the most part, developments of university lectures, include the following: (1) *Fondamenti per la teoria delle funzioni di variabili reali*, Pisa, 1878; (German edition, 1892); (2) *Sopra le serie di Fourier ed altre le representazioni analitiche per le funzioni di una variabile*, Pisa, 1880; (there is a type-written English translation of this work, by W. B. Ford, in the Library of Harvard University); (3) *Teoria delle funzioni elliticke* (autographed), 1893; (4) *Sugli sviluppi in serie per la rappresentazione analitica delle funzioni di una variabile reale date arbitrariamente in un certo intervallo*, Pisa, 1911; (5) *Lezioni di analisi infinitesimale*, Pisa, 1909–1915 (a forerunner of this thousand page work was the lithographed *Lezioni* of 1877–78).

A portrait and sketch of Dini are given in *Acta Mathematica, 1882–1912. Table générale des tomes 1–35*, Upsala, 1913].

BOLLETTINO DI BIBLIOGRAFIA E STORIA DELLE SCIENZE MATEMATICHE, volume 20, January–March, 1918: "Notizie su la Facoltà di scienze matematiche della R. Università di Modena" by E. Bortolotti, 1–11; Review by G. Loria of L. Braude's *Les coordonnées intrinsèques. Théorie et application* (Paris, 1914), 12; Review by C. Rosati of E. Ciani's *Il metodo delle coordinate proiettive omogenee nello studio degli enti algebrici* (Pisa, 1915), 13–15; Review by E. E. Levi of G. Fubini's *Lezioni di analisi infinitesimale.* 2a ed. (1916), 15–16; Review by G. Vivanti of U. Dini's *Lezioni di analisi infinitesimale*, Vol. 2: calcolo integrale (Pisa, 1909–15), 17–18; Review by G. Fubini of G. Vivanti's *Equazioni integrali lineari*, (Milano, 1916), 18–20; Review by G. Loria of P. Ruffini's *Opere matematiche*, Vol. 1 (Palermo, 1915) and Euclid's *Il primo libro degli Elementi, testo greco, versione italiana, introduzione e note a cura di* G. Vacca (Firenze, 1916), 20–21; Review by P. Burgatti of G. Colonnetti's *Principi di statica dei solidi elastici* (Pisa, 1916), 22–29; Notizie, 30–32.

L'ENSEIGNEMENT MATHÉMATIQUE, volume 20, no. 1, January, 1918: "L' approximation des fonctions d'une variable réelle" by C. de la Vallée Poussin, 5–29; "Deux recents ouvrages de

Géométrie" [by Darboux and d'Ocagne] by A. Buhl, 30–42; "Remarques sur la construction des courbes gauches avec application a la parabole cubique" by G. Loria, 43–47; "Théorie élémentaire de la toupic gyroscopique" by M. Zack, 47–62; Review by M. Plancherel of R. Fueter's *Synthetische Zahlentheorie* (Leipzig, 1917), 70–71; Review by A. Reymond of A. N. Whitehead's *The Organization of Thought* (London, 1917), 73–76.—No. 2, September, 1918: "Sur les congruences linéaires de cubiques gauches douées d'une seule courbe singulière" by L. Godeaux, 81–89; "Sur la gerbe de cubiques gauches passant par cinq points" by F. Gouseth, 90–93; "Sur les trajectoires d'un mobile soumis à une force centrale et a une resistance de milieu" by C. Cailler, 93–96; "Sur certaines identités vectorielles et leurs interprétations dans la géométrie sphérique et plane" by M. F. Daniëls, 97–122; "Pensée axiomatique" by D. Hilbert, 122–136; Reviews by A. Buhl of E. Borel's *Leçons sur les fonctions monogènes uniformes d'une variable complexe* (Paris, 1917) and L. Lecornu's *Cours de Mécanique* (tome 3) and M. d'Ocagne's *Cours de géométrie pure et appliquée de l'Ecole Polytechnique* (Paris, 1918), 143–144, 147–154; Review by M. Plancherel of E. Landau's *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale* (Leipzig, 1918), 146–147.—No. 3, December: "Notions d'arithmogéométrie" by E. Turrière, 161–174; "Sur la 'variété moyenne' de deux variétés convexes" by G. Tiercy, 175–189; "Contribution à la construction des éléments doubles d'une involution hyperbolique" by F. Redl, 190–193; "Extraction de la racine $n^{\text{ième}}$ d'une nombre réel par approximations successives" by M. T. Béritch, 194–198; "Note sur les permutations" by A. Aubry, 199–215; "Sur la rectification approchée d'un arc de cercle" by A. Pleskot, 215–218; "A propos d'un problème de Lagrange sur la construction des cartes géographiques" by L. Ballif, 219–221; "J. H. Graf (1852–1918)" by L. Crelier, 224–225; Review by H. Fehr of *Scritti matematici offerti ad Enrico D'Ovidio in occasione del suo LXXV genetliaco, 11 agosto, 1918* (Turin, 1918), 229–230.

L'Intermédiaire des Mathematiciens, volume 25, November–December, 1918: "Albert Gauthier-Villars," 121–122; "Questions," 122–126; "Reponses," 127–144.

Journal of the Indian Mathematical Society, volume 10, no. 6, December, 1918: "Infinite series and arithmetical functions" (conclusion) by F. Hallberg, 454–472; "Note on triangles inscribed in an ellipse" by A. N. Raghavachar, 473–475; "On poristic polygons" by F. H. V. Gulasekharam, 475–476; "A geometrical problem" [Given the circumcircle and the orthocenter of a triangle to find the locus of the in- and ex-centers"] by M. T. Naraniengar, 476–478; "In Memoriam" [R. J. Pocock], 479; Problems and solutions, 480–492.

Mathematical Gazette, volume 9, May, 1918: "The elementary theory of statical stability" by S. Brodetsky, 233–236; "Alice through the (convex) looking glass" by W. Garnett, 237–241; "The introduction to infinite series" by W. J. Dobbs, 242–246; "The teachers' library," i–vi. [This last article purports to give "a suggestive list of books suitable for inclusion in the mathematical library of a modern secondary school." In the list are scores of books such as: H. Weber, *Lehrbuch der Algebra,* 3 vols.; P. Muth, *Theorie und Anwendung der Elementarteiler;* E. Pascal, *Gruppi di trasformazioni;* Hamilton, *Elements of quaternions,* 2 vols., and Forsyth, *Theory of differential equations,* 6 vols.]—July: "Alice through the (convex) looking glass" (continued) by W. Garnett, 249–252; "The introduction to infinite series" (continued) by W. J. Dobbs, 253–256; "MacCullagh," 256; Note on the spherical triangle and haversines by E. M. Langley, 257–259; Review by G. Greenhill of D'Ocagne's *Cours de géométrie* (Paris, 1917–18), 261–262—October–December: "Mathematical Association teaching committee, report on the teaching of mechanics" by T. P. Nunn, A. W. Siddons, W. J. Dobbs, and G. Goodwill, 265–292 —January, 1919: "Alice through the (convex) looking glass" (concluded) by W. Garnett, 293–298; "Gleanings far and near" [from *Recollections of Mary Somerville*], 298, 307; "The introduction to infinite series" (concluded) by W. J. Dobbs, 299–301; "A letter from Sir William Rowan Hamilton," 302; "Notes on the life and works of Colin Maclaurin" by C. Tweedie, 303–305; "The equilibrium of jointed frameworks" by G. H. Bryan, 306–307; Review by H. P. Hilton of Darboux's *Principes de géométrie analytique,* 308.

Nouvelles Annales de Mathématiques, volume 77, November, 1918: "Sur les foyers rationnels d'une courbe algébrique plane ou gauche" by P. Appell, 401–402; "Sur les courbes algébriques planes" by R. Bouvaist, 403–417; "Sur deux points du plan d'un triangle et sur une généralisation des points de Brocard" by R. Goormaghtigh, 417–424; "Quelques applications géométriques de la théorie des infiniment petits" by M. Weill, 424–429; "Nouvelles identités" by G. Fontené, 430–431; "Sur la chainette d'égale résistance" by F. Balitrand, 431–433; Solutions and Questions, 434–440.

Quarterly Journal of Pure and Applied Mathematics, volume 48, no. 1, October, 1917: "An expansion related to Stirling's formula, derived by the method of steepest descents"

by G. N. Watson, 1–18; "On parallel curves and evolutes" by A. B. Basset, 19–39; "Some theorems in four-dimensional analysis" by C. E. Weatherburn, 39–58; "The attraction of equi-angular spirals" by S. Brodetsky, 58–76; "The normal number of prime factors of a number n" by G. H. Hardy and S. Ramanujan, 76–92; "On the representations of numbers as a sum of $2r$ squares" by L. J. Mordell, 93–96—no. 2, September, 1918: "On the representations of numbers as a sum of $2r$ squares" by L. J. Mordell, 97–104 (conclusion); "On two loci determined by con-current tangents to plane curves" by C. H. Sisam, 104–112; "Note on asymptotic formulæ for oscillating Dirichlet's integrals" by M. Kuniyeda, 113–135; "Moving axes with variable angles" by E. H. Neville, 136–141; "On a theorem in the theory of differentiation of functions defined by integrals" by G. Fichtenholz, 142–147; "Substitution groups on the terms of symmetric polynomials" by G. A. Miller, 147–150; "On early tables of logarithms and the early history of logarithms" by J. W. L. Glaisher, 151–192.

Revue de Métaphysique et de Morale, volume 26, no. 1, January–February, 1919: "L'entropie, extension conservative" by L. Selme [died January 3, 1919], 89–118; "Gaston Milhaud," 149–151; [Quotation: "Peu d'esprits donnaient au même degré que lui l'impression de la puissance spirituelle, qui s'impose par sa seule lumière, sans prestige matériel et sans contrainte. A toute sa philosophie on pourrait donner pour epigraphe les deux mots qui servaient déjà de conclusion à son premier ouvrage, et qu'il opposait 'aux subtilités d'analyse dont se joue la marche de l'esprit humain': Raison et Liberté"].

Revue générale des Sciences, volume 29, no. 23, December 15, 1918: "Gaston Milhaud" by A. Lalande. [Quotation: "Tous ceux qui l'approchaient étaient attirés et retenus par le charme de sa personne, la finesse de son esprit, la sûreté de son caractère et de ses affections. Jamais philosophe n'a discuté avec plus d'aimable courtoise; et cependant,—ou peut-être par là-même,—il trouvait le moyen de ne rien abandonner de son opinion et souvent de persuader son interlocuteur." Milhaud was born in 1858 and died October 1, 1918. At the age of 20 years he was admitted to both the Ecole polytechnique and the Ecole normale supérieure, but chose the latter and became agrégé ès science mathématique in 1881. He taught in several lycées before he was appointed professor at the University of Paris. He published his *Leçons sur les origines de la science grecque* in 1893. Soon afterwards he became doctor ès lettres at the University of Paris, his principal thesis being *L'essai sur les conditions et les limites de la certitude logique* (1894, 2e éd. 1898). Of his other books the following may be mentioned: *Les philosophes-géomètres de la Grèce: Platon et ses prédécesseurs* (1900); *Le positivisme et le progrès de l'esprit* (1902); *Etudes sur la pensée scientifique chez les Grecs et chez les modernes* (1906); *Nouvelles études sur l'histoire de la pensée scientifique* (1911).]—January 15, 1919: Review by J. Boyer, of Sedgwick and Tyler's *A Short History of Science* (New York, 1918), 28—February 28: "L'avenir du catalogue international de la littérature scientifique," supplément, 17; "Reprise de la publication de la Revue *Isis*" by G. Sarton, supplément, 17.

School Science and Mathematics, volume 19, no. 3, March, 1919: "Proceedings of the eighteenth meeting of the Central Association of Science and Mathematics Teachers," 197–199, 204–208, 265–268; "Progressive science and mathematics courses and teaching in France" by A. Barthelemy, 199–204; "Final report of subcommittee on content of course in first-year mathematics," 259–264; Problems and solutions, 275–281.

Science, volume 49, February 28, 1919: "Common numerals" by G. A. Miller, 215. [Quotation: "It is very interesting to note that during recent years available data relating to the origin of our common number symbols have been carefully reexamined by Carra de Vaux, who published in volume 21 of *Scientia* a brief summary of his results. Among the most surprising of these results are the following: Our common number symbol originated in Europe and from there were transmitted to the Persians. Both India and Arabia received them from Persia, so that the common term *Hindu-Arabic numerals* is decidedly misleading. The common numerals did not come from letters of the alphabet, but were formed directly for the purpose of representing numbers.

It does not appear likely that all of these conclusions reached by Carra de Vaux, who has made an extensive study of intellectual life among the Mohammedans, will be at once accepted, but they tend to exhibit the weak foundation upon which the history of our common numerals has thus far rested."]

Scientia, volume 82, February, 1919: "Sur l'introduction de la méthode des perturbations dans la mécanique générale" by J. M. Plans, 89–93; Reviews, by G. Scorza, of Darboux's *Leçons sur la théorie générale des surfaces*, 2e partie, 2 éd. (Paris, 1915), Darboux's *Principes de géométrie analytique* (Paris, 1917), and of Halphen's *Oeuvres*, tome 1 (Paris, 1916), 147–149.

Scientific Monthly, volume 8, no. 3, March, 1919: "Charles Edward Pickering" and full-page portrait, 284–286.

Sewanee Review, volume 27, January–March, 1919: "Edward Kidder Graham" [1876–1918, president of the University of North Carolina], by A. Henderson, 101–106.

Transactions of the Academy of Sciences of St. Louis, volume 23; all of the mathematical articles so far published in this volume are by the septuagenarian physicist, F. E. Nipher, author of the little book *Introduction to graphical algebra* published about twenty years ago. No. 4, July, 1916: "Disturbances impressed upon the earth's magnetic field," 153–162; "Gravitation and electrical action," 163–175—No. 5, November, 1917: "Gravitational repulsion," 177–192—No. 6, May, 1918: "Graphical algebra involving functions of the nth degree," 193–204—No. 7, January, 1919: "Graphical algebra," 205–212.

Transactions of the American Mathematical Society, volume 20, no. 1, January, 1919: Portrait frontispiece and notice of Maxime Bôcher; [This portrait may be obtained by sending twenty cents in postage stamps to the Society, 501 West 116th Street, New York City.] "Necessary conditions in the problems of Mayer in the calculus of variations" by G. A. Larew, 1–22; "Linear equations with unsymmetric systems of coefficients" by Anna J. Pell, 23–39; "On convex functions" by H. Blumberg, 40–44; "Projective transformations in function space" by L. L. Dines, 45–65; "On the order of primitive groups (IV)" by W. A. Manning, 66–78.

AMERICAN DOCTORAL DISSERTATIONS.

G. S. Counts, *Arithmetic tests and studies in the psychology of arithmetic* (Supplementary Educational Monographs, vol. 1, no. 4). University of Chicago Press, Chicago, 1917. 4 + 127 pp. (Chicago, 1916.)

G. James, 1882– , *Some theorems on the summation of divergent series.* New York, 227 West 17th St., W. D. Gray, 1917. 28 pp. (Columbia, 1917.)

W. S. Monroe, *Development of arithmetic as a school subject.* (Reprinted from *United States Bureau of Education,* Bulletin, 1917, no. 10.) Washington, D. C., 1917. 170 pp. (Chicago, 1915.)

Agnes L. Rogers, *Experimental tests of mathematical ability and their prognostic value.* (Teachers College, Columbia University. *Contributions to Education,* no. 89.) New York, Columbia University, 1918. 5 + 118 pp. (Columbia University, 1917.)

C. H. Yeaton, 1886– , *Surfaces characterized by certain special properties of their directrix congruences.* (Reprinted from *Annali di matematica,* series 3, vol. 26, 1916.) 3 + 33 pp. (Chicago, 1915.)

UNDERGRADUATE MATHEMATICS CLUBS.

Edited by U. G. Mitchell, University of Kansas, Lawrence.

CLUB ACTIVITIES.

The Mathematics Club of the University of Kansas, Lawrence, Kansas. [1918, 35–36, 450–1, 459.]

The officers of the club for the year 1918–19 are as follows: President, Wealthy Babcock '19; vice-president, Faye Dodderidge '19; secretary-treasurer, Edith Whitcher '19; reporter, Viola Engle '19; faculty adviser, Professor Charles H. Ashton; program committee, Rachel Bell '19, Viola Engle '19, Faye Dodderidge '19, Josephine Montague '19.

Below are given the programs for the winter and spring quarters.

December 30, 1918: "Chief contributions of mathematics to astronomy" by Professor Ellis B. Stouffer.

January 13, 1919: "The slide rule" by Evelina Watt '20.
January 27: "Valid aims of teaching mathematics in secondary schools"[1] by Josephine Montague '19.
February 10: "Russian peasant method of calculation" by Viola Engle '19; "Probability curves" by Beatrice Hagen '20.
February 24: "Abridged notation" by Jessie Craig '20.
March 10: "History of the calculus" by Vesta Shafer '19.
March 24: "Trilinear coördinates" by D'Estell Tremaine '19.
April 7: "Method of reciprocal polars" by Marie Brown '19.
April 21: "Origin of logarithms" by Carroll McDowell '19; "Polar planimeter" by Ruth Kelsey '20.
May 5: "Methods of projection" by Hazel Quick '19.
May 19: "Projective reflection" by Nellie Young '19.
June 2: Annual picnic.

THE JUNIOR MATHEMATICS CLUB, University of Minnesota, Minneapolis, Minn. [1918, 312].

The only officers of the club for the current year are an executive committee consisting of Professor Raymond W. Brink, chairman, Miss Ella Thorp, Instructor, and Laura Menk '19.

The programs given so far this year are as follows.

December 12, 1918: "Dimensionality" by Dr. Chester H. Yeaton, Instructor; "Leonhard Euler" by Ruth Stephens Gr.
February 6, 1919: "Gottfried Wilhelm Leibniz" by Lois Huney '19; "A geometrical method of summing a geometrical series" by Professor William H. Bussey.

Refreshments were served in the faculty parlor after each meeting.

TOPICS FOR CLUB PROGRAMS.

15. THE NUMBER π.

The fact that at least a dozen times in the club programs published during 1918 one finds topics relating to the number π such as "The number π," "Various definitions of π," "History of π," "Squaring the Circle" (which occurs six times), "Quadrature of the Circle," etc., is evidence that investigations of this remarkable number are still as interesting as they were in the time of Archimedes.

Since no member of a mathematics club is a "paradoxer," in De Morgan's sense of the term, little of this interest can be credited to the fact that the problem of the quadrature of the circle is, as De Morgan[2] put it, "connected with one of those propensities, the love of the marvellous, which, carried to an undue extent, tend more than others to throw the mind off its balance, and destroy the comfort

[1] *Cf.* "Valid aims and purposes for the study of mathematics in secondary schools," by A. Davis, *School Science and Mathematics*, vol. XVIII, pp. 112–123, 208–220, 313–324. (Committee Report, Mathematics Club of Chicago.)

[2] In his article "Quadrature of the Circle" in the *Penny Cyclopedia*, vol. 19, p. 186.

of the individual." In fact, the natural interest of mathematicians has undoubtedly been greatly lessened by the activities of "circle-squarers" of the type made famous by De Morgan's witty satire.

The desire for a better understanding of transcendental numbers leads the student of mathematics at the present time to a scientific interest in π, since π was the first transcendental number encountered by humanity (although not the first whose transcendence was proved) and it seems natural to think of it as the simplest and most familiar of the transcendental numbers. That π was first discovered in its relation to the circle is, of course, a mere accident. It was long since pointed out[1] that π is a number which occurs in various natural relations and would enter into analysis from whatever side the subject was approached.

When the ancient Greeks first attempted to find the length of the diagonal of a unit square, they were probably of the opinion that such a fraction (greater than unity, of course) existed, but that they were merely unable to determine it. Undoubtedly many a Greek mathematician had spent much time in fruitless testing of fractions, hoping to find one whose square was 2, before the suspicion arose that no such fraction existed and some one finally succeeded in making the proof to that effect as given by Euclid. Even then they did not conceive of a new number, different in kind from any they knew but no less definite, and invent a symbol to represent it.

Of a somewhat similar nature was the problem of determining the number which should represent the ratio of a circle to its diameter. The problem was, however, much more difficult, for two reasons. In the first place, it was much more difficult to test whether or not a given number was greater or less than the desired number and, in the second place, they were unable to prove that no such rational number existed. It must have been many centuries after the first approximate values were determined before any one suspected that there might be a number, definite and exact, but entirely different in nature from any integer or fraction, which represented the ratio of a circumference to its diameter. It was many more centuries before the first definite information regarding the true nature of π was established when J. H. Lambert, in 1761, communicated to the Berlin Academy an essentially rigorous proof[2] of the irrationality of π. Even

[1] *Cf.* DeMorgan, *Budget of Paradoxes* (London, 1872), pp. 171–172, second ed., edited by D. E. Smith (Chicago, 1915), vol. 1, pp. 284–286; also, Ball, *Mathematical Recreations and Essays*, 4th ed. (London, 1905), pp. 249–50, 5th ed. (London, 1911), p. 295.

[2] The usual citation for Lambert's proof is his "Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques," *Mémoires de l'Académie de Berlin* for 1761, Berlin, 1768, pp. 265–322. It is also cited as published in *Beiträge zum Gebrauche der Mathematik*, Bd. II, Berlin, 1770, S. 140–149.

The following paragraph from E. W. Hobson's *Squaring the Circle*, Cambridge, 1913, p. 44, is worthy of note in this connection.

"It has frequently been stated that the first rigorous proof of Lambert's results is due to Legendre (1752–1833), who proved these theorems in his *Eléments de Géométrie* (1794), by the same method, and added a proof that π^2 is an irrational number. The essential rigour of Lambert's proof has however been pointed out by Pringsheim (*Münch. Akad. Ber.*, Kl. 28, 1898), who has supplemented the investigation in respect of the convergence."

before Lambert's proof was given men were coming to believe that π was not only irrational but not an algebraic irrational. This belief was rendered the more probable by Liouville's proof[1] in 1840 of the existence of transcendental numbers, and finally confirmed by Lindemann's proof[2] in 1882 of the transcendence of π.

The literature of the subject is abundant and steadily increasing. For various articles in the periodical literature the reader will, of course, consult the mathematical encyclopedias, the Royal Society Index (especially pp. 233 and 434–436) and the volumes of the International Catalogue. Among the best recent special discussions of the subject are those of Hobson,[3] Young,[4] Beman and Smith,[5] Ball,[6] Enriques,[7] Teixeira,[8] Rudio,[9] Schubert,[10] and Tropfke.[11]

In the books of Beman and Smith, Young and Hobson the development of the subject is considered as falling into three periods, the last named author devoting a chapter to each period.

The first period may be characterized as the *empirical* period, extending from the earliest attempts at the quadrature of the circle to the invention of the calculus in the second half of the seventeenth century. During this period approximations for π are obtained by purely geometrical means until the limit of refinement of that method is reached.

The second period may be characterized as the *analytic* period, extending from the invention of the calculus about a century to the proof of the irrationality of π by Lambert in 1761. During this period, by means of the more powerful methods of the new analysis, π is expressed in terms of infinite products,

[1] The simpler of Liouville's methods is given by Hobson, *loc. cit.*, pp. 44–46.

[2] *Mathematische Annalen*, Bd. 20, S. 220–224, and *Berichte der Berliner Akademie*, 1882, Bd. 2, S. 679–682. Lindemann's proof is based upon Hermite's proof (*Comptes Rendus*, vol. 77, pp. 18–24, 74–79, 226–233, 285–292) of the transcendence of e.

Simplified forms of Hermite's and Lindemann's proofs are given in the books by J. W. A. Young (pp. 402–416) and Beman and Smith (pp. 61–67) cited below, and in various other books readily accessible.

[3] E. W. Hobson, *Squaring the Circle*, Cambridge University Press, 1913.

[4] J. W. A. Young, *Monographs on Modern Mathematics*, New York, 1911, Monograph IX, "The History and Transcendence of π," by D. E. Smith.

[5] W. W. Beman and D. E. Smith, *Famous Problems of Elementary Geometry*, Boston, 1897, English translation of Klein's *Vorträge uber ausgewählte Fragen der Elementar-Geometrie*, Leipzig, 1895, pp. 54–80.

[6] W. W. R. Ball, *Mathematical Recreations and Essays*, 5th ed., London, 1911, pp. 293–306; 4th ed., London, 1905, pp. 247–261.

[7] F. Enriques, *Fragen der Elementargeometrie*, Deutsche Ausgabe von H. Fleischer, II Teil, Leipzig, 1907.

[8] F. G. Teixeira, *Sur les problèmes célèbres de la géométrie élémentaire non résolubles avec la règle et le compas*, Coïmbre, 1915, pp. 83–104.

[9] F. Rudio, *Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über die Kreismessung*, Leipzig, 1892. See also articles "Der Bericht des Simplicius uber die Quadraturen des Antiphon und des Hippokrates" and "Zur Rehabilitation des Simplicius" by Rudio in *Bibliotheca Mathematica*, 1902, pp. 7–62, and 1903, pp. 13–18, 1907, pp. 13–22. Also a discussion of Rudio's first article by P. Tannery in the same volume (Bib. Math. for 1902), pp. 342–349.

[10] H. Schubert, *Mathematical Essays and Recreations*, translated by T. J. McCormack, Chicago, 1910, pp. 112–143.

[11] J. Tropfke, *Geschichte der Elementar-Mathematik*, Bd. 2, Leipzig, 1903, S. 108–138.

continued fractions and infinite series. Approximations[1] are obtained far beyond any conceivable practical need and mathematicians come to suspect that π is not only irrational but not an algebraic irrational.

The third period may be called the *critical* period, since it was devoted to "critical investigations of the true nature of the number π itself, considered independently of mere analytical representation."[2] The period extends from the middle of the eighteenth century late into the nineteenth century when the transcendence of π is finally definitely established.

William Jones[3] seems to have been the first to make use of the symbol π with its present special significance, but its permanent use as such was chiefly due to the influence of Euler.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

2773. Proposed by JOSEPH ROSENBAUM, Milford, Conn.

Point out the fallacy in the proof following the problem:

In the triangle $A_1B_1C_1$ let M be a point such that the sum of the distances from it to the sides is a maximum; also, let $A_2B_2C_2$ be a triangle formed by drawing lines through the vertices A_1, B_1, and C_1 parallel to their opposite sides. Then the sum of the distances from M to the sides of the triangle $A_2B_2C_2$ is a minimum.

Proof.—Because the sides of the two triangles are parallel in pairs, the sum of the distances from a variable point P in triangle $A_1B_1C_1$ to the six sides of the two triangles is constant. Now by hypothesis M is a point for which one part of this constant sum is a maximum, and hence it follows that the other part is a minimum.

2774. Proposed by FRANK IRWIN, University of California.

Evaluate the circulants

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & 1 & 2 & \cdots & n-2 & n-1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 3 & 4 & \cdots & n & 1 \end{vmatrix}, \qquad \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{vmatrix},$$

where, in the latter, $a_1, a_2, \cdots a_n$ form an arithmetical progression.

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[1] A condensed table of approximations of π as determined by various men is given by J. W. L. Glaisher, *Messenger of Mathematics*, vol. 2, pp. 122–128, and in vol. 3, pp. 45–46, some corrections are made of the table given on p. 122 of vol. 2. The approximation was carried to 707 places by William Shanks in 1873 (*Proceedings of the Royal Society of London*, vol. 21, p. 318 and vol. 22, p. 45). A considerable list of approximations is given by Ball, *loc. cit.*, 4th ed., pp. 250–261, 5th ed., pp. 296–306.

[2] Hobson, *l. c.*, p. 12.

[3] *Synopsis Palmariorum Matheseos*, London, 1706, pp. 243, 263, *et seq.* Cited by Ball, *loc. cit.*, 4th ed., p. 250, 5th ed., p. 296, and by others. Concerning its early use by others, see article "Sur le premier emploi du symbole π pour 3.14159 $\cdots$" by G. Eneström, *Bibliotheca Mathematica*, 1889, p. 28.

2775. Proposed by H. T. BURGESS, University of Wisconsin.

Solve in finite form, if possible, the differential equation

$$\frac{d^2y}{dt^2} + ay\frac{dy}{dt} + by = g,$$

when a and b are arbitrary constants and $g = 32.16$. When $t = 0$, $y = 0$, $dy/dt = 0$.

As solutions have not been received for more than 130 problems proposed since January, 1913, the number of *new* problems proposed each month is to be considerably reduced and *old* problems are to be reproposed. Solutions are desired for the following problems proposed before January, 1918:

Algebra—406, 411, 416, 417, 461, 481, 494.
Geometry—442, 446, 455, 463, 470, 472, 476, 477, 478, 499, 501, 510, 519, 523.
Calculus—348, 349, 353, 406, 415, 429, 432, 434, 436.
Mechanics—287, 291, 300, 308, 309, 313, 315, 332, 343, 344, 351.
Number Theory—198, 201, 202, 205, 231, 232, 234, 238, 245, 247, 263, 266, 270, 272, 273, 274, 275.

430 (Algebra) [March, 1915]. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the following equations both algebraically and graphically:

$$x^y + y^x = xy, \qquad x^x + y^y = x + y.$$

339 (Calculus) [June, 1913]. Proposed by T. H. GRONWALL, New York, N. Y.

To show that for any real value of x

$$\left|\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)\right| \leqq \frac{1}{n+1}, \quad \text{and} \quad \left|\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right)\right| \leqq \frac{1}{n+1}.$$

340 (Calculus) [June, 1913]. Proposed by C. N. SCHMALL, New York, N. Y.

A pencil of parallel rays of light is incident upon a lens whose faces have the radii r_1, r_2, respectively. Show that the distance of the principal focus from the center of the first face of the lens will be a maximum or a minimum when

$$\frac{r_1}{r_2} = \frac{(\mu - 1)^{1/2}}{1 + (\mu - 1)^{1/2}},$$

where μ has its usual meaning.

272 (Mechanics) [February, 1913]. Proposed by J. F. LAWRENCE, Stillwater, Okla.

A perfectly rough circular cylinder is fixed with its axis horizontal. A sphere is placed on it in a position of unstable equilibrium, and projected with a given velocity parallel to the axis of the cylinder. If the sphere be slightly disturbed in a horizontal direction perpendicular to the direction of the axis of the cylinder, determine at what point the sphere will leave the cylinder.

277[1] (Mechanics) [June, 1913]. Proposed by W. J. GREENSTREET, Editor of the Mathematical Gazette, England.

Around a smooth fixed circular pulley is wound a massless inextensible string, and straight portions go to two free ends A and B to which masses are fastened. The mass at A is initially projected perpendicular to the string while the other is initially at rest. The length of the straight portion to the first mass is initially l and subsequently is r. Find the velocity of the second mass at that moment.

[1] Incorrectly numbered 272 when first proposed.

279[1] (Mechanics) [September, 1913]. Proposed by W. W. LANDIS, Dickinson College.

A dam backs up the water for two miles. If the dam is raised 18 inches, will the water two miles up the stream be raised 18 inches, more or less?

191[2] (Number Theory) [June, 1913]. Proposed by L. E. DICKSON, University of Chicago.

Find an amicable number triple by solving one of the equations (other than the last) in the MONTHLY, March, 1913, page 92. Note that a solution a is to be excluded if not prime to the numbers in the same line.

192[3] (Number Theory) [June, 1913]. Proposed by the late ARTEMAS MARTIN.

Find rational values for v, w, and x that will satisfy simultaneously the conditions

$$(m^2+n^2)(v^2+w^2+x^2)^2-4m^2n^2v^2+m^2n^2(m^2+n^2)=0,$$
$$(m^2+n^2)(v^2+w^2+x^2)^2-4m^2n^2w^2+m^2n^2(m^2+n^2)=0,$$
$$(m^2+n^2)(v^2+w^2+x^2)^2-4m^2n^2x^2+m^2n^2(m^2+n^2)=0,$$

m and n being known quantities.

196[4] (Number Theory) [September, 1913]. Proposed by CHARLES MACAULAY, Chicago, Ill.

Combinations containing an even number of letters are formed of the letters a, b, c, d, etc. It is required to place the letters in two columns, so that half the letters in every combination are placed in one column and the other letters of the combination in the other column, and so that all the a's are placed in the same column; all b's in the same column; all the c's in the same column, etc.

SOLUTIONS OF PROBLEMS.

2667 [January, 1918]. Proposed by E. L. REES, University of Kentucky.

Given one diagonal of a parallelogram and the area of the rectangle whose sides are equal to those of the parallelogram, construct the parallelogram so that the diagonal shall make a given angle, α, with a given line and so that the sum of the angles that two adjacent sides make with this line shall be equal to a given angle, β.

2682 [March, 1918]. Proposed by E. L. REES, University of Kentucky.

Given the diagonal and the angle it makes with the bisector of one of the angles of a parallelogram. Construct the parallelogram so that the rectangle having sides equal to those of the parallelogram may have a given area.

SOLUTION BY THE PROPOSER.

If we place one end of the diagonal at the origin in the complex plane and let the fixed line be the real axis, it will be seen at once that what we have given is equivalent to the sum and product of the complex numbers represented by two of the vertices of the required parallelogram. Let this sum and product be denoted by a and b respectively, where mod a = length of diagonal, amp $a=\alpha$, mod b = given area, amp $b=\beta$.

The solution of our problem then requires merely the construction of the complex roots of the quadratic equation $x^2-ax+b=0$. This construction is effected by carrying out the operations indicated in the formula $x=a/2\pm\sqrt{[(a/2)^2-b]}$ all of which are possible with ruler and compasses.

It will be noted that problem no. 2682 is a special case of no. 2667 and hence the method of solution here suggested is applicable also to it.

Also solved by H. N. CARLETON.

[1] Incorrectly numbered 274 when first proposed.
[2] Incorrectly numbered 187 when first proposed.
[3] Incorrectly numbered 188 when first proposed.
[4] Incorrectly numbered 192 when first proposed.

2700 [May, 1918]. Proposed by the late ARTEMAS MARTIN.

In a factory 250 men are paid an average wage of \$15 each per week. The men are paid unequally, the wages being \$20, \$16, \$10, and \$8 per week, respectively, for different classes of work. How many are employed at each rate of pay?

NOTE.—I am told that this question was set in a Civil Service examination paper to be worked by arithmetic. 2,896 answers have been found. Are there any more?

SOLUTION BY H. S. UHLER, Yale University.

Let w, x, y, and z denote the number of men receiving \$20, \$16, \$10, and \$8 per week, respectively. The conditions of the problem lead at once to the following equations:

$$w + x + y + z = 250,$$

$$(20w + 16x + 10y + 8z)/250 = 15.$$

Elimination of z gives

$$6w + 4x + y = 875.$$

The last equation must be solved for positive integral values (zero included) of w, x, and y. This may be accomplished by assigning to w the values 0, 1, 2, 3, $\cdots$ and then discussing the number of possible solutions conditioned by the limited arithmetical progressions involving x, y, and z.

Since $y = 875 - 6w - 4x$ it is evident that the greatest value of w which will make y positive is 145. Let w' symbolize any number in the sequence 0, 1, 2, 3, $\cdots$, 144, 145. We may now imagine the following table filled out numerically:

w	x	y	z
w'	x'	$875 - 6w' - 4x'$	$5w' - 625 + 3x'$
w'	$x' - 1$	$875 - 6w' - 4x' + 4$	$5w' - 625 + 3x' - 3$
w'	$x' - 2$	$875 - 6w' - 4x' + 8$	$5w' - 625 + 3x' - 6$
.	. .	. . . . . .	. . . . . .

For a given value of w (w') the number of possible solutions is equal to the greatest number of rows that can be written in the above schematic table without introducing a negative value in one or more of the last three columns.

CASE 1.—*w' odd.* Let $w' = 2k + 1$, then

$$y = 1 + 4(217 - 3k - x'),$$

hence the greatest integral value of x' which will make y positive is $217 - 3k$. Accordingly, when w' is odd, the first or top row of the table will consist of the elements $2k + 1$, $217 - 3k$, 1, and $k + 31$ under the headings w, x, y, and z, respectively. Since the greatest value of w' is 145 the corresponding value of k equals 72, hence the least value of $217 - 3k$ is 1 so that all the elements of the above first row are positive.

Attention will now be directed to the last or bottom row of the table. When the total number of rows is $218 - 3k$ the elements of the last row will be $2k + 1$, 0, $869 - 12k$, and $10(k - 62)$, for the common differences of the second, third, and fourth columns are -1, 4, and -3, respectively. Therefore, as long as k does not fall below 62 the second column will limit the number of rows in the table. Under the conditions that w' be odd and $k \geqq 62$ the number of rows in a table equals $218 - 3k$ so that the number of solutions of the problem is the sum $2 + 5 + \cdots + 29 + 32 = 187$.

For values of k less than 62 the fourth column will limit the number or rows in the table, instead of the second column. Let k be of the form $3m + 2$, then the nth term of the fourth column may be written $z_n = 3(12 + m - n)$. This will vanish whenever $n = 12 + m$. Positive solutions for k, equal to $3m + 2$, and less than 62, are obtained when $m = 19, 18, \cdots$, or $k = 59, 56, \cdots$, or $w' = 119, 113, \cdots$. When $k = 3m + 3$ and $k = 3m + 4$ the $(12 + m)$th terms of the fourth column will be 1 and 2, respectively. On the other hand, the $(13 + m)$th terms will be -2 and -1, in the same order. Consequently, for odd values of w', corresponding to values of k less than 62, the tables may be collected in groups of three having the same number of rows or possible solutions. The number of rows is evidently $m + 12$. If we allow m to decrease from 19 to -1, or k from 61, $(3m + 4)$, to -1, $(3m + 2)$, we shall obtain not only

the number of possible solutions pertaining to values of w' from 123 to 1 but also one inadmissible solution belonging to $w' = -1$. Consequently, the number of possible solutions associated with odd values of w' and not included in the 187 given above is equal to $3(31 + 30 + \cdots + 12 + 11) - 11 = 1312$. Finally, the total number of possible solutions of the problem, when w' is odd, equals $187 + 1312 = 1499$.

CASE 2.—*w' even.* Let $w' = 2k$, then

$$y = 3 + 4(218 - 3k - x'),$$

hence $218 - 3k$ is the greatest integral value of x' that will make y positive. Accordingly, when w' is even, the first row of the table will comprise the constituents $2k$, $218 - 3k$, 3, and $29 + k$ under the headings w, x, y, and z, respectively. Since the greatest even value of w' is 144 the corresponding value of k equals 72, hence the least value of $218 - 3k$ is 2 so that all the elements of the above first row are positive.

When the total number of rows is $219 - 3k$ the elements of the last row will be $2k$, 0, $875 - 12k$, and $10k - 625$. Therefore, as long as k is not less than 63 the second column will limit the number of rows in the table. Under the present conditions the number of rows in a table equals $219 - 3k$ so that the number of solutions of the problem is the sum

$$3 + 6 + \cdots + 27 + 30 = 165.$$

For values of k less than 63 the fourth column will limit the number of rows in the table. Let k be of the form $3m + 1$, then the nth term of the fourth column may be written $z_n = 3(11 + m - n)$. This will vanish whenever $n = 11 + m$. Positive solutions for k, equal to $3m + 1$, and less than 63, are obtained when $m = 20, 19, \cdots$, or $k = 61, 58, \cdots$, or $w' = 122, 116, \cdots$. When $k = 3m + 2$ and $k = 3m + 3$ the $(11 + m)$th terms of the fourth column will be 1 and 2 respectively. On the other hand, the $(12 + m)$th terms will be -2 and -1, in the order named. Consequently, for even values of w', corresponding to values of k less than 63, the tables may be collected in groups of three having the same number of rows or possible solutions. The number of rows is obviously $m + 11$. If we allow m to decrease from 20 to 0, or k from 63, $(3m + 3)$, to 1, $(3m + 1)$, we shall include the case of $w' = 126$ which has already been disposed of and omit the number of solutions for $w' = 0$, namely 10. Consequently, the number of solutions not already accounted for equals $3(31 + 30 + \cdots + 12 + 11) - 31 + 10 = 1302$. Finally, the total number of possible solutions of the problem, when w' is even (or zero), equals $165 + 1302 = 1467$. Therefore, for all admissible values of w', the complete number of solutions is 2966. Hence 70 more solutions exist than the proposer states have already been found.

REMARK.—The number of solutions involving one or more zeros is 72.

N = number of solutions in the table.
N' = number of rows containing one or more zeros.
Only the first and last rows are given for each value of w.

w	x	y	z	N	N'	w	x	y	z	N	N'	w	x	y	z	N	N'
145	1	1	103	2	1	137	13	1	99	14	1	129	25	1	95	26	1
145	0	5	100			137	0	53	60			129	0	101	20		
144	2	3	101	3	1	136	14	3	97	15	1	128	26	3	93	27	1
144	0	11	95			136	0	59	55			128	0	107	15		
143	4	1	102	5	1	135	16	1	98	17	1	127	28	1	94	29	1
143	0	17	90			135	0	65	50			127	0	113	10		
142	5	3	100	6	1	134	17	3	96	18	1	126	29	3	92	30	1
142	0	23	85			134	0	71	45			126	0	119	5		
141	7	1	101	8	1	133	19	1	97	20	1	125	31	1	93	32	1
141	0	29	80			133	0	77	40			125	0	125	0		
140	8	3	99	9	1	132	20	3	95	21	1	124	32	3	91	31	0
140	0	35	75			132	0	83	35			124	2	123	1		
139	10	1	100	11	1	131	22	1	96	23	1	123	34	1	92	31	0
139	0	41	70			131	0	89	30			123	4	121	2		
138	11	3	98	12	1	130	23	3	94	24	1	122	35	3	90	31	1
138	0	47	65			130	0	95	25			122	5	123	0		

w	x	y	z	N	N'
121	37	1	91	31	0
121	7	121	1		
120	38	3	89	30	0
120	9	119	2		
119	40	1	90	31	1
119	10	121	0		
118	41	3	88	30	0
118	12	119	1		
117	43	1	89	30	0
117	14	117	2		
116	44	3	87	30	1
116	15	119	0		
115	46	1	88	30	0
115	17	117	1		
114	47	3	86	29	0
114	19	115	2		
113	49	1	87	30	1
113	20	117	0		
112	50	3	85	29	0
112	22	115	1		
111	52	1	86	29	0
111	24	113	2		
110	53	3	84	29	1
110	25	115	0		
109	55	1	85	29	0
109	27	113	1		
108	56	3	83	28	0
108	29	111	2		
107	58	1	84	29	1
107	30	113	0		
106	59	3	82	28	0
106	32	111	1		
105	61	1	83	28	0
105	34	109	2		
104	62	3	81	28	1
104	35	111	0		
103	64	1	82	28	0
103	37	109	1		
102	65	3	80	27	0
102	39	107	2		
101	67	1	81	28	1
101	40	109	0		
100	68	3	79	27	0
100	42	107	1		
99	70	1	80	27	0
99	44	105	2		
98	71	3	78	27	1
98	45	107	0		
97	73	1	79	27	0
97	47	105	1		

w	x	y	z	N	N'
96	74	3	77	26	0
96	49	103	2		
95	76	1	78	27	1
95	50	105	0		
94	77	3	76	26	0
94	52	103	1		
93	79	1	77	26	0
93	54	101	2		
92	80	3	75	26	1
92	55	103	0		
91	82	1	76	26	0
91	57	101	1		
90	83	3	74	25	0
90	59	99	2		
89	85	1	75	26	1
89	60	101	0		
88	86	3	73	25	0
88	62	99	1		
87	88	1	74	25	0
87	64	97	2		
86	89	3	72	25	1
86	65	99	0		
85	91	1	73	25	0
85	67	97	1		
84	92	3	71	24	0
84	69	95	2		
83	94	1	72	25	1
83	70	97	0		
82	95	3	70	24	0
82	72	95	1		
81	97	1	71	24	0
81	74	93	2		
80	98	3	69	24	1
80	75	95	0		
79	100	1	70	24	0
79	77	93	1		
78	101	3	68	23	0
78	79	91	2		
77	103	1	69	24	1
77	80	93	0		
76	104	3	67	23	0
76	82	91	1		
75	106	1	68	23	0
75	84	89	2		
74	107	3	66	23	1
74	85	91	0		
73	109	1	67	23	0
73	87	89	1		
72	110	3	65	22	0
72	89	87	2		

w	x	y	z	N	N'
71	112	1	66	23	1
71	90	89	0		
70	113	3	64	22	0
70	92	87	1		
69	115	1	65	22	0
69	94	85	2		
68	116	3	63	22	1
68	95	87	0		
67	118	1	64	22	0
67	97	85	1		
66	119	3	62	21	0
66	99	83	2		
65	121	1	63	22	1
65	100	85	0		
64	122	3	61	21	0
64	102	83	1		
63	124	1	62	21	0
63	104	81	2		
62	125	3	60	21	1
62	105	83	0		
61	127	1	61	21	0
61	107	81	1		
60	128	3	59	20	0
60	109	79	2		
59	130	1	60	21	1
59	110	81	0		
58	131	3	58	20	0
58	112	79	1		
57	133	1	59	20	0
57	114	77	2		
56	134	3	57	20	1
56	115	79	0		
55	136	1	58	20	0
55	117	77	1		
54	137	3	56	19	0
54	119	75	2		
53	139	1	57	20	1
53	120	77	0		
52	140	3	55	19	0
52	122	75	1		
51	142	1	56	19	0
51	124	73	2		
50	143	3	54	19	1
50	125	75	0		
49	145	1	55	19	0
49	127	73	1		
48	146	3	53	18	0
48	129	71	2		
47	148	1	54	19	1
47	130	73	0		

w	x	y	z	N	N'	w	x	y	z	N	N'	w	x	y	z	N	N'
46	149	3	52	18	0	30	173	3	44	15	0	14	197	3	36	13	1
46	132	71	1			30	159	59	2			14	185	51	0		
45	151	1	53	18	0	29	175	1	45	16	1	13	199	1	37	13	0
45	134	69	2			29	160	61	0			13	187	49	1		
44	152	3	51	18	1	28	176	3	43	15	0	12	200	3	35	12	0
44	135	71	0			28	162	59	1			12	189	47	2		
43	154	1	52	18	0	27	178	1	44	15	0	11	202	1	36	13	1
43	137	69	1			27	164	57	2			11	190	49	0		
42	155	3	50	17	0	26	179	3	42	15	1	10	203	3	34	12	0
42	139	67	2			26	165	59	0			10	192	47	1		
41	157	1	51	18	1	25	181	1	43	15	0	9	205	1	35	12	0
41	140	69	0			25	167	57	1			9	194	45	2		
40	158	3	49	17	0	24	182	3	41	14	0	8	206	3	33	12	1
40	142	67	1			24	169	55	2			8	195	47	0		
39	160	1	50	17	0	23	184	1	42	15	1	7	208	1	34	12	0
39	144	65	2			23	170	57	0			7	197	45	1		
38	161	3	48	17	1	22	185	3	40	14	0	6	209	3	32	11	0
38	145	67	0			22	172	55	1			6	199	43	2		
37	163	1	49	17	0	21	187	1	41	14	0	5	211	1	33	12	1
37	147	65	1			21	174	53	2			5	200	45	0		
36	164	3	47	16	0	20	188	3	39	14	1	4	212	3	31	11	0
36	149	63	2			20	175	55	0			4	202	43	1		
35	166	1	48	17	1	19	190	1	40	14	0	3	214	1	32	11	0
35	150	65	0			19	177	53	1			3	204	41	2		
34	167	3	46	16	0	18	191	3	38	13	0	2	215	3	30	11	1
34	152	63	1			18	179	51	2			2	205	43	0		
33	169	1	47	16	0	17	193	1	39	14	1	1	217	1	31	11	0
33	154	61	2			17	180	53	0			1	207	41	1		
32	170	3	45	16	1	16	194	3	37	13	0	0	218	3	29	10	10
32	155	63	0			16	182	51	1			0	209	39	2		
31	172	1	46	16	0	15	196	1	38	13	0						
31	157	61	1			15	184	49	2								

Also solved by L. P. Shidy, R. A. Johnson, F. H. Loud, H. N. Carleton, B. F. Yanney.

NOTES AND NEWS.

Edited by E. J. Moulton, Northwestern University, Evanston, Ill.

Mr. Warren Weaver has been appointed instructor in mathematics at the University of Wisconsin.

Mr. P. A. Fraleigh, of Cornell University, has been appointed instructor in mathematics at Dartmouth College.

Professor O. D. Kellogg, of the University of Missouri, has been appointed lecturer in Harvard University for the year 1919–20.

Dr. W. C. GRAUSTEIN, who has given up his work at the Aberdeen Proving Ground, and who has been teaching in Harvard University this spring, has been promoted to be an assistant professor of mathematics.

Professor W. H. GARRETT, of Baker University, is on leave of absence for this Spring, his duties being assumed by Mr. F. E. WOOD, who has been in the national service until quite recently.

Dr. C. E. WILDER, who is now at Clark University, has been appointed assistant professor of mathematics at Northwestern University.

At Wellesley College, Miss HELEN BARTON, now in charge of the department of Science and Mathematics at Salem College, and Miss MARION ELIZABETH STARK, head of the department of Mathematics at Meredith College, have been appointed instructors in mathematics.

At the U. S. Naval Academy at Annapolis the following promotions in the Department of Mathematics are announced as going into effect on April 1: to be Professor, Mr. PAUL CAPRON; to be Associate Professors, Mr. W. J. KING and Mr. J. B. EPPES; to be Assistant Professors, Mr. J. A. BULLARD, Mr. JOHN TYLER, Mr. ARTHUR KIERNAN, Mr. J. N. GALLOWAY, Mr. ALEXANDER DILLINGHAM, and Dr. G. R. CLEMENTS.

Many mathematicians who have called on Mr. ROBERT BOWES, of Bowes and Bowes, at No. 1 Trinity Street, Cambridge (the home of a succession of booksellers for more than four hundred years) will learn with deep regret of his death on February 9, 1919, in the eighty-fourth year of his age. About a year before, the University of Cambridge conferred on him the honorary degree of Master in Arts.

Mr. I. ROMAN, former instructor at Northwestern University, has received his discharge at the Aberdeen Proving Ground, and is studying this Spring at the University of Chicago.

Professor H. F. BLICHFELDT, of Leland Standford University, and Professor H. H. Mitchell, of the University of Pennsylvania, have completed their work at the Aberdeen Proving Ground and returned to their former duties.

Major OSWALD VEBLEN was abroad studying ordnance problems and methods in England, France and Italy, from November to March. He hopes to return to his University work at Princeton in the near future.

Major F. R. MOULTON has completed his work in the Ordnance Department at Washington and received his discharge. He has resumed his duties as professor of astronomy at the University of Chicago this Spring Quarter.

Major J. L. COOLIDGE of Harvard University is reported acting as a liaison officer in the office of one of the French Commissioners in Paris.

On March 1, WILLIAM BURNSIDE retired from his professorship of mathematics at the Royal Naval College, Greenwich. Fellow of Pembroke, second wrangler, first Smith prizeman, Royal Society medallist, and De Morgan medallist, he is known in America more particularly through his work on the *Theory of groups of finite order* (Cambridge, 1897; 2d ed., 1911).

Dr. H. S. WASHINGTON, of the geophysical laboratory, Carnegie Institution, has been elected a foreign member of the Accademia dei Lincei.

Science announces that JACQUES HADAMARD, professor in the Collège de France, has accepted an invitation from Yale University to be a Silliman Lecturer in the Spring of 1920. Professor Hadamard received the honorary degree of LL.D. at the Yale Bicentennial in 1901.

The Adams prize, of value £250, has been awarded to J. W. NICHOLSON, professor of mathematics at King's College, University of London. This prize, "awarded every two years for an essay on some branch of pure mathematics, astronomy, or other branch of natural philosophy" was founded by members of St. John's College, Cambridge, in 1848 in memory of "the first among mathematicians of Europe to determine from perturbations the unknown place of a disturbing planet exterior to Uranus." In 1907 the award was made to Professor E. W. BROWN. Earlier awards were made to J. C. MAXWELL, I. TODHUNTER and E. J. ROUTH.

Nature reports that at Cambridge University, the senate has approved of provision being made for the establishment of the degree of doctor of philosophy. "The syndicate dealing with this question recommends that, subject to certain exemptions, candidates for the degree, before submitting a dissertation, must have pursued a course of research for not less than three years, and the senate has determined that of this period one year, in the case of a graduate of the university, and two years, in the case of other students, must be spent in Cambridge."

In a recent letter to Professor H. E. SLAUGHT, Professor E. R. HEDRICK tells something of his new work in France in helping to organize educational work for the American Expeditionary Forces. The Army Educational Commission (Professors ERSKINE, BUTTERFIELD, and SPAULDING) acting with the new Army Educational Division of general headquarters under General REES (who was in charge of the American S. A. T. C.) has undertaken the establishment at Beaune, in the immense U. S. hospital buildings (with beds for 15,000) a real university to be constituted under the direction of American Heads of Departments with instructors drawn from officers of the A. E. F., and students (all high school

graduates) from the enlisted personnel of the A. E. F. The university was expected to open with 5,000 students and eventually to enroll nearly 20,000 students. About a hundred instructors in mathematics will be needed. Professor Hedrick reported the following as already available as instructors: Mr. JOSEPH ALLEN, Professor H. E. BUCHANAN, Major W. L. HART, Captain REHM, Captain CALDWELL, Captain FENTRESS, Captain SCHNAPP, Lieutenant NELSON, Lieutenant DILLON, and Miss MAY H. VANN.

Secretary of the Navy Daniels has approved an order establishing the usual collegiate grades of instructor, assistant professor, associate professor, and professor for the civilian teachers at the United States Naval Academy, with a schedule of salaries and promotions. The initial salaries for these grades are $2,000, $2,400, $3,000, and $3,600 per annum, respectively, with an automatic increase of $100 per year for each of the two lower grades, and of 10 per cent. of the initial pay of the grade for each five years of service at each of the two higher grades, with maxima of $2,400, $3,000, $3,600, and $4,500 respectively. All new appointments are to be at grade instructor, with promotion by selection to the next higher grade, subject to a requirement of two, five, and ten years respectively of continuous service at the Naval Academy for eligibility to the professorial grades, and subject to the further restriction that for each ten men or major part thereof in any department, there shall not be more than two assistant professors, two associate professors and one professor.

While this scheme does not have the force of statute, the civilian teachers at the Naval Academy being paid from a lump sum appropriation to be expended subject to the approval of the Secretary of the Navy, it is not anticipated that there will be any recessions from the provisions of this plan. There are several instructorships to be filled for the coming year. A competitive examination for the selection of these men (age, between 28 and 40 years) will be given at Annapolis, Md., June 10–12. Inquiries concerning the details of this examination should be addressed to Professor D. M. Garrison, head of the department of mathematics, U. S. Naval Academy.

In addition to the reports of summer courses in mathematics given in the March and April numbers of the MONTHLY, we have the following:

Northwestern University: June 23–August 15. Professor E. J. MOULTON: Differential calculus; Plane trigonometry. Mr. F. L. KERR: Plane analytic geometry; College algebra. Each course bears a credit of three semester-hours.

Cornell University: regular summer session, July 7–August 15. Professor VIRGIL SNYDER: Advanced geometry, credit 5 hours; Solid geometry, 3 hours. Professor D. C. GILLESPIE: Higher analysis, 5 hours; Algebra, 3 hours. Professor W. A. HURWITZ: Differential equations of physics, 3 hours; Analytics and a part of differential calculus, 5 hours. Professor CARVER: Trigonometry, 3 hours. Special session on account of the war, July 7–August 29. Professors F. R. SHARPE, C. E. CRAIG and F. W. OWENS: Integral calculus, 3 hours; Professor

OWENS: Advanced calculus III, 3 hours. Mrs. HELEN B. OWENS: Differential equations III, 3 hours. The last two courses are completions of those begun in January for a year's work.

University of Colorado: First term, June 30th to August 2d. Second term, August 4th to September 6th. *Instructors:* Associate Professor ABRAHAM COHEN, Johns Hopkins University; Assistant Professor LIGHT and Miss CLARIBEL KENDAL, University of Colorado. *Courses:* (1) Solid geometry (first term); (2) Trigonometry (throughout quarter); (3) College algebra (throughout quarter); (4) Plane analytic geometry (throughout quarter); (5) Differential calculus (throughout quarter); (6) History of mathematics (first term); (7) Fundamental concepts of mathematics (first term); (8) Differential equations (first term); (9) Least squares (first term); (10) Calculus of variations (first term); (11–16) Other advanced courses in mathematics. (*a*) Theory of algebraic equations, (*b*) Definite integrals, (*c*) Theory of a complex variable, (*d*) Elliptic integrals and functions, (*e*) An introductory course in analysis, (*f*) Differential geometry (first or second term).

University of Texas: First term, June 10–July 22. Professor H. Y. BENEDICT: The subject matter and teaching of High School mathematics; Algebra. Professor RICE: Solid analytic geometry; Analytic geometry. Professor E. L. DODD: Advanced statistics; Mathematics of investment. Dr. GOLDIE P. HORTON: Calculus; Introduction to analytic geometry. Miss M. E. DECHERD: Plane trigonometry; Solid geometry. Second term, July 22–August 30. Professor M. B. PORTER: Fundamentals in elementary mathematics; Calculus. Professor OGLESBY (of Williams and Mary College): Analytic geometry; Introduction to analytic geometry. Professor H. J. ETTLINGER: Calculus; Plane trigonometry.

Harvard University: There will be two sessions this year of the summer school of arts and sciences, the first extending from July 1 to August 9 and the second from August 11 to September 13. Separate courses in trigonometry and analytic geometry will be given during each session, during the first by Professor C. L. BOUTON and during the second by Professor G. D. BIRKHOFF. Professor BIRKHOFF will give a course beginning the differential and integral calculus during the first session, provided a sufficient number of students sent their names to Professor W. F. OSGOOD, chairman of the division of mathematics, before May 1, and subject to the same provision he will give, also during the first session, a slightly more advanced course in the calculus, presupposing a knowledge of the introductory portions of the calculus. Advanced work in mathematics for graduate students is offered during both sessions, during the first by Professor BOUTON and during the second by Professor BIRKHOFF. Each of the courses mentioned, carried satisfactorily for one session, counts as a half-course for a degree.

Columbia University: July 7–August 15. A. Graduate courses. Professor JAMES MACLAY Geometric constructions. Professor EDWARD KASNER: Graphical methods including nomography and applications of the calculus. Professor W. B. FITE: Functions of a complex variable. B. Undergraduate courses.

Professors FITE, MACLAY and L. P. SICELOFF, and Doctors C. A. FISCHER, LAMSON, G. W. MULLINS and J. F. RITT, and Mr. POST: Differential and integral calculus, Analytics, College algebra, Trigonometry, Plane and solid geometry, and Elementary algebra. Each graduate course is counted as 3 points credit, each undergraduate course as 2 points. In addition to the above, ten courses are offered in Teachers College on the teaching of mathematics and on applied mathematics by Professor C. B. UPTON, Mr. W. E. BRECKENRIDGE and Mr. W. S. SCHLAUCH.

At the April meeting of the *National Academy of Sciences* Major OSWALD VEBLEN and Professor EDWIN BIDWELL WILSON were elected Fellows.

At the celebration, on September 21, 1918, of the 250th anniversary of the founding of the University of Lund, degrees of doctor *honoris causa* were conferred on two mathematicians, Gustaf Eneström, the editor of *Bibliotheca Mathematica*, and J. L. W. V. JENSEN, telephone-engineer of Copenhagen.

The Rumford Committee of the American Academy of Arts and Sciences has voted the sum of $500.00 to Professor A. G. WEBSTER of Clark University, in aid of his researches in pyrodynamics and practical interior ballistics.

The Academy of Sciences of the Institute of France announces the award of the following prizes for the year 1918: Poncelet prize of 2000 francs to J. LARMOR, of Cambridge University, for the totality of his mathematical researches; the Francoeur prize of 1000 francs to P. MONTEL, of the University of Paris, for his work on sequences of analytic functions; a prize of 2000 francs to the late SAMUEL LATTÈS, professor at the University of Toulouse, for his work in mathematical analysis; a prize of 2000 francs to PAUL BARBARIN, professor at Lycée Saint Louis, Paris, for his work in non-euclidean geometry; a prize of 1500 francs to LOUIS FABRY, assistant astronomer at the Observatory of Marseilles, for his ephemerides of the minor planets; and a prize of 3000 francs to GASTON JULIA, editor of Bôcher's Sorbonne lectures, for a study of iteration submitted to the Academy in competition with two other memoirs, one of which, by the late Samuel Lattès, received honorable mention.

The twelfth regular meeting of the American Mathematical Society at Chicago was held on Friday and Saturday, March 28 and 29. About forty persons attended the meetings including thirty-one members of the Society. On Friday and Saturday mornings there were presented fifteen papers of the usual research nature. The Friday afternoon session was devoted to a symposium on the geometry of numbers. Professors H. F. BLICHFELDT and L. E. DICKSON presented the subject, basing their discussion largely on the works of Minkowski. On Friday evening twenty-nine members took dinner together at the Quadrangle Club. In the after-dinner informal speeches, Professor ALEXANDER ZIWET, of the University of Michigan, gave a most cordial invitation to the mathematical

public there represented to attend the summer meeting of the American Mathematical Society at Ann Arbor, Tuesday–Thursday, September 2–4, 1919. A more complete report of the meeting will be found in the *Bulletin of the American Mathematical Society.* President Morley appointed as members of the committee on arrangements for the summer meeting Professors BEMAN (chairman), BLISS, KARPINSKI, OSGOOD, and the Secretary.

The Association will hold its summer meeting by invitation at the University of Michigan on Thursday–Saturday, September 4–6, 1919, following and in conjunction with the summer meeting of the American Mathematical Society. President Slaught has appointed Professor W. B. FORD chairman of the Program Committee and Professor W. W. BEMAN chairman of the Committee on Local Arrangements. It is expected that a more detailed notice will be published in the June MONTHLY and that the final statement of the program will be mailed to members of the Association about the middle of August.

The finance committee of the Council of the Association has invested five hundred dollars of the reserve funds in a Victory Loan bond; this is in addition to a like amount invested in a Liberty Bond a year and a half ago, as reported at the annual meeting in December, 1917.

The University of Colorado has just announced that instead of the usual summer session of six weeks it will conduct, in the summer of 1919, a Summer Quarter of two terms, greatly increasing the scope and variety of courses. The first term will open on June 30 and close on August 2. The second term will open on August 4 and close on September 6.

Special Notice to All Members of the Association.

The Annual Register of the Association will be published early in the fall, containing such data as were listed in the Register issued in May, 1918. In order to make this list as serviceable as possible for the academic year 1919–1920, members are asked to assure its accuracy and completeness by sending to the Secretary as early as possible notice of any promotions, changes of address, etc., **affecting the next academic year.** All such additions and corrections will be used up to the time of printing the Register, but these should be in hand by September first at the latest. It is hoped that the usefulness of the Annual Register will be greatly enchanced by this plan, which is intended to make it a reliable guide for *the whole academic year.*

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

SECOND REPORT[1] OF THE COMMITTEE ON LIBRARIES—A LIST OF COLLEGE MATHEMATICAL TEXT-BOOKS.

Preliminary Statement. At the meeting of the Association held in Chicago in December, 1917, the Library Committee was requested to prepare a list of mathematical text-books used in colleges, thus bringing together in compact and accessible form the bulk of the available information pertaining to this field of instruction. The Committee, in following out the request, began by ascertaining the names and addresses of the various publishing houses of America, consulting for this purpose the so-called *Publisher's Trade List Annual.* A letter was then sent to each house asking for a full list of their mathematical texts of college grade. Replies were eventually received in nearly every instance, and the findings of the Committee as thus determined constitute the basis of the subjoined list. Books published independently, as by an author himself, have also been included whenever coming to the attention of the Committee.

It should be said that some omissions and errors doubtless occur, as would be the case whatever plan of procedure might be followed in attempting to prepare such a list, but since nearly every book listed has been personally inspected, it is hoped that errors are few. It is possible that a supplementary list which shall take account of omissions may be published later. The Chairman of the Committee will be glad to receive any information tending to make the report more accurate and more complete. It may be well to note that the list pertains only to text-books (as distinguished from books of reference) by American authors (or translators), and published in America, since even with such restrictions the list is of considerable length.

Florian Cajori, University of California;
E. S. Crawley, University of Pennsylvania;
Solomon Lefschetz, University of Kansas;
W. R. Longley, Yale University;
R. E. Root, U. S. Naval Academy;
W. B. Ford, University of Michigan, *Chairman.*

COLLEGE ALGEBRA.

A. First Course.

Ashton (C. H.) and Marsh (W. R.) *College Algebra.* Scribner, 1907.[2] 9 + 279 pp. $1.50.
Bowser (E. A.) *College Algebra.* Heath, 1890. 18 + 540 pp. $1.60.
Brenke (W. C.) *Advanced Algebra.* Century, 1917. 7 + 196 pp. $1.25.
Collins (J. V.) *Advanced Algebra.* American Book, 1913. 12mo. 10 + 342 pp. $1.00.
Crathorne (A. R.) See Rietz (H. L.).
Davisson (S. C.) *College Algebra.* Macmillan, 1910. 12mo. 16 + 243 pp. $1.50.
Dickson (L. E.) *College Algebra.* Wiley, 1902. 12mo. 7 + 214 pp. $1.50.
Downey (J. F.) *Higher Algebra.* American Book, 1901. 12mo. 416 pp. $1.50.
Fine (H. B.) *College Algebra.* Ginn, 1904. 12mo. 8 + 595 pp. $2.00.
Fisher (G. E.) and Schwatt (I.) *Higher Algebra.* Macmillan. 12mo. 615 pp. $2.00.
Fite (W. B.) *College Algebra.* Revised edition. Heath, 1918. 5 + 320 pp. $1.56.

[1] Cf. this Monthly, October, 1917.
[2] It is intended that here, and in the following titles, the date shall be that of the latest *edition* but not of the latest *reprint.*

HAWKES (H. E.) *Advanced Algebra.* Ginn, 1905. 8vo. 14 + 285 pp. $1.60.
Higher Algebra. Ginn, 1913. 8vo. 5 + 222 pp. $1.60.
MARSH (W. R.) See ASHTON (C. H.).
MERRILL, HELEN A., and SMITH, CLARA E. *A First Course in Higher Algebra.* Macmillan, 1917. 12mo. 16 + 247 pp. $1.50.
RIETZ (H. L.) and CRATHORNE (A. R.) *College Algebra.* Holt, 1910. 8vo. 14 + 261 pp. $1.60.
SCHULTZE (A.) *Advanced Algebra.* Macmillan, 1908. 12mo. 14 + 582 pp. $1.25.
SCHWATT (I.) See FISHER (G. E.).
SLAUGHT (H. E.) See WILCZYNSKI (E. J.).
SKINNER (E. B.) *College Algebra.* Macmillan, 1917. 12mo. 7 + 263 pp. $1.50.
SMITH, CLARA E. See MERRILL, HELEN A.
TAYLOR (J. M.) *College Algebra.* Allyn and Bacon, 1889. 12mo. 9 + 317 pp. $1.50.
WELLS (W.) *College Algebra.* Heath, 1890. 6 + 544 pp. $1.60
WENTWORTH (G. A.) *College Algebra.* Revised edition. Ginn, 1902. 12mo. 6 + 530 pp. $1.80.
Higher Algebra. Ginn, 1891. 12mo. 6 + 521 pp. $1.60.
WILCZYNSKI (E. J.) and SLAUGHT (H. E.) *College Algebra with Applications.* Allyn and Bacon, 1916. 20 + 507 pp. $2.00.

B. SECOND (ADVANCED) COURSE.

BÔCHER (M.) *Introduction to Higher Algebra.* Macmillan, 1907. 8vo. 11 + 321 pp. $1.90.

GEOMETRY (ANALYTICAL AND PROJECTIVE).

A. ANALYTICAL GEOMETRY.

ALLEN (J.) See TANNER (J. H.).
ASHTON (C. H.) *Plane and Solid Analytic Geometry.* Revised edition. Scribner, 1916. 12mo. 14 + 278 pp. $1.40.
BAILEY (F. H.) and WOODS (F. S.) *Plane and Solid Analytic Geometry.* Ginn, 1897. 8vo. 12 + 371 pp. $2.40.
BÔCHER (M.) *Plane Analytic Geometry.* Holt, 1915. 13 + 235 pp. $1.72.
BÔCHER (M.) See BRIGGS (G. R.).
BOWSER (E. A.) *An Elementary Treatise on Analytic Geometry.* Twenty-fifth edition. Van Nostrand, 1915. 319 pp. $1.75.
BRIGGS (G. R.) *Elements of Plane Analytic Geometry.* Seventh edition revised and enlarged by M. Bôcher. Wiley, 1907. 12mo. 5 + 191 pp. $1.00.
CANDY (A. L.) *Elements of Plane and Solid Analytic Geometry.* Heath, 1908. 8vo. 10 + 266 pp. $1.64. Edition with supplement, 1909. 12 + 368 pp. $2.20.
CRAWLEY (E. S.) and EVANS (H. B.) *Analytic Geometry.* E. S. Crawley, Philadelphia, 1918. 12mo. 14 + 239 pp. $1.60.
DOWLING (L. W.) and TURNEAURE (F. E.) *Analytic Geometry.* Holt, 1914. 12 + 266 pp. $1.80.
EVANS (H. B.) See CRAWLEY (E. S.).
FINE (H. B.) and THOMPSON (H. D.) *Coördinate Geometry.* Macmillan, 1910. 12mo. 10 + 300 pp. + 9 plates. $1.60.
GALE (A. S.) See SMITH (P. F.).
HARDY (A. S.) *Analytic Geometry.* Ginn, 1889. 8vo. 12 + 229 pp. $1.80.
HOPKINS (L. A.) See ZIWET (A.).
LAMBERT (P. A.) *Analytic Geometry for Technical Schools and Colleges.* Macmillan, 1897. 12mo. 11 + 216 pp. $1.50.
NICHOLS (E. W.) *Analytic Geometry.* Revised edition. Heath, 1918. 11 + 292 pp. $1.48.
PHILLIPS (H. B.) *Analytic Geometry.* Wiley, 1915. 12mo. 7 + 197 pp. $1.50.
RIGGS (N. C.) *Analytic Geometry.* Macmillan, 1910. 12mo. 11 + 294 pp. $1.75.
ROBERTS, MARIA M., and COLPITTS, JULIA T. *Analytic Geometry.* Wiley, 1918. 12 mo. 10 + 229 pp. $1.60.
SCHMALL (C. N.) *First Course in Analytical Geometry, Plane and Solid.* Van Nostrand, 1905. 7 + 318 pp. $1.75.

SICELOFF (L. P.) See WENTWORTH (G.).
SISAM (C. H.) See SNYDER (V.).
SMITH (D. E.) See WENTWORTH (G.).
SMITH (P. F.) and GALE (A. S.) *Elements of Analytic Geometry.* Ginn, 1904. 8vo. 12 + 424 pp. $2.40.
Introduction to Analytic Geometry. Ginn, 1905. 8vo. 8 + 217 pp. $1.40.
New Analytic Geometry. Ginn, 1912. 12mo. 10 + 342 pp. $1.80.
SNYDER (V.) and SISAM (C. H.) *Analytic Geometry of Space.* Holt, 1914. 11 + 289 pp. $2.50.
TANNER (J. H.) and ALLEN (J.) *An Elementary Course in Analytic Geometry.* American Book, 1898. 12mo. 20 + 418 pp. $2.00.
Brief Course in Analytic Geometry. American Book, 1911. 12mo. 10 + 282 + 24 pp. $1.50.
TRACEY (J. I.) See WILSON (W. A.).
WENTWORTH (G. A.) *Elements of Analytic Geometry.* Second edition. Ginn, 1891. 12mo. 12 + 273 + 27 pp. $1.60.
WENTWORTH (G.), SMITH (D. E.) and SICELOFF (L. P.) *Plane Analytic Geometry.* Ginn. (In press.)
WILSON (W. A.) and TRACEY (J. I.) *Analytic Geometry.* Heath, 1915. 10 + 212 pp. $1.28.
WOODS (F. S.) See BAILEY (F. H.).
ZIWET (A.) and HOPKINS (L. A.) *Elements of Analytic Geometry.* Macmillan, 1916. 12mo. 7 + 280 pp. $1.60.

B. PROJECTIVE GEOMETRY.

DOWLING (L. W.) *Projective Geometry.* McGraw-Hill, 1917. 12mo. 14 + 215 pp. $2.00.
EMCH (A.) *An Introduction to Projective Geometry and its Applications. An Analytic and Synthetic Treatment.* Wiley, 1905. 8vo. 7 + 267 pp. $2.50.
LEHMER (D. N.) *Synthetic Projective Geometry.* Ginn, 1917. 12mo. 13 + 123 pp. $1.12.
LING (G. H.) See WENTWORTH (G.).
SMITH (D. E.) See WENTWORTH (G.).
WENTWORTH (G.), SMITH (D. E.) and LING (G. H.) *Projective Geometry.* Ginn. (In press.)
VEBLEN (O.) and YOUNG (J. W.) *Projective Geometry.* Ginn. 8vo. Vol. 1, 1910. 10 + 342 pp. $4.00; Vol. 2, 1918. 12 + 511 pp. $5.00.
YOUNG (J. W.) See VEBLEN (O.).

TRIGONOMETRY.

ANDEREGG (F.) and ROE (E. D.) *Trigonometry.* Revised edition. Ginn, 1913. 12mo. 9 + 108 pp. $0.96.
ASHTON (C. H.) and MARSH (W. R.) *Plane and Spherical Trigonometry, with Tables.* Scribner, 1902. 12mo. 10 + 157 + 93 pp. $1.25.
BARKER (E. H.) *Plane Trigonometry with Tables.* Blakiston, 1917. 8vo. 7 + 172 pp. $1.00.
BAUER (G. N.) and BROOKE (W. E.) *Plane and Spherical Trigonometry.* Second revised edition. Heath, 1917. 12mo. 11 + 174 pp. $1.24. With Tables, 11 + 174 + 5 + 139. $1.60.
BÔCHER (M.) and GAYLORD (H. D.) *Trigonometry.* Holt, 1914. 12mo. 9 + 142 pp. $1.12.
BOWSER (E. A.) *Elements of Trigonometry.* Heath. 6 + 172 pp. $1.08. With Tables, $1.60.
A Treatise on Plane and Spherical Trigonometry and its Applications to Astronomy and Geodesy. Heath, 1892. 13 + 368 pp. $1.68.
BRENKE (W. C.) *Elements of Trigonometry.* Century, 1917. 6 + 121 + 39 pp. $1.25.
BROOKE (W. E.) See BAUER (G. N.).
BUCHANAN (A. H.) *Plane and Spherical Trigonometry.* Wiley, 1907. 8vo. 6 + 96 pp. $1.00.
CHAUVENET (W.) *A Treatise on Plane and Spherical Trigonometry.* Ninth edition. Lippincott, 1875. 8vo. 256 pp. $1.32.
CONANT (L. L.) *Plane Trigonometry.* American Book, 1909. 8vo. 183 pp. $0.90.
CRAWLEY (E. S.) *Elements of Plane and Spherical Trigonometry.* New and revised edition. E. S. Crawley, Philadelphia, 1910. 8vo. 4 + 186 pp. $1.10. With Tables, 4 + 186 + 32 + 76 pp., $1.50.
A Short Course in Plane and Spherical Trigonometry. E. S. Crawley, Philadelphia, 1902. 8vo. 116 pp. $0.90. With Tables, 116 + 28 pp. $1.25.
DURFEE (W. P.) *Elements of Plane Trigonometry.* Ginn, 1900. 12mo. 6 + 105 pp. $1.00.
FOERING (H. A.) See LAMBERT (P. A.).

FRINK (F. G.) See HALL (A. G.).
GAYLORD (H. D.) See BÔCHER (M.).
GODDARD (E. C.) See LYMAN (E. A.).
GORE (J. H.) *Elements of Plane and Spherical Trigonometry.* Putnam, 1907. 12mo. 6 + 122 pp. $1.20. With Tables, 6 + 122 + 17 + 62 pp. $1.20.
GRANVILLE (W. A.) *Plane and Spherical Trigonometry.* Ginn, 1909. 8vo. 11 + 264 pp. $1.00. With Tables, $1.45.
HALL (A. G.) and FRINK (F. G.) *Plane Trigonometry with Tables.* Holt, 1909. 8vo. 10 + 148 + 93 pp. $1.25.
Plane and Spherical Trigonometry. Holt, 1910. 8vo. 10 + 176 pp. $1.20
HARDING (A. M.) and TURNER (J. S.) *Plane Trigonometry.* Putnam, 1915. 12mo. 13 + 158 pp. $0.90. With Tables, 13 + 158 + 51 pp. $1.10.
HILL (G. A.) See WENTWORTH (G. A.).
HUN (J. G.) and MACINNES (C. R.) *The Elements of Plane and Spherical Trigonometry.* Macmillan, 1911. 8vo. 101 pp. $0.90. With Tables, 205 pp. $1.35.
INGOLD (L.) See KENYON (A. M.).
KENYON (A. M.) and INGOLD (L.) *Trigonometry.* Macmillan, 1913. 12mo. 11 + 132 pp. $1.20. With Tables, 11 + 132 + 18 + 124 pp. $1.50.
LAMBERT (P. A.) and FOERING (H. A.) *Plane and Spherical Trigonometry.* Macmillan, 1905. 12mo. 8 + 104 pp. $0.60.
LEIGH (C. W.) See PALMER (C. I.).
LYMAN (E. A.) and GODDARD (E. C.) *Plane and Spherical Trigonometry, with Tables.* Allyn and Bacon, 1900. 8vo. 7 + 139 + 66 pp. $1.20.
MARSH (W. R.) See ASHTON (C. H.).
MACINNES (C. R.) See HUN (J. G.).
MORITZ (R. E.) *Elements of Plane Trigonometry.* Wiley, 1911. 8vo. 14 + 361 + 91 pp. $2.00.
Plane and Spherical Trigonometry. Wiley, 1913. 8vo. 16 + 357 + 67 + 96 pp. $2.50.
Text Book of Spherical Trigonometry. Wiley, 1913. 8vo. 6 + 67 pp. $1.00.
MURRAY (D. A.) *Elements of Plane Trigonometry.* Longmans, 1911. 12mo. 9 + 136 pp.
NICHOLSON (J. W.) *Elements of Plane and Spherical Trigonometry.* Macmillan, 1898. 8vo. 8 + 101 pp. $0.90. With Tables, 8 + 101 + 44 pp. $1.10.
PALMER (C. I.) and LEIGH (C. W.) *Plane and Spherical Trigonometry.* Second edition revised and enlarged. McGraw-Hill, 1916. 8vo. 11 + 188 pp. $1.00. With Tables, 11 + 188 + 132 pp. $1.50.
PASSANO (L. M.) *Plane and Spherical Trigonometry.* Macmillan, 1918. 12mo. 15 + 140 pp. $1.25.
PURYEAR (C.) See TAYLOR (T. U.).
ROE (E. D.) See ANDEREGG (F.).
ROBBINS (E. R.) *Plane Trigonometry.* American Book, 1909. 8vo. 13 + 153 pp. $0.60.
ROTHROCK (D. A.) *Elements of Plane and Spherical Trigonometry.* Macmillan, 1910. 8vo. 11 + 147 pp. $1.20. With Tables, 11 + 147 + 15 + 99 pp. $1.50.
SLAUGHT (H. E.) See WILCZYNSKI (E. J.).
SMITH (D. E.) See WENTWORTH (G.).
SMITH (P. F.) See GRANVILLE (W. A.).
TAYLOR (J. M.) *Plane Trigonometry.* Ginn, 1904. 12mo. 8 + 171 pp. $0.90.
Plane and Spherical Trigonometry. Ginn, 1905. 12mo. 9 + 210 + 24 pp. $1.20.
TAYLOR (T. U.) and PURYEAR (C.) *Elements of Plane and Spherical Trigonometry.* Ginn, 1902. 8vo. 5 + 160 + 1 + 67 pp. $1.40.
TURNER (J. S.) See HARDING (A. M.).
WELLS (W.) *New Plane Trigonometry.* Heath, 1911. 5 + 110 pp. $0.76. With Tables, $0.96.
New Plane and Spherical Trigonometry. Heath, 1896. 8vo. 5 + 126 pp. $1.28. With Tables, $1.60.
Complete Trigonometry. Revised edition. Heath, 1911. 6 + 163 + 9 pp. $1.08. With Tables, $1.24.
WENTWORTH (G. A.) *Plane Trigonometry.* Second revised edition. Ginn, 1903. 12mo. 6 + 141 + 21 pp. $0.75. With Tables, 6 + 141 + 21 + 20 + 75 pp. $1.00.

Plane and Spherical Trigonometry. Second revised edition. Ginn, 1903. 8vo. 7 + 207 + 25 pp. $1.00. With Tables, 7 + 207 + 25 + 20 + 76. $1.20. [Tables by G. A. Wentworth and G. A. Hill.]

Plane Trigonometry, Surveying and Tables. Second revised edition. Ginn, 1903. 8vo. 7 + 238 + 23 + 20 + 75 pp. $1.48.

Plane and Spherical Trigonometry, Surveying and Tables. Second revised edition. Ginn, 1903. 8vo. 8 + 304 + 27 + 20 + 75 pp. $1.60.

Plane and Spherical Trigonometry, Surveying and Navigation. Second revised edition, 1903. 12mo. 452 pp. $1.48.

WENTWORTH (G.) and SMITH (D. E.) *Plane Trigonometry.* Ginn, 1914. 8vo. 5 + 188 pp. $1.08. With Tables, 5 + 188 + 5 + 104 pp. $1.30.

Plane and Spherical Trigonometry. Ginn, 1915. 8vo. 5 + 230 + 26 pp. $1.36. With Tables, 8vo. 5 + 230 + 26 + 5 + 104 pp. $1.60.

WILCZYNSKI (E. J.) and SLAUGHT (H. E.) *Plane Trigonometry and Applications, with Tables.* Allyn and Bacon, 1914. 8vo. 11 + 265 + 20 + 97 pp. $1.25.

GENERAL (COMBINED) COURSES.

A. ELEMENTARY.

BAILEY (F. H.) See WOODS (F. S.).

BENEDICT (H. Y.) See KARPINSKI (L. C.).

CALHOUN (J. W.) See KARPINSKI (L. C.).

KARPINSKI (L. C.), BENEDICT (H. Y.) and CALHOUN (J. W.) *Unified Mathematics.* Heath, 1918. 12mo. 8 + 522 pp. $2.80.

KELLER (S. S.) and KNOX (W. F.) *Mathematics for Engineering Students. Analytical Geometry and Calculus.* Van Nostrand, 1907. 8vo. 2 + 359 pp. $2.00.

KENYON (A. M.) and LOVITT (W. V.) *Mathematics for Collegiate Students of Agriculture and General Science.* Macmillan, 1917. 12mo. 7 + 337 pp. $2.00.

KNOX (W. F.) See KELLER (S. S.).

LOVITT (W. V. See KENYON (A. M.).

MCCLENON (R. B.) *Introduction to the Elementary Functions.* With the editorial coöperation of W. J. RUSK. Ginn, 1918. 8vo. 9 + 244 pp. $1.80.

MORGAN (F. M.) See YOUNG (J. W.).

PLANT (L. C.)

RANSOM (W. R.) *Freshman Mathematics.* Longmans, 1918. 12mo. 12 + 285 pp. $1.35.

RUSK (W. J.) See MCCLENON (R. B.)

SLICHTER (C. S.) *Elementary Mathematical Analysis. A Text Book for First Year College Students.* Second edition revised and entirely reset. McGraw-Hill, 1918. 12mo. 18 + 497 pp. $2.50.

SMITH (P. F.) and GRANVILLE (W. A.) *Elementary Analysis.* Ginn, 1910. 12mo. 10 + 223 pp. $1.80.

WEBBER (W. P.) and PLANT (L. C.) *Introductory Mathematical Analysis.* Wiley, 1919. 12mo. 13 + 304 pp. $2.00.

WOLFF (H. C.) *Mathematics for Agricultural Students.* McGraw-Hill, 1914. 12mo. 9 + 309 pp. $1.80.

WOODS (F. S.) and BAILEY (F. H.) *A Course in Mathematics.* Ginn. 8vo. Vol. 1, 1907. 12 + 385 pp. $2.75. Vol. 2, 1909. 11 + 410 pp. $2.75.

Analytic Geometry and Calculus. Ginn, 1917. 8vo. 11 + 516 pp. $3.60.

YOUNG (J. W.) and MORGAN (F. M.) *Elementary Mathematical Analysis.* Macmillan, 1917. 12mo. 13 + 548 pp. $2.60.

ZIWET (A.) and HOPKINS (L. A.) *Analytical Geometry and Principles of Algebra.* Macmillan, 1913. 12mo. 10 + 369 pp. $1.75.

B. ADVANCED

GOURSAT (E.) *Mathematical Analysis.* Translated from the French by E. R. Hedrick and O. Dunkel. Ginn. 8vo. Vol. 1, 1904, 548 pp., $4.00. Vol. 2, Part 1, 1916, 259 pp., $2.75. Part 2, 1917, 300 pp., $2.75.

STEINMETZ (C. P.) *Engineering Mathematics; a Series of Lectures.* Third edition revised and enlarged. McGraw, Hill, 1917. 8vo. 19 + 321 pp. $3.00.

CALCULUS (FIRST COURSE).

BALINE (R. G.) *The Calculus and its Applications, a Practical Treatise for Beginners, Especially Engineering Students.* Van Nostrand, 1909. 9 + 321 pp. $1.50.

BOWSER (E. A.) *An Elementary Treatise on the Differential and Integral Calculus.* Twenty-fourth edition enlarged. Van Nostrand, 1913. 12mo. 14 + 451 pp. $2.25.

BYERLY (W. E.) *Elements of the Integral Calculus.* Second edition revised and enlarged. Ginn, 1889. 8vo. 16 + 339 + 14 + 32 pp. $2.00.

CAMPBELL (D. F.) *The Elements of the Differential and Integral Calculus.* Macmillan, 1905. 12mo. 11 + 364 pp. $2.10.

CAIN (W.) *A Brief Course in the Calculus.* Third edition revised. Van Nostrand, 1911. 8vo. 11 + 281 pp. $1.75.

CHANDLER (G. H.) *Elements of the Infinitesimal Calculus.* Third edition rewritten. Wiley, 1907. 12mo. 6 + 319 pp. $2.00.

DAVIS (E. W.) *The Calculus.* Macmillan, 1912. 12mo. 21 + 384 + 63 pp. $2.10.

FRANKLIN (W. S.), MACNUTT (B.) and CHARLES (R. L.) *An Elementary Treatise on Calculus.* South Bethlehem, Pa., The Authors, 1913. 8vo. 10 + 253 + 41 pp. *Calculus Supplement.* To take the place of pages 1–41 of the 1913 edition. South Bethlehem, Pa., 1915. 8vo. 5 + 51 pp.

GOODENOUGH (G. A.) See TOWNSEND (E. J.).

GOULD (E. S.) *A Primer of the Calculus.* Sixth edition. Van Nostrand, 1918. 12mo. 91 pp. $0.50.

GRANVILLE (W. A.) *Elements of the Differential and Integral Calculus.* Ginn. 8vo. 463 pp. $2.80.

GUNTHER (C. O.) *Integration by Trigonometric and Imaginary Substitution.* Second edition revised. Van Nostrand, 1915. 8vo. 6 + 79 pp.

HUTCHINSON (J. I.) See SNYDER (V.).

JOHNSON (W. W.) *An Elementary Treatise on the Differential Calculus Founded on the Method of Rates.* Wiley, 1904. 8vo. 14 + 404 pp. $3.00.

An Elementary Treatise on the Differential Calculus Founded on the Method of Rates. (Abridged Edition.) Wiley, 1908. 8vo. 10 + 191 pp. $1.50.

An Elementary Treatise on the Integral Calculus Founded on the Method of Rates or Fluxions. Revised edition. Wiley, 1907. 8vo. 7 + 230 pp. $1.50.

A Treatise on the Integral Calculus Founded on the Method of Rates. Wiley, 1907. 8vo. 14 + 440 pp. $3.00.

HULBURT (L. S.) *Differential and Integral Calculus.* Longmans, 1912. 8vo. 18 + 481 pp. $2.25.

LAMBERT (P. A.) *Differential and Integral Calculus for Technical Schools and Colleges.* Macmillan, 1898. 12mo. 10 + 245 pp. $1.50.

LEIB (D. B.) *Problems in the Calculus with Formulas and Suggestions.* Ginn, 1915. 12mo. 12 + 224 pp. $1.40.

LOVE (C. E.) *Differential and Integral Calculus.* Macmillan, 1917. 8vo. 18 + 343 pp. $2.10.

MARCH (H. W.) and WOLFF (H. C.) *Calculus.* McGraw-Hill, 1917. 16 + 360 pp. $2.00.

MURRAY (D. A.) *An Elementary Course in the Integral Calculus.* American Book, 1898. 14 + 288 pp. $2.00.

NICHOLS (E. W.) *Differential and Integral Calculus.* Revised edition. Heath, 1918. 19 + 394 pp. $2.20.

OSBORNE (G. A.) *Differential and Integral Calculus.* Revised edition. Heath, 1907. 12mo. 12 + 404 pp. $2.20.

OSGOOD (W. F.) *A First Course in the Differential and Integral Calculus.* Revised edition. Macmillan, 1909. 12mo. 15 + 462 pp. $2.00.

PHILLIPS (H. B.) *Differential Calculus.* Wiley, 1916. 12mo. 5 + 162 pp. $1.25.

Integral Calculus. Wiley, 1917. 12mo. 5 + 194 pp. $1.25. Combined edition, two volumes in one, 1917. $2.00.

SMITH (P. F.) *Elementary Calculus.* American Book, 1902. 89 pp. $1.25.

SNYDER (V.) and HUTCHINSON (J. I.) *Elementary Text-Book on the Calculus.* American Book, 1912. 12mo. 384 pp. $2.20.

Differential and Integral Calculus. American Book, 1902. 16 + 320 pp. $2.00.

TAYLOR (J. M.) *Elements of the Differential and Integral Calculus.* Revised edition. Ginn, 1898. 8vo. 13 + 269 pp. $2.50.

TOWNSEND (E. J.) and GOODENOUGH (G. A.) *Essentials of Calculus.* Second edition revised. Holt, 1914. 12mo. 12 + 355 pp. $2.00.
First Course in Calculus. Holt, 1910. 8vo. 12 + 466 pp.
WOLFF (H. C.) See MARCH (H. W.).

MISCELLANEOUS.

BAKER (A. L.) *Elliptic Functions.* Wiley, 1890. 8vo. 5 + 118 pp. $1.50.
BARTON (S. M.) *An Elementary Treatise on the Theory of Equations.* Heath, 1899. 12 + 199 pp. $1.60.
BEMAN (W. W.) See KLEIN (F.).
BLICHFELDT (H. F.) *Finite Collineation Groups with an Introduction to the Theory of Groups of Operators and Substitution Groups.* University of Chicago Press, 1917. 12mo. 11 + 194 pp. $1.50.
BLICHFELDT (H. F.) See also MILLER (G. A.).
BURKHARDT (H.) *Theory of Functions of a Complex Variable.* Translated from the fourth German edition by S. E. Rasor. Heath. 8vo. 13 + 432 pp. $4.00.
BYERLY (W. E.) *Elementary Treatise on Fourier's Series.* Ginn, 1895. 8vo. 9 + 288 pp. $3.00.
Harmonic Functions. Fourth edition. (Monograph Series.) Wiley, 1906. 8vo. 5 + 66 pp. $1.00.
Introduction to the Calculus of Variations. (Mathematical Tracts for Physicists.) Harvard University Press, 1917. 8vo. 48 pp. 75 cents.
Introduction to the Use of Generalized Coördinates in Mechanics and Physics. Ginn, 1916. 8vo. 7 + 118 pp. $1.25.
CAJORI (F.) *A History of Mathematics.* Macmillan. New edition in press.
Introduction to the Modern Theory of Equations. Macmillan. $1.75.
CAMPBELL (D. F.) *A Short Course on Differential Equations.* Macmillan, 1907. 12mo. 123 pp. $1.10.
CARMICHAEL (R. D.) *The Theory of Numbers.* (Monograph Series.) Wiley, 1914. 8vo. 94 pp. $1.00.
CLEMENTS (G. R.) *Problems in the Mathematical Theory of Investment.* Ginn, 1916. 12 mo. 2 + 24 pp. $0.40.
COFFIN (J. G.) *Vector Analysis. An Introduction to Vector Methods and their Applications in Physics and Mathematics.* Second edition. Wiley, 1911. 12mo. 22 + 262 pp. $2.50.
COHEN (A.) *Elementary Treatise on Differential Equations.* Heath, 1906. 9 + 271 pp. $2.00.
An Introduction to the Lie Theory of One-Parameter Groups. Heath, 1911. 7 + 248 pp. $2.00.
COMSTOCK (G. C.) *An Elementary Treatise upon the Method of Least Squares.* Ginn, 1890. 8vo. 6 + 68 pp. $1.40.
DICKSON (L. E.) *Algebraic Invariants.* (Monograph Series.) Wiley, 1914. 8vo. 10 + 100 pp. $1.25.
Elementary Theory of Equations. Wiley, 1914. 8vo. 5 + 184 pp. $1.75.
Introduction to the Theory of Algebraic Equations. Wiley, 1903. 8vo. 4 + 104 pp. $1.25.
DURÈGE (H.) *Elements of the Theory of Functions of a Complex Variable, with Especial Reference to the Methods of Riemann.* Translated from the German by G. E. Fisher and I. J. Schwatt. Philadelphia, G. E. Fisher and I. J. Schwatt, 1896. 8vo. 13 + 288 pp. $2.00.
EISENHART (L. P.) *A Treatise on the Differential Geometry of Curves and Surfaces.* Ginn, 1909. 8vo. 11 + 474 pp. $4.50.
FIELD (P.) See ZIWET (A.).
FISHER (G. E.) See DURÈGE (H.).
FISHER (L. S.) See HUNTINGTON (E. V.).
FISKE (T. S.) *Functions of a Complex Variable.* Fourth edition. (Monograph Series.) Wiley, 1906. 8vo. 5 + 99 pp. $1.00.
GAUSS (K. F.) *General Investigations of Curved Surfaces of 1827 and 1825.* Translated with notes and bibliography by J. C. Morehead and A. M. Hiltebeitel. Princeton University Press, 1902. 4to. 6 + 127 pp. $1.75.
GLENN (O. E.) *A Treatise on the Theory of Invariants.* Ginn, 1915. 8vo. 10 + 245 pp. $2.75.

HANCOCK (H. H.) *Elliptic Integrals.* (Monograph Series.) Wiley, 1917. 104 pp. $1.25.
Theory of Maxima and Minima. Ginn, 1917. 8vo. 14 + 193 pp. $2.50.
HANUS (P. H.) *An Elementary Treatise on the Theory of Determinants.* Ginn, 1886. 8vo. 8 + 217 pp. $1.80.
HARDY (A. S.) *Elements of Quaternions.* Ginn, 1881. 8vo. 8 + 234 pp. $2.00.
HEDRICK (E. R.) and KELLOGG (O. D.) *Applications of the Calculus to Mechanics.* Ginn, 1909. 8vo. 6 + 116 pp. $1.25.
HUNTINGTON (E. V.) and FISHER (L. S.) *Handbook of Mathematics for Engineers; with Tables of Weights and Measures* by L. S. Fisher; reprint of sections 1 and 2 of L. S. MARK's *Mechanical Engineers' Handbook.* McGraw-Hill, 1918. 12mo. 7 + 191 pp. $1.50.
INGERSOLL (L. R.) and ZOBEL (O. J.) *The Mathematical Theory of Heat Conduction.* Ginn, 1913. 8vo. 6 + 171 pp. $2.00.
JOHNSON (W. W.) *Curve Tracing in Cartesian Coordinates.* Wiley, 1884. 12mo. 6 + 86 pp. $1.00.
Differential Equations. (Monograph Series.) Fourth edition. Wiley, 1906. 8vo. 6 + 72 pp. $1.00.
Theory of Errors and the Method of Least Squares. Wiley, 1892. 10 + 152 pp. $1.50.
Theoretical Mechanics—An Elementary Treatise. Wiley, 1901. 15 + 434 pp. $3.00.
Treatise on Ordinary and Partial Differential Equations. Third edition. Wiley, 1889. 8vo. 12 + 368 pp. $3.50.
KLEIN (F.) *Famous Problems of Elementary Geometry.* Translated by W. W. Beman and D. E. Smith. Ginn, 1897. 12mo. 9 + 80 pp. $0.60.
KELLOGG (O. D.) See HEDRICK (E. R.).
KING (W. I.) *The Elements of Statistical Method.* Macmillan, 1916. 12mo. 250 pp. $1.50.
KOCH (E. H.) *The Mathematics of Applied Electricity.* Wiley, 1912. 8vo. 316 pp. $3.00.
LESTER (O. C.) *The Integrals of Mechanics.* Ginn. 8vo. 67 pp. $1.00.
LILLY (S. B.) See MILLER (J. A.).
LIPKA (J.) *Graphical and Mechanical Computation.* Wiley, 1919. 8 vo. 9 + 264 pp. $4.00.
LONGLEY (W. R.) See SMITH (P. F.).
MACFARLANE (A.) *Vector Analysis and Quaternions.* Fourth edition. (Monograph Series.) Wiley, 1906. 8vo. 5 + 50 pp. $1.00.
MCMAHON (J.) *Hyperbolic Functions.* Fourth edition. (Monograph Series.) Wiley, 1906. 8vo. 6 + 77 pp. $1.00.
MARTIN (L. A.) *Text-Book of Mechanics.* Wiley. 12mo. Vol. 1, 1906. *Statics.* 12 + 142 pp. $1.25; Vol. 2, 1907. *Kinematics and Kinetics.* 14 + 203 pp. $1.50; Vol. 3, 1911. *Mechanics of Materials.* 13 + 229 pp. $1.50; Vol. 4, 1913. *Applied Statics.* 12 + 198 pp. $1.50; Vol. 5, 1914. *Hydraulics.* 12 + 223 pp. $1.50; Vol. 6, 1916. *Thermodynamics.* 18 + 313 pp. $1.75.
MAURUS (E. J.) *An Elementary Course in Differential Equations.* Ginn, 1917. 12mo. 51 pp. $0.72.
MAURER (E. R.) *Technical Mechanics.* Fourth edition. Wiley, 1917. 8vo. 16 + 382 pp. $2.75.
MERRIMAN (M.) *The Solution of Equations.* (Monograph Series.) Fourth edition enlarged. Wiley, 1906. 8vo. 5 + 47 pp. $1.00.
A Text-Book on the Method of Least Squares. Eighth edition revised. Wiley, 1911. 8vo. 7 + 230 pp. $1.80.
MERRILL (G. A.) *An Elementary Text-Book of Theoretical Mechanics.* American Book, 1905. 8vo. 267 pp. $2.20.
MILLER (G. A.), BLICHFELDT (H. F.) and DICKSON (L. E.) *Theory and Applications of Finite Groups.* Wiley, 1916. 8vo. 17 + 390 pp. $4.00.
MILLER (J. A.) and LILLY (S. B.) *Analytic Mechanics.* Heath, 1915. 8vo. 15 + 297 pp. $2.20.
OSBORNE (G. A.) *Examples of Differential Equations.* Ginn, 1886. 12 mo. 7 + 50 pp. $0.80.
OSGOOD (W. F.) *Introduction to Infinite Series.* Third edition. Harvard University Press, 1910. 8vo. 2 + 71 pp. $0.75.
PEIRCE (B. O.) *Elements of the Theory of the Newtonian Potential Function.* Third edition revised and enlarged. Ginn, 1902. 13 + 490 pp. $3.75.
PIERPONT (J.) *Lectures on the Theory of Functions of a Real Variable.* Ginn. 8vo. Vol. 1, 1905, 12 + 560 pp. $4.50. Vol. 2, 1912, 13 + 645 pp. $5.00.
Functions of a Complex Variable. Ginn, 1914. 8vo. 14 + 583 pp. $5.00.

RASOR (S. E.) See BURKHARDT (H.).
RUNNING (T. R.) *Empirical Formulas.* (Monograph Series.) Wiley, 1917. 8vo. 144 pp. $1.40.
SKINNER (E. B.) *The Mathematical Theory of Investment.* Ginn, 1913. 8vo. 9 + 245 pp. $2.50.
SMITH (D. E.) See KLEIN (F.).
SMITH (P. F.) and LONGLEY (W. R.) *Theoretical Mechanics.* Ginn, 1910. 8vo. 10 + 288 pp. $3.00.
TOWNSEND (E. J.) *Functions of a Complex Variable.* Holt, 1915. 8vo. 7 + 384 pp. $4.00.
WELD (L. D.) *The Theory of Errors and Least Squares.* Macmillan, 1916. 12mo. 12 + 190 pp. $1.25.
WELD (L. G.) *A Short Course in the Theory of Determinants.* Second edition. Macmillan, 1893. 12mo. 13 + 238 pp. $1.90.
Determinants. Fourth edition. (Monograph Series.) Wiley, 1906. 8vo. 37 pp. $1.00.
VEBLEN (O.) and LENNES (N. J.) *Introduction to Infinitesimal Analysis, Functions of One Variable.* Wiley, 1907. 8vo. 7 + 227 pp.
WEST (C. J.) *Introduction to Mathematical Statistics.* Columbus, Ohio, R. G. Adams and Co., 1918. 8vo. 150 pp. $2.50.
WILSON (E. B.) *Advanced Calculus.* Ginn, 1912. 8vo. 9 + 566 pp. $5.00.
Vector Analysis. Scribner, 1901.[1] 8vo. 18 + 436 pp. $5.00.
WOODWARD (R. S.) *Probability and Theory of Errors.* Fourth edition. (Monograph Series.) Wiley, 1906. 8vo. 5 + 47 pp. $1.00.
ZIWET (A.) and FIELD (P.) *Introduction to Analytic Mechanics.* Macmillan, 1912. 12mo. 9 + 378 pp. $1.60.
ZOBEL (O. J.) See INGERSOLL (L. R.).

N. B.—Reprints of the Report given above may be procured, by members of the Association, on application to Professor W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

REPORT OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS.

This Committee was appointed in 1916, by the Mathematical Association of America, with Professor J. W. YOUNG of Dartmouth College as chairman. Progress reports were published in this MONTHLY for October, 1916 and December, 1917. In the latter it was indicated that the following topics had been chosen for study: "The valid aims and purposes of mathematical study"; "Recent criticisms of mathematics"; "Formal discipline and the transfer of training"; "Desirable topics of algebra and their treatment"; "Scientific investigations"; "Questionnaires"; "World experience as to mathematical curricula and the training of teachers." Other topics for investigation have since been decided upon. The extraordinary importance of an exhaustive national inquiry in connection with such questions can hardly be overestimated. It is therefore with great satisfaction that the Committee announces that funds, sufficient for carrying on its work in a very thorough manner, have been secured.

At a meeting held in New York on May 22, 1919, the General Education Board appropriated the sum of $16,000 for the use of the National Committee. The plan involves the appointment of one college man and one high school man

[1] Taken over and issued by the Yale University Press in 1913.

to devote their full time to the work of the Committee, their salaries for one year to be paid out of the fund mentioned. There is also ample provision for stenographic and other clerical help, and for printing, stationery, and postage, as well as for travelling expenses. It is hoped that a more extended account of the proposed work of the Committee published in the next issue of the MONTHLY. By that time, also, the organization of the Committee under the new plan should have been effected.

FOURTH ANNUAL MEETING OF THE OHIO SECTION.

The fourth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, on Friday, April 18, 1919, in connection with the meetings of the Ohio College Association. Professor C. N. Moore occupied the chair, being relieved by Professor K. D. Swartzel for an interval.

The following thirty-eight persons were registered, all but the last ten being members of the Association:

R. B. Allen, Kenyon College; W. E. Anderson, Miami University; G. N. Armstrong, Ohio Wesleyan University; C. L. Arnold, Ohio State University; C. B. Austin, Ohio Wesleyan University; Grace M. Bareis, Ohio State University; Mrs. W. E. Beckwith, Western Reserve University; R. D. Bohannan, Ohio State University; R. L. Borger, Ohio University; J. B. Brandeberry, Toledo University; A. G. Caris, Defiance College; T. M. Focke, Case School of Applied Science; M. E. Graber, Heidelberg University; Harris Hancock, University of Cincinnati; William Hoover, Columbus; Emma L. Konantz, Ohio Wesleyan University; H. W. Kuhn, Ohio State University; C. N. Moore, University of Cincinnati; C. C. Morris, Ohio State University; Anna H. Palmié, Western Reserve College for Women; S. E. Rasor, Ohio State University; Hortense Rickard, Ohio State University; W. G. Simon, Adelbert College; S. A. Singer, Capital University; K. D. Swartzel, Ohio State University; R. B. Wildermuth, Capital University; F. B. Wiley, Denison University; D. T. Wilson, Case School of Applied Science. Non-members: H. M. Beatty, Ohio State University; W. S. Beckwith, Ohio Northern University; Clara F. Brumbach, Denison University; C. A. Hahn, Otterbein College; G. W. McCoard, Ohio State University; F. W. Parsons, Ohio Northern University; Anna B. Peckham, Denison University; F. J. Shollenberger, Mt. Union College; C. H. Skinner, Ohio Wesleyan University; S. E. Slocum, University of Cincinnati.

The following program, slightly changed in order, was carried out as arranged by the executive committee:

General theme: Mathematics and Warfare.

1. Chairman's Address: The rôle of mathematics in world progress by Professor C. N. MOORE, University of Cincinnati.

2. The mathematical features of navigation by Professor D. T. WILSON, Case School of Applied Science.
3. Discussion of the preceding by Professor W. E. ANDERSON, Miami University.

Business and Intermission.

4. Ballistics as applied mathematics by Professor M. E. GRABER, Heidelberg University.
5. The mathematics of aviation by Professor S. E. SLOCUM, University of Cincinnati. (Introduced by Professor C. N. Moore.)

About twenty of those present dined together at the Ohio Union in the evening. A recess was taken during the evening program to attend the lecture given before the coöperating societies by the representative of the Carnegie Foundation, on "The Pension Problem for College Teachers." Many remained over Saturday to attend meetings of the Ohio College Association.

ABSTRACT OF PAPERS.

1. In speaking of the rôle of mathematics in world progress, Professor Moore said that the great importance of mathematics in various war activities was due to its extreme usefulness in many technical and scientific labors. Hence, the war record of mathematics shows in general that knowledge of this subject is of equal value in connection with the tasks of peace. In addition to this, a number of uses of mathematics during the war, such as the application of mathematical analysis to the study of wounds, may be regarded as positive contributions to world progress. This application of mathematics, together with other recent applications in such fields as biology, sociology and economics, illustrates the fact that the domain of usefulness of mathematics is as wide as the domain of science. It is the duty of mathematicians to make some effort to bring this fact home to the general public and to strive to make wider use of the applications of mathematics in connection with their courses. In this way many more college students will be induced to continue their mathematical studies through a first course in calculus, and the better trained workers in other fields will thus be able to discover many new applications of mathematics.

2. Mathematics is essential to the navigator in determining the angle of departure, in getting the details of his course, and in the maintenance of his course. The track is usually either a loxodrome or a great circle. For the loxodrome the angle of departure and distance between ports are computed from the principles of the Mercator projections. For a great circle these are computed from spherical triangles of which the vertices are the two ports, the pole and the vertex of the arc. In order to maintain his course, the navigator must be able to determine his position at sea by the principles of practical astronomy.

3. Professor Anderson noted that the principles enunciated in the theory of navigation may be utilized in creating an interest in the applications of mathematics, and thus in mathematics itself. The astronomical principles should be used as a means of creating an interest in astronomy, which should be made

available to the great majority of our students. The apparently complex formulæ of spherical trigonometry can be shown to be reducible to three simple formulæ, thus illustrating the fact that the fabric of mathematical structure is exceedingly simple, failure to grasp which fact is the source of much of the student's difficulty in his mathematical work.

4. In this paper the mathematical foundations of interior and exterior ballistics were outlined by Professor Graber. The applications of mathematics to the theory of explosives, principal stresses and strains and the design of artillery were mentioned and a typical case of design considered. Sarau's modified formulae of design were given and their application indicated. Then followed a comparison of the Mayevski velocity-resistance graph and the corresponding graph embodying the latest French results. An analytic expression was exhibited for the velocity-resistance relation between the velocities 1700 ft./sec. and 700 ft./sec. In the field of rational exterior ballistics, Siacci's method was developed and a graphic solution of problems in direct fire explained from constructed graphs. Graphs of different trajectories were also exhibited with their corresponding calculated elements.

5. The purpose of the paper by Professor Slocum was to call attention to the possibilities of aëronautics as a field of mathematical research. Thirty years ago there were authoritative proofs that mechanical flight was impossible, and it was only fifteen years ago that this was disproved by actual demonstration. In this brief period, aviation has attained a remarkable practical development and its theoretical development must follow in the near future. The present content of aëronautics was outlined under three heads: experimental, theoretical, and structural. In the second of these, the extent of mathematical analysis was indicated, and the need for an extension of theory to supplement practice pointed out. Attention was called to the future possibilities of aviation, and the duty and privilege of mathematicians to aid in the conquest of the air.

At the Friday evening round table, held in the Ohio Union and attended by twenty-three persons, the discussion, widely participated in, centered upon experience and lessons from S. A. T. C. work, supervised study, and war activities. Professor Swartzel spoke of his experiences as an inspector of S. A. T. C. work. Professor Hancock gave results of extensive mathematics tests given S. A. T. C. applicants at the University of Cincinnati, which indicated very inadequate preparation on the part of the men.

The following were elected as officers for 1919–20:

Chairman: R. L. Borger, Ohio University.
Secretary-Treasurer: G. N. Armstrong, Ohio Wesleyan University.
Third member of the Executive Committee: S. E. Rasor, Ohio State University.

G. N. Armstrong, *Secretary-Treasurer.*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION.

The Rocky Mountain Section of the Mathematical Association of America held its annual meeting at Ft. Collins, Colorado, April 12, 1919; the members were royally entertained by President Lory and the professors of Colorado Agricultural College. The papers were extremely interesting and were discussed very thoroughly. The twenty-four persons in attendance included the following thirteen members of the Association: I. M. DeLong, University of Colorado, J. C. Fitterer, University of Wyoming, W. H. Hill, High School, Greeley, Col., Claribel Kendall, University of Colorado, O. C. Lester, University of Colorado, G. H. Light, University of Colorado, S. L. Macdonald, Colorado Agricultural College, O. A. Randolph, University of Colorado, C. B. Ridgaway, University of Wyoming, C. H. Sisam, Colorado College, C. S. Sperry, University of Colorado, T. O. Walton, Colorado School of Mines, and J. W. Woodrow, University of Colorado.

The following papers were read at morning and afternoon sessions:

Generalization of the addition formulæ in trigonometry by Professor I. M. DeLong, University of Colorado; New experiments for the laboratory course in general physics, and Positions and properties of the cardinal points of a lens system from the standpoint of the wave theory, by Professor J. W. Woodrow, University of Colorado; Probable errors of Mendelian class frequencies by Mr. Breeze Boyack, Colorado A. and M. College; Calculation of high frequency resistance of wires by Professor O. C. Lester, University of Colorado; Development of empirical formulas for the solution of problems in hydraulics by Mr. L. R. Parshall, Colorado A. and M. College; Electromagnetic waves on wires by Professor O. A. Randolph, University of Colorado; On non-ruled optic surfaces whose plane sections are elliptical by Professor C. H. Sisam, Colorado College; Some characteristics of the mercury arc by Professor J. W. Woodrow, University of Colorado; An application of hyperbolic functions of a complex quantity to the determination of the performance of long distance alternating current transmission lines by Professor L. S. Foetz, Colorado A. and M. College.

An invitation from Colorado College to hold the next meeting there was received and accepted. The following were elected officers for the ensuing year:

Chairman: C. H. Sisam, Colorado College.

Vice-chairman: T. O. Walton, Colorado School of Mines.

Secretary-Treasurer: G. H. Light, University of Colorado.

G. H. Light, *Secretary-Treasurer.*

ELECTION OF MEMBERS.

The Council of the Association has elected to membership the following persons and institutions:

To individual membership.

W. S. BECKWITH, A.M. (Harvard). Prof., Ohio Northern Univ., Ada, Ohio.

B. H. BROWN, A.M. (Brown). Instr., Harvard Univ., 1919–20.

MRS. THEODOSIA T. CALLAWAY, B.S. (Columbia). Prof., Stephens Coll., Columbia, Mo.

JESSIE R. CAMPBELL, A.B. (Syracuse). Instr., Hollywood Junior Coll., Los Angeles, Calif.

W. P. DOBSON, M.A.Sc. (Toronto). Lab. Engr., Hydro Elec. Power Commission of Ontario, Toronto, Can.

PHILIP FRANKLIN, B.S. (Coll. of the City of New York). Ballistic Computer, Ord. Dept., Aberdeen Proving Ground, Md.

ELIZABETH FREAS, A.B. (Lake Erie). Computer, Coast and Geod. Survey, Washington, D. C.

J. S. GEORGES. Student, Maryville Coll., Maryville, Tenn.

ADELE C. HOLTWICK, A.M. (Washburn). Instr., Henry Kendall Coll., Tulsa, Okla.

ROBERT MASSEY, B.A. (Queen's Univ., Belfast). Prin., Prevocational Sch., Calgary, Alb., Canada.

REV. L. F. OTT, S.J. Prof. Canisius Coll., Buffalo, N. Y.

F. W. PARSONS, B.S. in E. E. (Ohio Northern). Asst., Ohio Northern Univ., Ada, Ohio.

ANNA B. PECKHAM, A.M. (Denison). Asso. Prof., Denison Univ., Granville, Ohio.

R. V. PRITCHARD, B.S. (Centr. Normal Coll.) Instr., Case Sch. of Applied Sc., Cleveland, Ohio.

REV. PATRICK RAFFERTY, S.J. Prof., Coll. of the Holy Cross, Worcester, Mass.

J. C. ROGERS, B.S. (Earlham). Dean, Piedmont Coll., Demorest, Ga.

MAX SHAMOS. Engr., Cleveland, Ohio.

MARGARETHE M. SMITH, A.M. (Radcliffe). Instr., Wilson Coll., Chambersburg, Pa.

EUGENE STEPHENS, St. Louis, Mo.

J. H. TANNER, Ph.D. (New Hampshire Coll.). Prof., Cornell Univ., Ithaca, N. Y.

A. P. WHISENHUNT, A. M. (Lenoir; Catawba). Head of Dept. of Math., Catawba Coll., Newton, N. C.

To institutional membership (with the official representative).

UNIVERSITY OF ARIZONA, Tucson, Ariz. Pres. R. B. von Kleinsmid.

ST. OLAF COLLEGE, Northfield, Minn. Prof. P. G. Schmidt.

ADELPHI COLLEGE, Brooklyn, N. Y. Dr. F. D. Blodgett.

CANISIUS COLLEGE, Buffalo, N. Y. Pres. M. J. Ahern.

BETHANY COLLEGE, Bethany, W. Va. Prof. E. H. Vance.

W. D. CAIRNS, *Secretary-Treasurer.*

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

21. For the diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$$\begin{array}{lrrrrrrrr} x = & 3, & 4, & 5, & 9, & 23, & 282, & 375, & 378661, \\ y = & -2, & -1, & 2, & 4, & 8, & 43, & 52, & 5234. \end{array}$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given diophantine equation?[1] How may all the solutions of this equation be found by a systematic procedure?

REPLY BY E. B. ESCOTT, Chicago, Ill.

A complete solution of questions of this kind does not seem to be possible. In default of a complete solution, I give four methods by which all of the given solutions may be found.

I. Let $x = a$, $y = b$ be a solution found by trial. From

$$x^2 - y^3 = 17, \tag{1}$$

and

$$a^2 - b^3 = 17, \tag{2}$$

we get

$$(x + a)(x - a) = (y - b)(y^2 + by + b^2). \tag{3}$$

Let $\dfrac{x-a}{y-b} = m$; or $x = m(y - b) + a$.

Substituting in (3), removing factor $y - b$ from both members and arranging,

$$y^2 - (m^2 - b)y + bm^2 - 2am + b^2 = 0. \tag{4}$$

The roots of (4) will be rational if its discriminant is a perfect square, i.e., if

$$m^4 - 6bm^2 + 8am - 3b^2 = r^2. \tag{5}$$

Solutions of this equation may be found by the following method, given by Euler in his *Elements of Algebra*, part II, Chapters 8 and 9.

For example, let $a = 3$, $b = -2$; then (5) becomes

$$m^4 + 12m^2 + 24m - 12 = r^2. \tag{6}$$

By trial, we find that $m = 1$ is a solution.

Let $m = p + 1$; then

$$p^4 + 4p^3 + 18p^2 + 52p + 25 = r^2. \tag{7}$$

Let

$$r = ep^2 + fp + g.$$

Substituting in (7) and combining terms

$$(e^2 - 1)p^4 + (2ef - 4)p^3 + (2eg + f^2 - 18)p^2 + (2fg - 52)p + (g^2 - 25) = 0. \tag{8}$$

In (8) we may choose e, f, and g so that the equation reduces to one of the first degree in p.

(*a*) If $e^2 - 1 = 0$, $2ef - 4 = 0$, $2eg + f^2 - 18 = 0$, we have $e = 1$, $f = 2$, $g = 7$. Then

$$p = 1, \text{ and } m = 2.$$

(*b*) If $e^2 - 1 = 0$, $2ef - 4 = 0$, $g^2 - 25 = 0$, we have $e = 1$, $f = 2$, $g = \pm 5$.

[1] In *Mathematical Questions and Solutions from the "Educational Times,"* new series, Vol. 8, 1905, p. 53, R. F. Davis gives all of the solutions indicated above and purports to show that "any single solution leads to an interminable chain of other solutions." The question is also discussed by A. Cunningham, *Mathematical Questions*, etc., *l.c.*, p. 54; and vol. 14, 1908, pp. 107–108. See also *L'Interméd. des Math.*, 1905–13, Qu. 2512, 4057.—Editor-in-Chief.

Then

$$p = -8, \text{ and } -3; \text{ and } m = -7, \text{ and } -2.$$

(c) If $e^2 - 1 = 0$, $2fg - 52 = 0$, $g^2 - 25 = 0$, we have $e = 1, f = \pm 26/5$, $g = \pm 5$.
Then

$$p = -119/40, \text{ and } -1/15; \text{ and } m = -79/40, \text{ and } 14/15.$$

(d) If $2eg + f^2 - 18 = 0$, $2fg - 52 = 0$, $g^2 - 25 = 0$, we have $e = -113/125$, $f = 26/5$, $g = 5$.

Then $p = -8725/119$, and $m = -8606/119$.

Substituting these values of m in (4), we get the following solutions:

m	y		x	
2	8	-2	23	3
-7	52	-1	-375	-4
-2	4	2	-9	-5
$-\frac{79}{40}$	$\frac{137}{64}$	$\frac{94}{25}$	$-\frac{2651}{512}$	$-\frac{1047}{125}$
$\frac{14}{15}$	$-\frac{8}{9}$	$\frac{94}{25}$	$\frac{109}{27}$	$\frac{1047}{125}$

Since one solution just found is $x = 5$, $y = 2$, we can take $a = 5$, $b = 2$.
Substitute these values in (5).

$$m^4 - 12m^2 + 40m - 12 = r^2. \tag{9}$$

Since $m = 2$ is a solution, let $m = p + 2$ and proceed as before. We find the following values of m, and the corresponding solutions:

m	y		x	
2	4	-2	9	-3
3	8	-1	23	-4
-7	4	43	-9	-282

Similarly, putting $a = 375$, $b = 52$, we find $m = 73$; whence $y = 43$, and 5234;
$x = -282$, and 378661.

The above method will give an indefinite number of rational solutions but apparently only a limited humber of integral solutions.

II. Consider the curve of the third degree,

$$x^2 - y^3 = 17. \tag{1}$$

The tangent at (a, b) is

$$2ax - 3b^2y + a^2 - 51 = 0. \tag{2}$$

This cuts the curve again in the point

$$x = -\frac{(a^2 + 153)^2 - 31212}{8a^3}, \quad y = \frac{b}{4a^2}(a^2 - 153). \tag{3}$$

Taking any rational solution (a, b) of (1), we get a second rational solution from (3). From this solution we get another, etc.

Example. From $a = 3$, $b = -2$, we get from (3)

$$x = 23, \quad y = 8.$$

III. Under the first method, equation (4) is a quadratic equation in y. Calling the roots y_1 and y_2 we have

$$y_1 + y_2 = m^2 - b.$$

If we have for a given value of m, one value of y, the other value may be found from this relation.

Considering the same equation as a quadratic in m, and calling the roots m_1 and m_2 we have

$$m_1 + m_2 = -\frac{2a}{y - b}.$$

In the same way given a value of y and one value of m, we can find another value of m.

In this way we can find very rapidly a chain of values of m and y.

Example: Let $a = 3$, $b = -2$.

$$y_1 + y_2 = m^2 + 2, \tag{1}$$

$$m_1 + m_2 = -\frac{6}{y+2}. \tag{2}$$

Starting with	$y = -2$,	$m = 2$,	(1) gives $y = 8$,
			(2) gives $m = \infty$;
	$y = 8$,	$m = 2$,	(2) gives $m = -\frac{13}{5}$;
	$y = 8$,	$m = -\frac{13}{5}$,	(1) gives $y = \frac{19}{25}$;
	$y = \frac{19}{25}$,	$m = -\frac{13}{5}$,	(2) gives $m = -\frac{49}{115}$;
. . . .			

IV. Solutions of $x^2 - y^3 = 17$ in integers may be found by Gauss's "Method of Exclusion" by which *all* the solutions under any desired limit may be found.

Since
$$y^3 + 17 = x^2,$$

the residues of $y^3 + 17$ to any modulus must be quadratic residues. Therefore, we can exclude all values of y for which $y^3 + 17 \equiv$ a non-quadratic residue. This gives us the following:

$y \not\equiv 0 \pmod 3$;
$y \not\equiv 1, 5 \pmod 8$, $y \not\equiv 7 \pmod 8$, excepting $y \equiv 15 \pmod{16}$;
$y \not\equiv 0, 1 \pmod 5$, $y \not\equiv 2 \pmod 5$, excepting $y \equiv 2 \pmod{25}$;
$y \not\equiv 0 \pmod 7$;
$y \not\equiv 0, 1, 5, 6, 7 \pmod{11}$, $y \not\equiv 3 \pmod{11}$, excepting $y \equiv -30 \pmod{121}$;
$y \not\equiv 1, 3, 9 \pmod{13}$;
$y \not\equiv 0, 3, 5, 6, 7, 10, 11, 12, 14 \pmod{17}$;
$y \not\equiv 1, 7, 10, 11, 13, 15 \pmod{19}$;
. . . .

Rejecting all the cases above and others (mod 23 and 29) we find that the only possible solutions less than 1,000 are

$$y = 2, 4, 8, 43, 52, 934, 988.$$

Trying these values, we find the first five give solutions. The last two do not.

Remark. It is worth while noting the essential difference between the equations

$$x^2 - a = y^3,$$

and

$$x^2 + a = y^3.$$

The latter equation has only a finite number of solutions, which can be found by a perfectly general method,[1] but it is quite different with the first equation. The reason is that the properties of quadratic forms with positive determinant are quite different from those with a negative determinant. (The word "determinant" is used in its original sense of discriminant.)

NOTE. In the last section of an article[2] entitled "The Diophantine Equation $y^2 - k = x^3$," L. J. Mordell indicates the results obtained as to the existence of an infinite number of solutions, a finite number, or none, for all values of k between ± 100. In accordance with his list, the case $k = 17$ should yield an infinite set of integral solutions. The question under discussion thus still presents as a challenge to our readers the alternative of either disproving Mr. Mordell's result for the case $k = 17$, or else verifying it and producing one or more new solutions.—EDITOR.

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[1] Gerono showed (*Nouvelles Annales de Mathématiques*, 1877, pp. 325–326) that there are *no* integral solutions of the equation $x^2 + 17 = y^3$.—Editor-in-Chief.

[2] *Proceedings of the London Mathematical Society*, series 2, vol. 13 (1914), pp. 60–80.

32. In a discussion of the Peaucellier[1] Cell by analytic methods the following equations are obtained:

$$(1)\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - b^2 = 0; \qquad (2)\ (x_3 - x_1)^2 + (y_3 - y_1)^2 - b^2 = 0;$$

$$(3)\ (x_2 - X)^2 + (y_2 - Y)^2 - b^2 = 0; \qquad (4)\ (x_3 - X)^2 + (y_3 - Y)^2 - b^2 = 0;$$

$$(5)\ x_2^2 + y_2^2 - K^2 = 0; \qquad (6)\ x_3^2 + y_3^2 - K^2 = 0;$$

$$(7)\ x_1^2 + y_1^2 - 2cx_1 = 0.$$

The result of eliminating x_1, y_1, x_2, y_2, x_3, y_3 gives an equation of the first degree, which establishes that the linkage will trace a straight line. There are various ways of effecting this elimination.

1. What element of the situation is left unused by the following procedure in the elimination?
   (*a*) From equations (1), (3), (5) eliminate x_2 and y_2 and obtain an equation

$$(8)\ f_1(x_1, y_1) = 0.$$

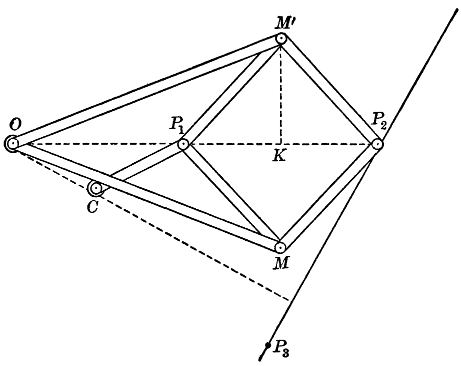
   (*b*) From equations (2), (4), (6) eliminate x_3 and y_3 and obtain an equation

$$(9)\ f_2(x_1, y_1) = 0.$$

   (*c*) From equations (7), (8), (9) eliminate x_1 and y_1 and obtain the desired equation.
2. How should this procedure be supplemented to secure the result?

Reply by G. H. Ling, University of Saskatchewan.

The configuration discussed has a line of symmetry; the equations developed show that the points (x_2, y_2) and (x_3, y_3) are *either coincident or symmetrical with respect to the line of symmetry*, but they fail to express that the essential feature of the configuration is that the two points mentioned are *not* coincident.



The suggested procedure for elimination cannot take account of this unexpressed fact. Equations (8) and (9) are identical and the elimination of x_1 and y_1 from (7), (8) and (9) halts because of the lack of three independent equations.

The procedure may be supplemented as follows:

If two of the three equations (1), (3), (5) be solved for x_2 and y_2 two distinct solutions are obtained and these must be the values for x_2 and y_2, and for x_3 and y_3. Both of these sets of values must satisfy that one of the equations (1), (3), (5) which is not employed in the solution just mentioned. The substitution of both of these sets of values in this third equation yields two equations (8) and (9) which associated with the equation (7) yield a single resultant equation free from x_1 and y_1. This last equation is the equation of the desired locus. (See May Monthly, p. 188.)

It is easy to see that the case here discussed can be generalized, though in the more general cases the treatment of the problem would not be nearly so simple.

DISCUSSIONS.

Every teacher of trigonometry, analytic geometry, and the calculus has at times experienced the difficulty of inducing students to regard the radian measure of an angle as a pure number. How is this difficulty to be met? Professor Car-

[1] If reference is made to the article on "Linkages" in the December, 1915, Monthly, by Mr. Leavens, the following coördinates may be applied to his figure (reproduced above): O (0, 0); C (c, 0); P_1 (x_1, y_1); $M(x_2, y_2)$; M' (x_3, y_3); P_2 (X, Y).

ver's discussion below, covering substantially the ground of a paper presented by him at the summer meeting of the Mathematical Association of America in September, 1918, should be suggestive and stimulating, and will, it is hoped, give rise to further expressions of opinion.

Trigonometric Functions—of What?[1]

By W. B. Carver, Cornell University.

The first idea which our students get of a trigonometric function—say $\sin x$—is that the argument x is an *angle*, a geometric entity. According to this conception, the sine of a right angle is 1 whether one thinks of it as an angle of 90° or of $\pi/2$ radians; while $\sin 2$ has different values according as the 2 means 2 radians, 2 degrees, or 2 right angles.[2] Later the student needs the notion of the number $\sin x$ as a function of the *number* x, a functional relation which is not dependent upon any sort of geometric ideas or units of geometric measurement. This second point of view is needed not only by those who specialize in mathematics, but also by the large class of students who go no further than a first course in the calculus, and whose purpose may be entirely utilitarian.

It is the writer's conviction that, certainly in our text-books, and possibly in our teaching, we are not doing as much as we might to help the student across from the one point of view to the other.

In our courses in trigonometry the first point of view must prevail: but the way may be prepared for the second by insisting upon familiarity with the circular measurement of angles. The radian unit should be introduced early and *used frequently* throughout the course. The tables of functions should have a column giving the angle in radians adjacent to the column reading degrees and minutes.[3] In the problems for solution in both right and oblique triangles, the given angles should, in at least a few cases, be expressed in radians. The relation of the number π to this method of measuring angles should be made clear. That there is confusion on this point is indicated by such questions as "Why does π mean 3.1416 in one place, and in another place 180°?"

Analytic geometry should bring us nearer to the idea of a trigonometric function of a number. Should a student be permitted to draw the curve represented by the equation $y = \sin x$ with the wave-length and amplitude in any ratio that pleases him? If so, may he plot $4x^2 + y^2 = 1$ as a circle, or $x^2 + y^2 = 1$ as an ellipse? In drawing such trigonometric curves, should we not insist that the units of length on the x and y axes shall be the same, and that $\sin x$ means the sine of an angle of x radians? Queerly enough, in polar coördinates the trouble arises when we do *not* have trigonometric functions of θ rather than

[1] Read before the Mathematical Association of America, September 6, 1918.

[2] Many of the text-books state explicitly the convention that the radian is to be assumed when no other unit is indicated.

[3] Such tables are surprisingly scarce; as are also protractors reading radians and decimal parts of radians.

when we do. Why does the curve $\rho = \theta^2$ raise the question of the unit of angle measurement while the curve $\rho = \sin\theta$ does not? Is the θ of our polar coördinates a number or an angle? The plotting of the curve $\sin\rho = \theta$ presents the difficulty in even more acute form. The best way out of all these difficulties would seem to be to emphasize the fact that ρ and θ are numbers; and that in polar coördinates ρ is represented graphically by ρ units (any convenient unit) of length, while θ is represented by an angle of θ *radians*.

It is in the course in calculus that the student must be brought to a full realization of the significance of $\sin x$ as a function of a number. In deducing the formula $d\sin x/dx = \cos x$, the idea of x as an angle is still in evidence; but it should be emphasized that the validity of the formula depends upon the fact that x means the number of *radians* in the angle. One good method of impressing this point is *first* to deduce a formula assuming that x means an angle of x degrees, so that the student may see clearly the advantage of the use of the radian unit. Being thus committed to this unit of angle measurement, it will follow later that in the formula

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C,$$

$\sin^{-1} x$ means the number of radians in the angle[1] whose sine is x. How many of our students have any reason for a choice between $\pi/4$ and $45°$ as a value for the integral $\int_0^1 \sqrt{1-x^2}dx$, other than the fact that the latter value does not seem to give the "right answer" for the area of the quarter circle?

When we have expanded $\sin x$ in a power series, convergent (happily) for all values of x, we have finally the basis for a definition of the function which is independent of geometric notions. An analogy may be helpful to the student at this point. The geometric notions of the area and length of side of a square may be used to exhibit the relation between the numbers x and $\sqrt{x}$; but nevertheless this number relation is independent of such geometric considerations. And the student of fairly keen mathematical insight will be interested to see that he now has a relation between the numbers x and $\sin x$ which is similarly independent of geometric considerations.

RECENT PUBLICATIONS.

REVIEWS.

Mathematics for Freshmen.

1. *Introduction to the Elementary Functions.* By R. B. McClenon with the editorial coöperation of W. J. Rusk. Boston, Ginn, 1918. 8vo. 9 + 244 pp. Price $1.80.

[1] It is also important, of course, that $\sin^{-1} x$ should have been so defined as to make it a single-valued and continuous function.

2. *Freshman Mathematics.* By W. R. Ransom. New York, Longmans, 1918. 12mo. 12 + 285 pp. Price $1.50.

3. *Elementary Mathematical Analysis. A textbook for first-year college students.* By C. S. Slichter. Second edition revised and entirely reset. New York, McGraw-Hill, 1918. 12mo. 18 + 497 pp. Price $2.50.

4. *Unified Mathematics.* By L. C. Karpinski, H. Y. Benedict, and J. W. Calhoun. Boston, Heath, 1918. 12mo. 8 + 522 pp. Price $2.80.

What shall be the content of the course in freshman mathematics? This is a question that has been occupying the minds of many of the teachers of mathematics in our colleges for the past five or six years. The traditional course in mathematics for the freshman year has held sway for a long period of time and the assaults that have been directed against it as a required subject have, in the main, left it secure.

Why should the content and arrangement of the course be changed? The fact that its security might be at stake is neither a sufficient nor an intelligent reason. So long as technique is regarded as the all-important aspect of the subject, there is no reason why there should be any change from the traditional course, except possibly toward unification and even this may be questioned. On the other hand, in view of the fact that freshman mathematics is a required course in the majority of our colleges, there is a feeling that all the emphasis should not be put on technique, but that insight and understanding should receive their due share of emphasis, thus affording a clearer perception or recognition of the worth of the subject in practical application. The four books which we shall now examine attempt to answer the question confronting us.

1. The subject matter which consists of trigonometry, elementary analytic geometry, and an introduction to the differential calculus, is presented in ten chapters (230 pages). This does not include the appendix, containing nine pages of review work in algebra and geometry. The authors take functionality as the unifying principle and endeavor to keep this idea foremost throughout the book.

Chapters I and II deal with graphical representation and functional relations. These two chapters are largely introductory. Much of the material contained in the 32 pages of Chapter III on graphic algebra seems somewhat disconnected and rather elementary. The quadratic equation with which the student should be familiar is taken up in too great detail. It also seems a waste of time and space to devote three pages to the factor theorem for the quadratic equation, and an even greater waste to discuss in this chapter the maximum and minimum values of the quadratic function, which comes up again in the differential calculus. Chapters IV and IX consisting of 52 pages are devoted almost entirely to trigonometry, the solution of triangles by logarithms being deferred until the exponential and logarithmic functions are taken up in Chapter X. This separation of trigonometry into three parts seems rather artificial and causes one to wonder if functionality, which is the unifying principle, does not necessitate an arrangement somewhat unnatural and not conducive to the best results. There seems

to be no good reason why Chapters IV and IX should not be combined. In the beginning of Chapter IV the trigonometric functions are defined at once for any angle. In this chapter the work is well arranged. Although velocities and forces are considered under the application of the trigonometric functions, radian measure, and angular velocity are nowhere mentioned.

Chapter V deals with simple irrational functions and the locus problem, and the three following chapters are devoted to the consideration of the straight line, the circle and the conics. In Chapters VI and VII the geometric aspect of the work receives emphasis, while graphic algebra predominates in Chapter VIII.

Some disappointment is felt over the fact that in Chapter X, the logarithmic and exponential functions seem to be introduced solely for the use of logarithms in the solution of triangles. In our opinion not enough consideration is given to the functional relation $y = \log x$. The important functions $y = \log_e x$, $y = e^x$ and $y = e^{-x^2}$ are not mentioned. The concluding chapter of the book, which is on the differential calculus, presents in 32 pages, some necessary theorems on limits, the definition of the derivative of a function, rules for finding the derivative of some of the simpler functions, rates, and maxima and minima. There is also a well-chosen set of problems. The material of this chapter is well selected and clearly presented. In our opinion the notation for the derivative, D_xy, is most confusing to the beginner and for this reason should not be employed.

2. If the object of the freshman course in mathematics is to give the student interesting material with which to work and from which may be derived useful information, and if mathematical coherence and completeness are of no great importance to the freshman, except as demanded for such information, this book can be highly praised. But the well-selected material is not the only merit the book has. The work is well presented and the explanations are clear. It is quite surprising too, to find that so much can be put into 285 pages without undue congestion.

There are twenty-six short chapters in the book. The more elementary part of the work is given in the first seven chapters. The leading topics of these chapters are arithmetical computation, the application of trigonometry to right triangles, and an introduction to coördinates. The trigonometric functions are defined for acute angles and the definitions for any angle are not given when coördinates are first introduced but are deferred until the consideration of vectors comes up (Chapter XII) toward the close of the work on trigonometry. This causes an awkward situation on page 96 where the relation, $\sin A = \sin (180° - A)$, must be given as a definition.

The next seven chapters of the book are devoted entirely to trigonometry and its applications. The use of the slide rule is explained and a chapter on spherical trigonometry and its applications is added.

Determinants, variation, and the solution of numerical equations are the topics of algebra which are taken up in Chapters XIV–XVII.

The last eight chapters of the book deal primarily with differential and integral calculus and their applications. The derivative is made use of in plotting

exponential and other special curves; in the study of rates and in finding maximum and minimum values. The fifteen pages of the integral calculus are devoted to its application in finding areas and volumes.

The book on the whole is interesting and stimulating. The exercises and problems have been selected and graded with care, and no demand is made on the students' ability to handle complicated algebraic expressions. On the other hand, the book lacks coherence. The analytic geometry that is given is scattered throughout the book, and one has a feeling that this very important subject has not received the attention it merits. The conic sections are inadequately treated and many of their important properties are not mentioned. Since one of the most important problems that confronts the author of a unified course in mathematics, is the proper selection from a wealth of material, it seems unwise to devote twelve pages to spherical trigonometry and deal scantily with analytic geometry.

3. Despite the author's assertion in the preface that "this book is not intended to be a text on 'Practical Mathematics,' in the sense of making use of scientific material and of fundamental notions not already in the possession of the student, or in the sense of making the principles of mathematics secondary to technique," the book has the appearance of a manual on practical mathematics.

On the first reading one is inclined to say that the book has been written from the engineering point of view. But while the author deals elaborately with the practical applications of all of the mathematical material that the student has on hand or acquires throughout the course, he none the less lays emphasis on the proper understanding of mathematical principles and, as regards technique and application, strives to maintain the proper balance. Just how well he has succeeded in doing this it is hard to say.

The book will certainly arouse interest and provoke thought. It shows breadth of thought both in conception and execution. It is impossible to pass judgment, without a fair trial, on its usefulness as a textbook.

The first three chapters of the book deal with scales, rectangular coördinates, variables and functions, and the power function. In our opinion this part of the work is given in too great detail. The author has an admirable way of presenting transformations early in the course. Certain theorems on loci are stated in connection with the graphs of functions and these theorems explain in a simple but satisfactory way translation, rotation, expansion and shear.

In Chapter IV the trigonometric functions are defined for any angle and the treatment of the functions is taken up in connection with the analytic geometry of the circle. While Chapter V, on the ellipse and hyperbola has some connection with Chapter IV, there is a rather abrupt change in passing to the selected topics of algebra in Chapters VI–VIII. After this there is another change in taking up the logarithmic and exponential functions of Chapter IX and the remainder of the work in trigonometry in Chapter X. A better arrangement seems possible. Why can not Chapter IX follow Chapter III and Chapter X follow Chapter IV? This seems more desirable and more natural from the point of view of coherence. The functionality idea would thus be predominant in

the first four chapters and there would be a natural shift through the circular functions to the two following chapters in trigonometry.

A review of the most important topics of secondary school algebra is found in the twenty-four pages of the appendix.

4. To what extent have the authors succeeded in unifying algebra, trigonometry and analytic geometry and what is the process of unification? They have not chosen the way of functionality. The notion of a function first appears on page 58, and while it is constantly in use throughout the book, it is not predominant and does not determine the arrangement of the subject matter.

The part of the book that has been unified in a splendid way is the first seventeen chapters (280 pages). This includes the application of algebra to arithmetic, trigonometry and its applications, and the more elementary part of analytic geometry. The graphical method serves in a most natural way as a connecting link. A characteristic feature of this part of the book is the "timing exercises." Wherever a set of exercises is given for the purpose of developing the students' mechanical ability, a tentative time is specified. This will undoubtedly stimulate interest and challenge the students' mental activity. The next seven chapters are devoted to the conics and their applications. The most important properties of the conics are derived and the subject is enlivened by a splendid collection of practical problems. In connection with the practical problems on elliptic and parabolic arches, there are several photographs of bridges where these arches are used. There are also photographs showing the use of elliptic gears.

The solution of numerical equations, wave motion, laws of growth, polar coördinates, complex numbers, and solid analytic geometry are the subjects considered in the last eight chapters. The two most interesting chapters in this part of the book are those on wave motion and laws of growth. The authors have succeeded splendidly in presenting, in connection with the more formal part of the work, interesting problems that have practical application. Indeed this is a distinctive feature of their book. The problem about the big gun that bombarded Paris and the curve of healing of a wound, give evidence that they are up-to-date. While they have placed much emphasis on the practical application of the subject matter, they have not neglected the more formal part of the work.

There is some doubt as to how far they have succeeded in unifying the later material. The latter part of the book seems to lack the coherence that is characteristic of the first part. Perhaps this is due to the fact that trigonometry forms the body of the first part.

There are many historical notes throughout the book.

The regularity of the type gives it an attractive appearance.

The following tables are bound in with the text: squares, cubes, square roots, cube roots, and reciprocals (from 1 to 100); logarithms; natural functions; radian measure of angles; tables for e^x and e^{-x}; tables of interest functions.

Having examined these four books in detail let us now take a broader view of them.

So far as unification is concerned, they have a common purpose, though Ransom does not give as much attention to coherence as the others do. There are differences however in the subject matter as well as in the way by which unity may be accomplished. There is also a certain difference in the spirit of the books. The chief purpose of McClenon and Rusk seems to be to unify the subject matter with which they are concerned. The other three books breathe a different spirit. They, too, seek coherence but, in addition to this, they agree that a prominent feature of the work of the freshmen, should be the use of interesting and helpful problem material—material that brings together the workaday world and mathematical principles. They have succeeded admirably in collecting such material.

As regards subject matter, there is rather close agreement on the work in trigonometry. Has enough attention been given to trigonometry or has it been cut down to an irreducible minimum? In McClenon there are 73 pages on trigonometry and its application; in Ransom, 76 pages; in Slichter, 90 pages; in Karpinski, Benedict and Calhoun 114 pages. In our opinion the best arrangement of the trigonometry is given by Karpinski, Benedict and Calhoun. Ransom scatters the subject throughout the book in six different parts.

There is disagreement about the topics in college algebra. McClenon and Rusk confine their attention to graphic algebra. Ransom devotes 32 pages to determinants and the solution of numerical equations; Slichter gives 17 pages to permutations and combinations; Karpinski, Benedict and Calhoun give 14 pages to the solution of numerical equations.

Analytic geometry with its applications is given far greater emphasis by Karpinski, Benedict and Calhoun, and Slichter than by the other two authors. Since Slichter and Karpinski lay so much emphasis on the practical application of mathematics, it is rather surprising that they do not give the fundamental ideas of the differential and integral calculus, and their applications to the simplest functions. In Karpinski the work on poles and polars, transformations, polar coördinates, complex numbers, and solid analytic geometry covers about 90 pages. In our opinion it would be more profitable to devote this space to the calculus. Slichter could also find time for the calculus by not going into such elaborate detail in different parts of the book. Many of his directions, specifications and remarks could be left to the teacher.

The tendency to make as wide use as possible, of practical problems in illustrating mathematical principles is a step in the right direction. When the teachers of mathematics know more of its applications and make wider use of them, and when the teachers of allied subjects call into use the mathematics that the student may have in his possession, there will then be a greater recognition of its supreme importance.

It means little to praise or criticize these books without first giving them a fair trial. Whatever their faults, this new arrangement of material should be tested in the class room. We welcome the books as a contribution towards the solution of our problem. They will be of service to every mathematics teacher of college freshmen.

George W. Mullins.

Analytic Geometry. By MARIA M. ROBERTS and JULIA T. COLPITTS. New York, Wiley, 1918. 10 + 229 pp. Price $1.60.

The following extracts from the preface serve to indicate the general plan of the book: "This book is the result of several years of experience in teaching mathematics to students of engineering and science. . . . Emphasis has been placed on those portions of analytic geometry in which experience has shown the student of calculus to be most frequently deficient. In this connection, in particular, polar coördinates have received more than usual attention and transcendental and parametric equations considerable space. . . . The material is so arranged that the first ten chapters together with a portion of Chapter XIII include those subjects ordinarily offered to such freshman classes as cover in the first year the three subjects, college algebra, trigonometry and analytic geometry. The addition of Chapter XIV will round out a good course of five hours a week for a semester. The entire book should easily be covered in a three-hour course throughout a year."

Chapter I (17 pages) is entitled Cartesian Coördinates. After the usual introduction, the formulæ for distance between two points, the slope of a line, the point of division, and the area of a triangle, are introduced and applied to numerical examples. Some familiar theorems from plane geometry are introduced for proof by analytic methods.

Chapter II (29 pages) is devoted to Loci. Here algebraic equations only are considered. In deriving the equation of a locus emphasis is placed on the fact that the student must show "why the coördinates of all points off the locus fail to satisfy the equation." Plotting the locus of an equation is considered at length. Most of the examples involve conic sections, although a few higher plane curves are introduced. The outline for the discussion of an equation is very good, particularly the section on extent of the curve. Curves with asymptotes parallel to one of the coördinate axes are introduced. The chapter closes with an interesting section on loci through the intersections of two given loci.

Chapter III (26 pages) is devoted to the straight line. Here the proof for the normal form of the equation, which is made to depend on the intercept form, is not satisfactory since it does not hold when the line passes through the origin. Moreover there seems to be no good reason why p in this equation should not be considered as always positive, instead of positive above the x axis and negative below as suggested by the authors. In fact, in the later chapter on solid analytic geometry, the perpendicular distance from the origin to a plane is always considered positive.

The fourth Chapter (17 pages) introduces polar coördinates. The treatment here is excellent but the subject is one which offers serious difficulties to the student. Since polar coördinates are not used elsewhere in the book, save in the chapter on transformation of coördinates, Chapter IV might well come later, after the student has acquired facility in methods of analytic geometry.

Transformation of coördinates comes in the fifth chapter, very appropriately preceding the chapters on the various conic sections. The treatment here is

brief (10 pages) but adequate. Translation as well as rotation of axes are considered and equations are simplified by each of these transformations. Examples involving the removal of the xy term are limited, however, to cases where a rotation of 45° is called for. The chapter concludes with a section on transformation from rectangular to polar coördinates and vice versa.

Chapters VI (20 pages), VII (10 pages), VIII (12 pages) and IX (14 pages) take up the circle, parabola, ellipse and hyperbola, respectively. In the case of the circle it seems hardly necessary to call $(x - h)^2 + (y - k)^2 = r^2$ the "First standard equation of a circle" and $x^2 + y^2 = r^2$ the "Second standard equation." Similarly the four standard equations of the ellipse and of the hyperbola could well be reduced to two, the others being merely special cases.

The chapter on the parabola begins with a derivation of the rectangular equation of a conic section from the ratio definition. The ratio definition is used also in deriving the equations of the ellipse and of the hyperbola, the familiar properties involving respectively the sum and difference of the focal radii being proved as theorems, not assumed as definitions. This seems on the whole the best method of approach. The treatment here is brief, only the fundamental properties of the various conics being discussed.

Tangents and normals are considered in the tenth chapter of twelve pages. The slope of a tangent is derived by means of the disguised calculus method with h and k substituted for Δx and Δy. This method is applied in particular to the general equation of the second degree, the resulting rule being very convenient in numerical examples. The chapter concludes with a paragraph on the lengths of tangents, normals, subtangents and subnormals and one on the equation of the tangent when the slope is given.

The remaining chapters take up topics of a more advanced nature and are not intended, as the preface indicates, for freshman classes. Poles, polars, diameters and confocal conics are discussed in the eleventh chapter of thirteen pages. The notion of pole and polar is introduced by means of harmonic division. The other topics are treated in the usual way.

A brief chapter (the twelfth) of only six pages follows. This takes up the general equation of the second degree. Methods for the classification of conics are developed and examples involving conics through five given points solved.

Chapter XIII (20 pages) involves transcendental and parametric equations. The usual logarithmic and exponential curves are plotted as well as the graphs of trigonometric and inverse trigonometric functions. Under parametric equations, some of the more important curves considered are the cycloid, the epicycloid, the hypocycloid, the path of a projectile, the witch and the cissoid.

The concluding chapter, numbered fourteen, is one of the best of the book. The material here is well selected and well presented. In the brief space of thirty-three pages one finds practically all the material regarding solid analytic geometry necessary for a first course in the calculus. In addition to the standard proofs and examples concerning the line and plane, the chapter includes discussions of cylindrical surfaces, spherical surfaces, surfaces of revolution, the more

important forms of quadric surfaces and the equations of a space curve. The statement that the polar coördinates of a point in space are ρ, α, β and γ (ρ being the radius vector and α, β and γ the direction angles) is not in accordance with the usual notation for polar coördinates. Moreover the use of four coördinates for a point in some cases and three in others is confusing to the student, even though he is told that the four are not independent.

The problems in the book are almost entirely from the field of pure mathematics, few from engineering or other sources being used. The answers are given for most of the problems, either immediately after the statement of the problem or in a list of answers at the end of the book. Sets of miscellaneous examples inserted from time to time furnish an opportunity for review.

As already indicated in quotations from the preface, the book is conveniently arranged for use either in a course for a term, a semester or a year. The traditional over emphasis on conic sections is relieved by the excellent chapters on polar coördinates, transcendental equations and parametric equations.

CLINTON H. CURRIER.

PROVIDENCE, R. I.,
March 1, 1919.

Lezioni sulla teoria dei gruppi continui finiti di transformazioni. By L. BIANCHI. Pisa, E. Spoerri, 1918. 8vo. 6 + 590 pp. Price 25 lire.

This volume was written for the students of the Italian universities and is a reproduction, with few changes, of the course by the same author which appeared in lithographed form about fifteen years ago. It constitutes the second introductory treatise on continuous groups in the Italian language, an earlier one having been published by G. Vivanti under the title *Teoria dei gruppi di transformazioni*, 1898; translated into French by A. Boulanger, 1904.

The interest of Italian mathematicians in the general field of group theory is reflected by the fact that they have now in their own language at least five books on this subject. Two of these relate to finite groups, viz., the translation into Italian of Netto's *Substitutionentheorie* by G. Battaglini, 1885, and the excellent introduction entitled *Lezioni sulla teoria dei gruppi di sostituzioni*, 1910, by the author of the work now under review. Their other books on group theory are devoted to the groups of analysis and geometry, and include the well-known *Teoria dei gruppi discontinui e delle funzioni automorfe* by G. Fubini, 1908, in addition to the two works mentioned in the preceding paragraph.

L. Bianchi is well known to American mathematicians as the author of good books relating to each of the three great fields of mathematics—algebra, analysis and geometry. His wide range of mathematical knowledge and his extensive experience as a writer have qualified him admirably for the production of a work having such varied contact as a treatise on continuous groups should exhibit. In the present work he has restricted himself largely to a lucid exposition of the theories contained in the first volume of the *Theorie der Transformationsgruppen* by Lie-Engel, 1888–1893.

The book is divided into thirteen chapters. The first of these is devoted to a brief exposition of auxiliary theories relating mainly to systems of differential equations. In chapters II to XI inclusive there is given a very clear exposition of the fundamental concepts of continuous groups, exclusive of the important theory of contact transformations. The last two chapters are devoted to applications. In the former of these, Chapter XII, the author gives the fundamental concepts relating to the applications of the theory of continuous groups in the theory of differential equations. The final chapter is devoted to geometric applications and bears the heading "Applicazioni alla teoria degli spazi pluridimensionali con gruppi continui di movimenti."

Although the present work is considerably more concise than the *Vorlesungen über Continuierliche Gruppen*, 1893, by Lie-Scheffers it is almost equally elementary and leads the student much more rapidly to the fundamental theorems of the subject. The book is unusually free from errors and constitutes a decidedly useful aid for acquiring quickly an insight into the abstract theory of finite continuous groups. While the founder of the theory of continuous groups was a Norwegian he published most of his extensive systematic developments in the German language, and the present work will be especially welcomed by those who desire to acquire a knowledge of this field but do not read German comfortably.

G. A. MILLER.

An Introductory Treatise on Dynamical Astronomy. By H. C. PLUMMER. Cambridge: at the University Press, 1918. Royal 8vo. 19 + 343 pp. Price 18 shillings.

Contents—Chapter I: The law of gravitation, 1–10; II: Introductory propositions, 11–20; III: Motion under a central attraction, 21–32; IV: Expansions in elliptic motion, 33–48; V: Relations between two or more positions in an orbit and the time, 49–64; VI: The orbit in space, 65–72; VII: Conditions for the determination of an elliptic orbit, 73–84; VIII: Determination of an orbit, Method of Gauss, 85–93; IX: Determination of parabolic and circular orbits, 94–102; X: Orbits of double stars, 103–114; XI: Orbits of spectroscopic binaries, 115–128; XII: Dynamical principles, 129–141; XIII: Variation of elements, 142–157; XIV: The disturbing function, 158–176; XV: Absolute perturbations, 177–191; XVI: Secular perturbations, 192–206; XVII: Secular inequalities, Method of Gauss, 207–217; XVIII: Special perturbations, 218–235; XIX: The restricted problem of three bodies, 236–253; XX-XXI: Lunar theory, 254–291; XXII: Precession, nutation and time, 292–311; XXIII: Libration of the moon, 312–322; XXIV: Formulæ of numerical calculation, 323–340; Index, 341–343.

Quotations from the preface: "This book is intended to provide an introduction to those parts of Astronomy which require dynamical treatment. To cover the whole of this wide subject, even in a preliminary way, within the limits of a single volume of moderate size would be manifestly impossible. Thus the treatment of bodies of definite shape and of deformable bodies is entirely excluded, and hence no reference will be found to problems of geodesy or the many aspects of tidal theory. Already the study of stellar motions is bringing the methods of statistical mechanics into use for astronomical purposes, but this development is both too recent and too distinct in its subject-matter to find a place here. . . . Since the main object in view has been to cover a wide extent of ground in a tolerably adequate way rather than to delay over critical details, the absence of mathematical rigour may sometimes be noticed. Very little attention is given to such questions as the convergence of series. It is not to be inferred that these points are unimportant or that the modern astronomer can afford to disregard them. But apart from a few simple cases where the reader will either be able to supply what is necessary for himself, or would not benefit even if a critical discussion were added, such questions are extremely

difficult and have not always found a solution as yet. It is precisely one of the aims of this book to increase the number of those who can appreciate this side of the subject and will contribute to its elucidation."

Mr. Plummer is professor of astronomy in the University of Dublin and Royal Astronomer of Ireland.

Handbook of Mathematics for Engineers. By E. V. HUNTINGTON. With Tables of Weights and Measures by L. A. FISCHER. New York, McGraw-Hill, 1918. 12mo. 5 + 191 pp. Price $1.50.

This volume is a reprint of sections 1 and 2 of the *Mechanical Engineers' Handbook* edited by L. S. Marks (1916). Mr. Fischer's tables occupy pages 70–85. The rest of the work is by Professor Huntington.

The "mathematical tables" (pages 2–69) are: squares of numbers, cubes of numbers, square roots of numbers, cube roots of numbers, three-halves powers of numbers, reciprocals of numbers, circles (areas, segments, etc.), spheres (volumes, segments, etc.), regular polygons, binomial coefficients, common logarithms, degrees and radians, trigonometric functions, exponentials, hyperbolic (Napierian) logarithms, hyperbolic functions, multiples of 0.4343 and 2.3026, residuals and probable errors, compound interest and annuities, and decimal equivalents.

The contents of the rest of the work are as follows: Arithmetic (Numerical computation, logarithms, the slide rule, computing machines, financial arithmetic), 88–98; Geometry and Mensuration (Geometrical theorems, geometrical constructions, lengths and areas of plane figures, surfaces and volumes of solids), 99–111; Algebra (Formal algebra, solution of equations in one unknown quantity, solution of simultaneous equations, determinants, imaginary or complex quantities), 112–127; Trigonometry (Formal trigonometry, solution of plane triangles, solution of spherical triangles, hyperbolic functions), 128–135; Analytic Geometry (The point and the straight line, the circle, the parabola, the ellipse, the hyperbola, the catenary, other useful curves), 136–156; Differential and Integral Calculus (Derivatives and differentials, maxima and minima, expansion in series, indeterminate forms, curvature, table of indefinite integrals, definite integrals, differential equations), 157–172; Graphical representation of functions (Equations involving two variables, equations involving three variables, equations involving four variables), 173–184; Vector Analysis, 185–186; Index, 187–191.

The volume contains a great amount of useful and interesting information admirably edited.

The American Society of Mechanical Engineers, New York. *The Weights and Measures of Latin America*[1] by F. A. HALSEY. New York, 1918. 8vo. 34 pp.

This is a report based on the replies received after distributing five hundred copies of a questionnaire throughout South and Central America and the West Indies. Four of the six questions of the questionnaire were as follows:

1. What are the units of weight and measure commonly used with relation to the buying and selling at retail of the following products?—Groceries, fruits, milk, butter and cheese, other farm products, hardware, fish, meat, flour, tea and coffee, dry goods, fuel, tobacco, miscellaneous.

2. What are the units of measure commonly used with relation to buying and selling articles of clothing, as follows?—Ready made clothing, hats, collars, underwear and hosiery, shoes, gloves, corsets, miscellaneous.

3. What are the units of measure commonly used with relation to the sale of lands and filing of paper and deeds as follows?—In the farming districts, in the smaller towns, in the cities.

6. What are the units of weight and measure commonly used with relation to transportation tariffs?—Railway tariff for passengers and freight (load and distance), loads and rates for city transportation, loads and rates for transportation by muleback across the mountains, railway track gages and length of lines, railway equipment (units used in the construction and repairing of locomotives, cars, etc.).

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[1] Paper presented at the annual meeting of The American Society of Mechanical Engineers, December, 1918.

The inquiry sets forth the extent to which the metric system has been adopted by Latin America.

Cours d'analyse mathématique. Par E. Goursat. Troisième édition revue et augmentée. Tome 2, Paris, 1918. 8vo. 2 + 672 pp. Price (unbound) 36 francs.

Quotation from the preface: "Cette nouvelle édition ne diffère de la précédente que par quelques additions, dont la plus importante est relative à une proposition célèbre de M. Picard. Ce théorème a fait l'objet d'un grand nombre de travaux, qui ont conduit à une démonstration presque élémentaire, ne faisant appel qu'à des inégalités classiques de la théorie des séries entières. Il m'a semblé qu'une démonstration de cette nature avait sa place marquée dans un Cours d'Analyse."

The third edition of tome 2 contains 24 pages more than the second edition.

NOTES.

In *A Calendar of Leading Experiments* by W. S. Franklin and B. MacNutt (South Bethlehem, Pa., Franklin, MacNutt, and Charles, 1918), Part I (pages 1–68) is entitled, "Mechanics," and Part VI (pages 163–205), "The dynamics of wave-motion."

In *The California Alumni Fortnightly,* volume 12, page 131, May 3, 1919, there is a portrait of Professor Frank Morley, who is to teach in the Summer School of the University of California.

The Carnegie Institution of Washington has published the first volume of Professor L. E. Dickson's monumental *History of the Theory of Numbers* (Paper, $7.50; cloth, $8.00). The second volume is in the press.

An Italian translation of J. W. Young's *Lectures on Fundamental Concepts of Algebra and Geometry* has been recently published by Luigi Pierro, Naples.[1] The translated text has neither been enlarged nor diminished in the new edition. The translator has, however, added a large number of explanatory and bibliographical notes.

Wiley has just published *Lectures on Ten British Physicists* by the late Alexander Macfarlane. It is a companion volume to his *Lectures on Ten British Mathematicians* published in 1916 and it is No. 20 in the series of "Mathematical Monographs." The physicists considered are: James Clerk Maxwell (1831–1879); William John Macquern Rankine (1820–1872); Peter Guthrie Tait (1831–1901); Sir William Thomson, First Lord Kelvin (1829–1907); Charles Babbage (1791–1871); William Whewell (1794–1866); Sir George Gabriel Stokes (1819–1903); Sir George Biddell Airy (1801–1892); John Couch Adams (1819–1892); Sir John Frederick William Herschel (1792–1871).

In *The American Year Book . . . 1918* (New York, Appleton, 1919) are articles on "Mathematics" (pages 615–617) by G. A. Miller, and on "Astronomy"

[1] *I Concetti fondamentali dell'Algebra e della Geometria,* Versione e Note di Domenico Mercogliano, Con Prefazone di Gino Loria 1919. 418 pp. Price 8 lire.

(pages 617–622) by R. S. Dugan. The five divisions of the article on mathematics are entitled: 1. History of Mathematics; 2. Effects of the War on Mathematical Advances; 3. Advances in Pure Mathematics; 4. Teaching of Mathematics; 5. Personal Notes. It is stated that Professor Florian Cajori "enjoys the unique distinction of being the first regular university professor of this history [of mathematics] in the world." There is no mention of America's mathematical contributions to ballistics and other branches of applied mathematics. Under Personal Notes there are references to one foreigner, G. Cantor, "founder of the theory of aggregates," who died January 6, and to eight Americans.

Encyclopædia of Religion and Ethics. Edited by JAMES HASTINGS with the assistance of J. A. Selbie and L. H. Gray. Edinburgh, Clark, Volume 9, 1917; Volume 10, 1919. These volumes include the following articles of interest to the mathematician:

"Numbers"—Introductory by T. Davidson, 406–407; Aryan by A. B. Keith, 407–413; Semitic by W. Cruickshank, 413–417.

"Pascal" by W. F. Cobb, 652–658; one of the three sections of the biography is entitled "His mathematical aptitude."

"Greek Philosophy" by Paul Shorey, 859–865.

"Plato and Platonism" by Henry Jackson, 54–61.

"Points of the Compass" by T. D. Atkinson, 73–88.

"Pythagoras and Pythagoreanism" by John Burnet, 520–530.

A New English Dictionary on Historical Principles (Oxford, Clarendon Press) —the greatest dictionary, in any language, ever published—is nearing completion. The part issued in March, 1918, contains discussions of the following words of interest to the mathematician: Supplement, surd, surdesolid and sursolid, surface, surveying and surveyor, suversed, and swan-pan (*also* souan-, shwan-, swam-, suan-). The Chinese word Soroban, which occurs frequently in English writings, and is another equivalent of swan-pan, is not given in the '*N.E.D.*'

The earliest quotations given in connection with some of these words as mathematical terms are as follows: 1570, Billingsley's *Euclid*, Book 1, Theorem 32, "In every parallelograme, the supplementes of those parallelogrammes which are about the diameter, are equall the one to the other"—1551, Recorde's *Path. Knowl.* II. Pref., "Quantitiees partly rationall, and partly surde." [Surd is derived from the Latin word *surdus* meaning deaf. "The mathematical sense 'irrational' arises from Latin *surdus* being used to render Greek ἄλογος (Euclid, bk. 10, Def.), apparently through the medium of Arabic *açamm*, deaf."]—1557, Recorde's *Whetst.* G iiib, "That root is a Sursolide roote, that yieldeth a Sursolide nomber"—[Suversed = supplement + versed; suversed sine: the versed sine of the supplement.[1]] 1827, Airy in *Encycl. Metrop.* (1845) I, 674, "The versed sine

[1] That is, suvers $x = 1 + \cos x$. The term suversed was used before 1827; for example in: (1) J. de Mendoza Rios, *A complete Collection of Tables for Navigation and Nautical Astronomy*, London, 1805; (2) W. Lax, *Tables to be used with the Nautical Almanac for finding Latitude and Longitude at Sea*, London, 1821; (3) E. Riddle, *Treatise on Navigation and Nautical Astronomy . . . with all the Tables requisite* . . . London, 1824.

of one is the suversed sine of the other"—1736 tr. *Du Halde's Hist. China,* III, 70," "In casting up Accounts they [the Chinese] make use of an Instrument called Souan pan." [The term is derived from the Chinese and means, literally, reckoning board.]

ARTICLES IN CURRENT PERIODICALS.

American Journal of Mathematics, volume 41, no. 1, January, 1919: "Groups generated by two operators whose relative transforms are equal to each other" by G. A. Miller, 1–4; "A classification of general (2, 3) point correspondences between two planes" by T. R. Hollcroft, 5–24; "The classification of plane involutions of order (3)" by Anna M. Howe, 25–48; "On surfaces containing a system of cubics that do not constitute a pencil" by C. H. Sisam, 49–59; "An isoperimetric problem with variable end-points" by A. S. Merrill, 60–78—No. 2, April: "Asymptotic satellites near the straight-line equilibrium points in the problem of three bodies" by D. Buchanan, 79–110; "Concerning the invariant theory of involutions of conics" by W. Sensenig, 111–122; "Note on seminvariants of partial differential equations" by A. L. Nelson, 123–132; "On a method for determining the non-stationary state of heat in an ellipsoid" by B. Datta, 133–142; "Nilpotent algebras generated by two units, i and j, such that i^2 is not an independent unit" by G. W. Smith, 143–164.

Annaes Scientificos da Academia Polytechnica do Porto, volume 11, 1916, no. 1: "Sur différents procédés d'approximation" by A. Aubry, 5–35; "Timorenses de Okussi e Ambeno" by A. A. Mendes Corrêa, 36–51; "Palavras proferidas na sessão do lançamento da primeira pedra da Escola de Pharmacia da Universidade do Porto no dia 1 de fevereiro de 1915," 52–57; "Quelques considérations sur les systèmes de formes linéaires" by Miss Velleda Gradara, 58–64—No. 2: "Recherches des involutions de genres zéro, bigenre un, appartenant à une surface de genres un" by L. Godeaux, 65–78; "Discurso proferido na sessão solemne do lançamento da primeira pedra a construcção do edificio da Escola de Farmacia do Porto" by N. F. Dias Salgueiro, 79–88; "Sôbre a correlação de certos índices mandibulares com o índice cefálico" by A. Pires de Lima, 89–103; "Essai d'une théorie analytique des lignes non-euclidiennes" (continuation) by G. Pirondini, 104–124; "L. Orlando" by G. T[eixeira], 125–126; Review by A. Freire de Andrade of L. F. Navarro's *Cristalografia geométrica elemental* (Madrid, 1915), 127–128—No. 3: "Sur l'aire d'un segment de courbe convexe" by E. Turrière, 129–140; "Essai d'une théorie analytique des lignes non-euclidiennes" (continued), by G. Pirondini, 141–146; "Notas sobre Vernier" by A. Cardoso Pereira, 147–154; Reviews by Gomes Teixeira, of Lecornu's *Cours de Mécaniques* (tomes 1–2, Paris, 1914–1915), Richardson and Landis's *Fundamental Conceptions of Modern Mathematics* (Chicago, 1916), Archibald's *Euclid's Book on Divisions of Figures* (Cambridge, 1915), and L. O. de Toledo's *Elementos de Aritmética Universal,* tome 2 (Madrid, 1916), 185–188; Review by Mendes Corrêa of Frassetto's *Diagnosi e valutazione numerica delle curve in antropometrica ed in biometrica* (Rome, 1916), 189–190—No. 4: "Extrait d'une lettre adressée à F. Gomes Teixeira" by E. Turrière, 193–195; "Sur les formes isoclines et le problème Diophantique qui en découle" by V. Souza Brandão, 196–223—Volume 12, No. 1, 1917: "Sur l'aire d'une courbe plane générale" by E. Turrière, 5–12; "Sur une intégrale définie dont l'élément est une exponentielle de degré 4" by P. Appell, 12–13; "Sobre a construcção das tangentes á cissoide obliqua que passam por um ponto exterior á curva" by F. Gomes Teixeira, 14–17; "L'équation tangentielle polaire des courbes de Cesàro" by L. Braude, 18–26; Reviews by G. T[eixeira] of Rey Pastor's *Fundamentos de la geometria proyetiva superior* (Madrid, 1916) and *Introduccion á la matematica superior* (Madrid, 1916), P. Burgatti's *Lezioni di meccanica razionale* (Bologna), Halphen's *Oeuvres* (tome 1, Paris, 1916), Bôcher's *Leçons sur les méthodes de Sturm* (Paris, 1917), and Boutroux's *Les principes de l'analyse mathématique* (Paris, 1914), 53–59—No. 2, 1918: "Los problemas de la mecánica" by E. Terradas, 94–125; "Rapport de M. Appell sur les travaux de M. F. Gomes Teixeira, 126–128, [Reprinted from *Comptes rendus de l'académie des sciences de Paris,* 1917, tome 165, p. 907; Quotation: "En dressant un catalogue raisonné de ces courbes, en donnant leur histoire dans un important ouvrage, M. F. Gomes Teixeira a rendu à la science un grand service, que la Commission propose de reconnaître en lui décernant le prix Binoux"]—No. 4, 1918: "Congresso de sevilha celebrado pela Associação Espanhola para o progresso das sciencias (6–11 de Maio de 1917)" by A. F. de Lacerda, "Sciências matemáticas" 201–204; "Sur la réprésentation géométrique de la torsion d'une courbe gauche," 218–224.

Annals of Mathematics, 2nd series, volume 20, no. 3, March, 1919: "On quaternions and their generalization and the history of the eight square theorem" by L. E. Dickson, 155–171; "Non-symmetric kernels of positive type" by Caroline E. Seely, 172–176; "Elementary properties of the Stieltjes integral" by H. E. Bray, 177–186; "A kinematical property of ruled surfaces" by J. K. Whittemore, 187–190; "Systems of linear inequalities" by L. L. Dines, 191–199; "On the shortest line between two points in non-Euclidean geometry" by T. H. Gronwall, 200–201; "The generalized gamma functions" by E. L. Post, 202–217; "On the most general plane closed point-set through which it is possible to pass a simple continuous arc" by R. L. Moore and J. R. Kline, 218–223; "Repeated integrals" by D. C. Gillespie, 224–228.

Athenaeum, London, 1919, April 18: "Dreams and facts," Part I by Bertrand Russell, 198–199; Review of Russell's *Introduction to Mathematical Philosophy* (London, 1919), 209–210—April 25: "Dreams and facts," Part II by B. Russell, 232–233.

Biometrika, Cambridge, Eng., volume 12, parts 1–2, November, 1918: "On the standard deviations of adjusted and interpolated values of an observed *polynomial function* and its constants and the guidance they give towards a proper choice of the distribution of observations" by Kirstine Smith, 1–85; "On the product-moments of various orders of the normal correlation surface of two variates" by K. Pearson and A. W. Young, 86–92; "The correlation coefficient of a polychroic table" by A. Ritchie-Scott, 93–133; "On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables" by L. Isserlis, 134–139; "On the mathematical expectation of the moments of frequency distributions" by A. A. Tchouproff, 140–169; "Sur les moments de la fonction de corrélation normale de n variables" by S. Bergström, 177–183; "Formulæ for determining the mean values of products of derivatives of mixed moment coefficients in two to eight variables in samples taken from a limited population" by L. Isserlis, 183–184.

Columbia University Quarterly, Volume 21, No. 2, April 1919: "The new wisdom" by C. J. Keyser, 118–124. [First paragraph: "It is evident that to meet the great demands of the coming time men and women must needs have a spiritual equipment—intelligence, imagination, sympathy, understanding—far surpassing that which has hitherto been deemed sufficient. For in the time to come men and women cannot be, as hitherto, mere dwellers in a province or mere members of a commonwealth, a state, or a nation. Whether they will or no, they are destined to be, in respect of their interests and obligations, citizens of the world. As competent citizens of the world they will have to be either competent leaders in the affairs of world citizenship or—what is not less important—competent judges of such leadership. Provincial wisdom, however precious or fine, cannot longer suffice. What is demanded is a certain large intelligence—a certain wisdom, as we may call it—about the world."]

Education, Boston, volume 39, no. 8, April, 1919: "A review book in mathematics" by R. R. Goff, 471–472.

Journal of Education, London, volume 51, February 1, 1919: "Mathematical problem papers" by C. Davison, 90–91; "Mathematical Association," 130.

Journal of the Washington Academy of Sciences, volume 9, no. 3, February 4, 1919: "A contribution to quantitative epidemiology" by A. J. Lotka, 73–77.

Matematisk Tidsskrift, Copenhagen, 1919, no. 1, February, *A*: "H. G. Zeuthen" (Portrait frontispiece), 1–2 [In honor of the 'Nestor of Danish mathematicians' whose eightieth birthday occurred on February 15, 1919. He was editor of the *Tidsskrift,* 1871–1889]; "Trekantens Vinkelsum" by J. Hjelmslev, 3–11; "De matematiske Opgaver ved Mellemskoleeksamen og Realeksamen. En kritisk Vurdering" by F. Friss-Petersen, 11–25; Review of *Tidsskrift för elementär Matematik, Fysik och Kemi,* Stockholm, 1917–18, 28–30; "Danske Eksamensopgaver," 30–37—*B*: "Til Tidsskriftets Laesere og Medarbejdere" by H. Bohr and T. Bonnesen, 1–2; "Det ojeblikkelige Drejningspunkt" by J. Hjelmslev, 2–14; "Om den Hadamard'ske 'Hulsætning'" by H. Bohr, 15–21; Review by T. B[onnesen] of Hjelmslev's Lærebog i Geometri til Brug ved den polytekniske Laereanstalt (København, 1918), 21–26; "Del af polyteknisk Eksamen, 1918- Deskriptiv Geometri" 26–28.

Mathematics Teacher, volume 11, no. 3, March, 1919: "Introductory course in mathematics" by D. E. Smith, 105–114; "First-year algebra, as developed in the academic high school, New Britain, Conn." by R. R. Goff, 115–117; "Indeterminate forms in trigonometry" by M. O. Tripp, 118–120; "The Courtis tests in arithmetic" by P. A. Boyer, 121–132; "Report of the committee to recommend a suitable program in mathematics for Junior High Schools" (H. D. Gaylord, chairman), 133–140; "Platform of the allied associations of public school teachers of Baltimore," 141; "Book Reviews" and "Notes and News" 142–144.

MIND, London, volume 43, July, 1918: "A general notation for the logic of relations" by C. D. Broad, 184–303; "A proof that any aggregate can be well-ordered" by P. E. B. Jourdain, 386–388. [First sentence: "That any aggregate can be well-ordered was stated by Georg Cantor, but he never succeeded in proving this theorem (*cf.* Cantor's *Contributions to the Founding of the Theory of Transfinite Numbers*, English translation, Chicago and London, 1915, pp. 60, 62–63, 66, 90, 109, 204–206)."]—No. 108, October: "On the relation between induction and probability (Part I)" by C. D. Broad, 389–404; "Note" by C. D. Broad, 508. [The "note": "I see, on looking through my paper in *Mind*, N.S., No. 107, that, although I at first speak of 'Whitehead and Russell' I later generally refer to pieces of notation contained in *Principia Mathematica* as 'Russell's.' I did not mean by this to ascribe them to Mr. Russell rather than to Dr. Whitehead. I have no idea which of the authors is responsible for any given part of the book, and I only used 'Russell' as an abbreviation for 'Russell and Whitehead.' I should be sorry indeed to appear unfair to Dr. Whitehead, and my only excuse is the extreme difficulty of putting a phrase like 'Russell and Whitehead' into the possessive in English. Perhaps Dr. Whitehead will provide me with a suitable notation for this purpose, and Mr. Russell will guarantee it to be 'ethically neutral.'"] Volume 44, January, 1919: "Notes on Zeno's arguments on motion" by P. E. B. Jourdain, 123–134; [Quotation: "The following notes have to do with two points. The first is to call attention to an argument used by Mr. R. A. P. Rogers; [In his paper 'On transfinite numbers, and some problems relating to the structure of actual space and time,' *Hermathena*, Vol. 15, 1910, pp. 397–415.] the second is to bring out the force of some remarks attributed to the shade of Zeno on pages 52–55 of the number of *Mind* for January, 1916,[1] and which do not seem to have been expressed clearly enough."] "Note on C. D. Broad's article in the July *Mind* by Bertrand Russell, 124. [The "note": "Mr. Broad's very interesting article . . . attributes to me (for what reason I cannot guess) a number of notations employed in *Principia Mathematica*. As far as my memory serves me, all these were invented by Dr. Whitehead, who, in fact, is responsible for most of the notation in that work. My original notation, before he came to my assistance, may be found in Peano's *Revue de Mathématiques*, vols. 7 and 8."]

NATIONAL REVIEW, London, volume 72, November, 1918: "The position of mathematics" by C. H. P. Mayo; [Reprinted in *Educational Review*, volume 57, March, 1919, pp. 194–204.].

NATURE, volume 103, March 13, 1919: Review by H. S. A. of Bryant's Galileo (London, 1918), 23; "Graphic methods in nautical astronomy" by A. Hutchinson, 25; [Quotations: "In the issue of *Nature* published on October 24 last (vol. 102, p. 155) there appeared an account of an ingenious chart devised by Mr. G. W. Littlehales, of the United States Hydrographic Department, for dealing rapidly with certain problems in nautical astronomy which involve the solution of a spherical triangle when the three sides, or the two sides and the included angle are known. The article is entitled 'A New Graphic Method in Nautical Astronomy,' but it would appear that the idea has been familiar in France for more than five-and-twenty years. The possibility of constructing a chart like that made by Mr. G. W. Littlehales was demonstrated by Maurice d'Ocagne so long ago as 1891 in his work *Nomographie: les calculs usuels effectués au moyen des abaques*, p. 84 and an abacus devised by him on these lines was described in W. Dyck's *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente*, published in 1892, p. 163. A figure of the chart can be found in a paper by d'Ocagne which appeared in *Journal de l'Ecole Polytechnique* (second series, 4th cahier, 1898, p. 224), and also in his *Traité de Nomographie*, 1899, p. 328. In a modified form the chart was employed by E. Collignon in 1898. (See his 'Note sur la détermination de l'heure du passage du soleil dans un plan vertical,' *Journal de l'Ecole Polytechnique, loc. cit.*, pp. 123–135. . . . The particular cases in the solution of spherical triangles it [the chart] is designed to deal with, frequently occur in the reduction of crystal measurements, and the use of the chart can be confidently recommended to crystallographers"]; "Professor E. C. Pickering," 28–29; "Roger Bacon (1214–94)" by C. Singer, 35–36; [Quotations: ". . . Roger Bacon, great as are his titles to remembrance, was neither the inventor nor introducer of the mariner's compass. . . . His work on this subject [optics] was a text-book for the next two centuries. He saw the importance of lenses and concave mirrors, and showed a remarkable grasp of mathematical optics. He described a system which is equivalent to a two-lens apparatus, and there is trustworthy evidence that he actually used a compound system of lenses equivalent to a telescope. . . . Astronomy was Bacon's perpetual interest. He spent the best part of twenty years in the construction of astronomical tables. His letter to the Pope, in favour of the correction

[1] "The flying arrow: an anachronism," pp. 42–45.

of the calendar, though unsuccessful in his own days, was borrowed and reborrowed, and finally, at third-hand, produced the Gregorian correction. . . . Suggestions described by him include the automatic propulsion of vehicles and vessels. He records also the working out of a plan for a flying-machine. . . . His insistence on the supreme value of mathematics as a foundation for education recalls the attitude of Plato. It was an insistence that the method of thought was more important than its content.

"Summed up, his legacy to thought may be regarded as accuracy of method, criticism of authority, and reliance on experiment—the pillars of modern science. The memory of such a man is surely worthy of national recognition."]—March 20: "Graphical methods in nautical astronomy" by H. B. Goodwin, 44; [Quotations: "As the author of the article in *Nature* of October 24, 1918, in which the diagram referred to by Dr. Hutchinson last week was first brought to the notice of your readers, may I be permitted to supplement the information as to previous efforts in the same direction? . . . M. d'Ocagne . . ., so far as appears at present, is clearly entitled to the credit claimed for him as first in the field. . . . The share of Mr. Littlehales, however, is marked by two features of interest:—(1) That he seems to have been the first to prepare and publish the diagram in a form that promises to be useful in the navigation of air and ocean, and (2) that the simplicity of treatment which deduced the principle and graduation of the chart directly from a formula of spherical trigonometry renders the theory of the matter intelligible to many nautical persons to whom the mysteries of 'Nomographie' are as a sealed book."] "A proof that any aggregate can be well-ordered" by P. E. B. Jourdain, 45; [Quotation: "All the critics of my method sketched or described in my two letters to *Nature* (vol. 101, pp. 84 and 304, 1918), in my two notes in *Comptes rendus* (vol. 166, pp. 520–523 and 984–986, 1918), in *Mind* for July, 1918, and in *Science Progress* for October, 1918, wish to see a certain particular case solved in detail. Although this case does not throw so much light on the problem as the equally simple method of dealing with the general case, which I happen to have discovered long before I applied it to special cases, I here give the treatment of the particular case referred to."] "Ludvig Sylow" by G. B. M[athews], 49.

Nyt Tidsskrift for Matematik, volume 29, no. 3, January, 1919; *A*: "Några aritmetiska satser" by F. de Brun, 49–54; "Om tal som tallägen" by A. Arwin, 54–63; Review by N. E. Nørlund of A. F. Andersen, H. Bohr and J. Mollerup's *Nyere Undersøgelser over Integralregningen* (København, 1917), 64–68; Review by C. Hansen of E. Landau's *Einführung in die . . . Theorie der algebraischen Zahlen und der Ideale* (Leipzig, 1918), 68–73; "Studentereksamen i Maj 1918—For den matematisk-naturvidenskabelige Linie," 73–75; "Kronik," 77–79—*B*: "Einige Sätze über die ganzen, rationalen Funktionen" by T. Nagel, 53–62; "Skoleembedseksamen Juni 1918," 62–65; "Løninger" by N. Nielsen, 65–68.

Proceedings of the American Academy of Arts and Sciences, Boston, Volume 54, No. 5, April, 1919: "A new geometrical model for the orthogonal projection of the cosines and sines of complex angles" by A. E. Kennelly, 369–378, 4 plates. [Geometrical constructions for producing plane-vector representations of $\cosh(\theta_1 + i\theta_2)$, $\sinh(\theta_1 + i\theta_2)$, and hence also $\cos(\theta + i\theta_2)$, $\sin(\theta_1 + i\theta_2)$, where $(\theta_1 + i\theta_2)$ is a complex argument, or a complex 'angle,' have been available for some time.[1] The constructions involve a rectangular hyperbola and an associated circle, in one and the same plane, which is the plane of the drawing. By making certain projections in this plane, followed by a rotation through a quadrant, a plane-vector is produced from the origin, corresponding to the complex hyperbolic or circular sine or cosine required. This process is open to the objection that it is somewhat forced and artificial, lacking the simple projective property that a sine or cosine of any real angle possesses in either circular or hyperbolic trigonometry.

More recently, a method of deriving the hyperbolic cosine or sine of a complex angle has been obtained,[2] which enabled the new three-dimensional model here described to be prepared. In this model, it will be seen that the cosine or sine of a complex angle, either hyperbolic or circular, can be produced, by two successive orthogonal projections on to the XY plane, one projection being made from a rectangular hyperbola, and the other projection being made from a particular circle definitely selected among a theoretically infinite number of such circles," concentric but not coplanar.]

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[1] "Two elementary constructions in hyperbolic trigonometry" by A. E. Kennelly, *Annals of Mathematics*, 2d series, Vol. 5, pp. 181–184, July, 1904; reproduced in A. E. Kennelly, *Tables of complex hyperbolic and circular functions*, Harvard University Press, 1914, pp. 165–168.

[2] A. E. Kennelly, *Artificial Electric Lines*, New York, 1917, pp. 120–121.

Proceedings of the National Academy of Sciences of the United States of America, volume 5, no. 3, March, 1919: "Tables of the zonal spherical harmonic of the second kind $Q_1(z)$ and $Q_{11}(z)$" by A. G. Webster and W. Fisher, 79–82.—No. 4, April: "On the real folds of Abelian varieties" by S. Lefschetz, 103–106; "Covariants of binary modular groups" by O. E. Glenn, 107–110; "The general solution of the indeterminate equation: $Ax + By + Cz + \ldots = r$" by D. N. Lehmer, 111–114.

Record of the American Institute of Actuaries, Chicago, volume 7, June, 1918: Review by E. B. Escott of Running's *Empirical Formulæ* (New York, 1917), 55–60.

School Science and Mathematics, volume 19, no. 4, April, 1919: "The case method of teaching mathematics" by G. A. Miller, 344–349; "On the relations of mathematics to commerce" by R. E. Moritz, 350–357; "Plane geometry for the ninth and tenth grades" by R. R. Goff, 357–358; "Notes on 'Bolshevik Multiplications,'" 359–361; "Problems and solutions," 372–376.

Science, new series, Volume 44, March 14,1 919: "Cross-section lines on blackboards and their illumination" by P. F. Gaehr, 265. ["Those who wish cross-section rulings on blackboards temporarily, thus leaving the board free for other work after the curve-plotting is finished, can do so by a simple device. On a sheet of white paper make a ruling of lines, 2 cm. apart, the whole grid 16 x 24 cm., and the lines not quite one mm. thick. Take a photograph of this, making the camera image the size of a lantern-slide. Mount the negative in a lantern, projecting the image on the blackboard. A lantern equipped with a 400-watt Mazda lamp will make the lines sufficiently visible for plotting even in a well-lighted room. The lines are erased by turning off the lamp."]—May 9: "Apropos of the proposed historical science section" [of the American Association for the Advancement of Science] by G. A. Miller, 447–448.

Scientia, Bologna, volume 25, no. 3, March, 1919: "The place of Tycho Brahe in the history of astronomy" by J. L. E. Dreyer, 177–185, (French translation, supplément, 39–47); Review by P. E. B. Jourdain of Forsyth's *Theory of Functions of a complex variable*, Third Edition (Cambridge, 1918), 232–233.

Sigma Xi Quarterly, volume 6, no. 3, September, 1918: "In memoriam, Henry Shaler Williams (1847–1918)" (Portrait), 53–74. [Professor Williams was the founder of the Sigma Xi Society, and was elected the first president of the first chapter which was established at Cornell University in 1887.]

Texas Mathematics Teachers' Bulletin, volume 4, no. 2, February, 1919: "The State Teachers' Association" by J. W. Calhoun, 3; "A neglected opportunity" by H. Y. Benedict, 4–7; "The mathematics club in the high school" by C. E. LaMaster, 8–12; "Military applications of mathematics" by E. J. Oglesby, 13–15; "Factoring in first year algebra" by Mattie R. Watson, 16–18; "Geometry notebooks" by Mrs. Helen Bolton, 19–21; "Combining arithmetic and algebra in the high school" by L. V. Stockard, 22–24; "Can I interest all the pupils in geometry" by Mildred Watkins, 25–28; "The straight edge," 29.

Transactions of the Royal Society of Canada, series 3, volume 12, December, 1918, section 3: "Rational plane anharmonic cubics" by A. M. Harding, 185–195.

AMERICAN DOCTORAL DISSERTATIONS.

L. C. Cox, *The finite groups of birational transformations of a net of cubics.* [Reprinted from *American Journal of Mathematics*, volume 39, 1917.] Pp. 59–74. (Cornell, 1915.)

J. V. De Porta, *Irrational involutions on algebraic curves.* [Reprinted from *American Journal of Mathematics*, volume 40, 1918.] Pp. 47–68. (Cornell, 1916.)

T. R. Hollcroft, *A classification of general* (2, 3) *point correspondences between two planes.* [Reprinted from *American Journal of Mathematics*, vol. 41, 1919.] Pp. 5–24. (Cornell, 1917.)

Anna M. Howe, *The classification of plane involutions of order* (3). [Reprinted from *American Journal of Mathematics*, volume 41, 1919.] Pp. 25–48. (Cornell, 1917.)

J. H. Minnick, *An investigation of certain abilities fundamental to the study of geometry.* Lancaster, Pa., New Era, 1918. Sm. 4to. 8 + 108 pages. (University of Pennsylvania, 1918.)

UNDERGRADUATE MATHEMATICAL CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence, Kan.

CLUB ACTIVITIES.

MATHEMATICS CLUB OF COLUMBIA UNIVERSITY, New York, N. Y. [*1918*, 227–228].

Officers 1918–19: President, Israel Koral '19; secretary, Charles P. Davis '19; faculty adviser, Professor Lewis P. Siceloff.

Programs of meetings for the current semester are as follows:

February 24, 1919: "Groups of operations" by Professor Cassius J. Keyser.

March 10: "Philosophy of mathematics"[1] by Charles P. Davis '19.

March 24: A visit to the mathematical museum, conducted by Professor James Maclay.

April 7: "Projectiles" by August B. Kinzel '20.

THE MATHEMATICAL CLUB, Harvard University, Cambridge, Mass. [*1918*, 186–7, 449–50].

At the first meeting of the current academic year, held in the Common Room of Conant Hall, February 12, Professor William F. Osgood spoke on "Professor Bôcher's Scientific Start in Life." The following is an extract from the minutes of the meeting.

"Professor Bôcher's student days lay in the years just following the establishment of the principal courses in the Department of Mathematics, as we know them today. Instruction in the Calculus at Harvard, it is true, goes back at least to the middle of the eighteenth century, as is clear from the Commencement programmes, and the first catalogues which print the titles of the courses offered show that formal instruction in the subject was given at least as early as 1830. In the years from 1841 to 1846 Benjamin Peirce published his *Curves and Functions*, which for more than three decades served as a text-book in analytic geometry and the calculus.

"With the coming of Professor Byerly to Harvard in 1876, new and more efficient methods of undergraduate instruction were brought into the Department. The solving of problems by the student as a part of the work of each day's assignment is due to him. He found an ardent and efficient supporter in Professor Benjamin Osgood Peirce, who came into the Department in 1881, and these men gave to Mathematics 2 and 5 [the first and the second course in the Calculus] essentially the character they have today. Both men were interested primarily in the applications of the calculus, and the problems they used bear witness to the fact that they believed the calculus should be taught in its relation to physics. It has been so taught at Harvard ever since.

"Mathematics 4 [Mechanics] was Professor B. O. Peirce's work. On the other hand, Professor Byerly created Mathematics 3 [Modern Geometry]. Advanced courses in geometry had been given in the seventies by Professor James Mills Peirce, who was a pupil of Chasles in Paris in the fifties. But he assigned few problems, and the student found the presentation abstract and difficult to understand. Later, Professor Bôcher and Professor Bouton made useful contributions to the subject matter of the course. But its plan and organization had been completed before Bôcher elected it as a student.

"Two further courses which were to be determinative for Professor Bôcher's scientific work in life were Mathematics 10 [Potential Functions and Developments into Series] and 13 [Theory of Functions of a Complex Variable]. The first of these was created jointly by Professor Byerly and Professor B. O. Peirce, and it took a powerful hold on Bôcher while he was still an undergraduate. Indeed, his thesis for Final Honors in Mathematics—he received Highest Final Honors at graduation—was in this field. It was entitled: 'Three Systems of Parabolic Coördinates.'

[1] Cf. Shaw, J. B., *Lectures on the Philosophy of Mathematics*, Chicago, 1918.

"The other course, Mathematics 13, was given by Mr. (at that time not yet Dr.) Frank Nelson Cole, who had just returned from Germany and was aglow with the enthusiasm which Felix Klein inspired in his students. Cole was not the first to give a formal course of lectures at Harvard on the theory of functions of a complex variable, Professor James Mills Peirce having lectured on this subject in the seventies. That presentation was, however, solely from the Cauchy standpoint, being founded on the treatise of Briot et Bouquet, *Fonctions elliptiques*. Cole brought home with him the geometric treatment which Klein had given in his noted Leipzig lectures of the winter of 1881–82. Cole also gave a course in Modern Higher Algebra, with its applications to geometry. The enthusiasm which he felt for his subject was contagious. Interesting as were the other courses I have mentioned, they stood as the Old over against the New, and of the latter Cole was the apostle. The students felt that he had seen a great light. Nearly all the members of the Department,—Professor J. M. Peirce, Dr. B. O. Peirce, and, I think, Professor Byerly,—attended his lectures. It was the beginning of a new era in graduate instruction in mathematics at Harvard, and mathematics has been taught here in that spirit ever since.

"These were the influences which moulded Bôcher's scientific life when he was yet an undergraduate; this was the atmosphere in which he lived. From here he went to Göttingen, where he studied for six semesters under Klein. An account of his work there will be found in the speaker's forthcoming article on "The Life and Services of Maxime Bôcher," in the *Bulletin of the American Mathematical Society*, May, 1919."

Mathematics Club of the University of Maine, Orono, Me. [*1918*, 132, 453–454].

During the Fall Term, October to December, 1918, the club was inactive, since the University of Maine was given over to the work of the Student's Army Training Corps, and regular university work was suspended. Many of the members most active in the club last year did not return on account of graduation or war conditions.

In January, 1919, the club was reorganized and the following officers elected. President, Grace H. Hodgdon '19; vice-president, Flavia L. Richardson '20; secretary-treasurer, Edith I. Deering '21; faculty member of the program committee, Professor Myron O. Tripp.

Programs since January, 1919, have been as follows:

January: Social meeting at the home of Dean James N. Hart.

February: "The training of teachers for mathematics" by Professor Myron O. Tripp; "War Savings Stamps as an investment" by Flavia L. Richardson '20.

March: No meeting was held in March.

The Mathematical Club of the University of Nebraska, Lincoln, Nebr. [*1918*, 313–315].

At the University of Nebraska, as at many other universities, the mathematics club was inactive during the first semester on account of war activities. Soon after the opening of the second semester, the club resumed its activities under the leadership of the officers given below.

Officers, 1919: President, Frances R. Botkin '19; secretary, Mervyn C. Kimberly '20; faculty member of executive committee, Professor Albert Babbitt.

February 13, 1919: "The game of Nim"[1] by Professor Meyer G. Gaba.

[1] A brief discussion of the game of Nim, with references to related literature, was included in Club Topic No. 7, published in the March, 1918, issue of this Monthly (Vol. XXV, pp. 139–142).

March 20: "Complex numbers" by Josiah Brooks '21; "Roots of unity" by Paul J. White '21.

FLATLANDERS—A MATHEMATICAL PLAY IN ONE ACT.[1]

Persons in the Play.

MR. CUBE.
MASTER RATIO—A Schoolmaster, inclined to be fanatical.
CYCLUS—A young nobleman just entering school.
BARON MULTILATUS—Father of Cyclus.

Prologue.

Spoken by Cube.

Dear People of the Third Dimension,
I have to bring to your attention
A place that's hard to understand—
A country that is called *Flatland.*
The people here, as you will see,
Are long or wide—as the case may be,
But one thing they are wont to slight—
They never heard of having *height!*
That woman is a fashion-plate
Whose form's a line that's thin and straight.
And, lest some man should fail to see
A line that's drawn so daintily
And e'er he'd time to step aside
With horrid bump they should collide.
She hums a note that's thin and clear
To let him know she's drawing near.
Since Flatlanders' nobility
Is *sides* instead of ancestry
The King's a circle, and his Prides
Are polygons of many sides.
The Triangle's prestige is small.
The Angle has no name at all.
And yet, unto a Cube like me
The case is grievous as can be.
For, though I'll fight 'gainst circumstance
To pull them *up* from ignorance,
I fear in flight they may not revel—
For Flatlanders are on the *level!*

[1] Performed before The Mathematics Club of Vassar College, February 20, 1919. The Prologue and Epilogue were written by Kathleen Millay '21, and the Scene was adapted by Lucile Free '21. This adaptation of E. A. Abbott's *Flatland* was suggested by a description of a dramatization, in seven scenes, performed at the Haberdashers' Aske's Girls' School, Acton, England, in June, 1913. The description in *School Science and Mathematics*, October, 1914, volume 14, pp. 583–587, was reprinted from *The Mathematical Gazette*, January, 1914, volume 7, pp. 228–231.

The Scene.

Setting: The schoolroom. Around the walls are hung various shaped cardboard figures—inhabitants of Flatland. There is the Teacher's desk, and the benches for the children. In the front row of these are—in line—a cardboard: Triangle; Pentagon; Straight Line; Octagon. Problems treating Areas are on the blackboard. Cyclus is just entering the school. Ratio enters on the other side.

The costumes may be all black and white with cardboard printed names hung around the necks of the characters. Cube may be inside a paper cube if desired.

Time: To-day, in Flatland.

Ratio. Good-Morning, Cyclus. So this is our new pupil.

Cyclus. Good-morning, Master Ratio. Father said you were to teach me my angles very well this year so that I will not make the mistake of associating with people beneath my rank.

Ratio. My students always know their angles very well. But first, let's see how much you know already. You look like a bright lad, and doubtless the son of the great Baron Multilatus has had practice in discovering the rank of his playfellows. Begin over there on the first row, young man, and tell me the class of each child.

Cyclus [*Goes to cardboard figures*]. This, this—let me see—why, this is a Triangle! How stupid of you to admit Triangles to such an exclusive school. I'm sure my father won't approve of it. And this next one—this is a Pentagon.

Ratio. Right, sir.

Cyclus. And this—oh, this is only a woman. I won't even have to bother with it. But this next person seems to trouble me quite a good deal. It couldn't be! Why, yes, you're a Decagon. I would like to walk home from school with you sometime, Mr. Decagon. I'm sure we'll have very much in common. Your father must be a Count.

Ratio. That will do, Cyclus. I see where your trouble lies. By the way, that last gentleman was not a Decagon, but an Octagon. I'm sure we can correct your faults very easily. Now, sir, have you any questions before we begin the lesson for the day?

Cyclus. Please, sir, what is a straight line?

Ratio. A Straight Line is formed by a moving Point.

Cyclus. And what is an Area?

Ratio. An Area is formed by moving a Straight Line.

Cyclus. [*Musing to himself.*] Then I wonder why they couldn't move to the right or left.

Ratio. I beg pardon, sir. You wonder why *who* couldn't move to the right or left?

Cyclus. I was just thinking about a very strange dream I had last night. I was in a strange land. The people all seemed to me to be very unhappy because they were so crowded together. They could not run about and play as we do, but had to move all the time in a Straight Line. They could never pass each other, but when they met all they could do was to go back again the way they had come. It was all so tiresome and stupid. I insulted their king by taking

him for a woman, although how they could tell him from anyone else is more than I can see. A funny Straight Line bumped in to me, and I told him to go to the right. He looked at me in a blank, helpless sort of way, and began to back up. Master Ratio, what do you suppose was the matter?

Ratio. That was a very peculiar dream, Cyclus, but it is all quite clear to me. You see those people only knew one dimension. How much happier they could have been if they had only known that they could move to the right or left in areas. Your dream proves to you, sir, what a lucky person you are. Your life is not hindered by ignorance of the possibilities you have.

Cyclus. [*Hesitating.*] But, Master Ratio, sometimes I think I *am* hindered. I wonder what would happen if you should move an Area.

Ratio. [*Calmly.*] Nothing at all would happen. See, I will illustrate for you. [*He pulls a piece of paper around on the top of his desk.*] There, I have moved it to the right and it is still in the same shape and size. And now to the left—backward,—forward. You cannot change an area. [*There is a knock at the door.*] Come in! [*Cube enters. Master Ratio goes up and feels of or inspects his angles. Of course, all he can see of the cube is a cross section.*] Ah, yes, Mister Square, How do you do? And what can I do for you today?

Cube. I am not a Square; I am a Cube.

Ratio. A what?

Cube. A Cube.

Ratio. And where do you come from?

Cube. I come from the Land-of-the-Third-Dimension.

Ratio. Third Dimension! What a peculiar language you have. According to the meaning of our word *Dimension* there is no third Dimension. And you call yourself a Cube. And yet, you are exactly what we call a Square.

Cube. My language is just the same as yours, Master Ratio, only the range of my vocabulary is much larger. And I tell you again that I am not a Square. If you take a Square and move it up, you will discover what I am.

Ratio. Up?—Up?—Now what can *up* mean? Not this way.—[*Illustrates again pulling his paper over the top of his desk, right, left,* etc.]—nor this, this or this. Cyclus, get me the dictionary, for I must find this word. And what do you mean, Mister Cube, when you say *Third* Dimension?

Cyclus. Maybe we're as ignorant as the people in my dream, after all.

Ratio. [*Irritated.*] No, no, Cyclus! What an absurd idea! What would your father, magistrate of our village, say if he knew you entertained such thoughts! No, no! [*Ratio and Cyclus search dictionaries. Cube inspects figures on the wall. Baron Multilatus enters without knocking.*] What a fruitless search, *Up* is not in any of our dictionaries.

Baron Multilatus. [*Sarcastically.*] Well, well, Master Ratio! I thought I should some day catch you unawares! Of course, *up* is not in the dictionaries. There is no such word. Do you think that you can put any two letters together and make a word? Why not look for *m-i*, or *k-t*, or *p-b*, *l-l*, anything instead of using *u-p*.

Ratio. But Mister Cube, this fellow here whom we commonly call Mister Square, wants me to move a Square *up* and find the Third Dimension. Baron! [*Dramatically.*] I feel as though I were on the point of making some wonderful new discovery! Something that will help our future generations to live freer, happier lives than ours have been. Something that will ——

Baron Multilatus. Enough of this, my man! I fear you are becoming mentally unbalanced. You'll only end by being burned as a heretic. And if you continue in this fanatical idea of yours I myself shall see that you get your just punishment. Come along, Cyclus—We'll seek a better master, and a saner one.

[*Exeunt Baron Multilatus with Cyclus.*]

Ratio. [*Musing.*] He thinks I'm crazy, does he? Well, well, perhaps I am,—but we'll see. We'll see. [*To Cube.*] And now, friend Cube, I'm off to my fellow-masters to see if I can get any help in this miraculous problem you have suggested. Doubtless they, too, will think me mad. Oh, when will the world learn to respect a man who is seeking after new truths? But—I'll meet you later, sir, and perhaps then we can bring happiness to my poor people. [*Exit Ratio.*]

Epilogue.

Spoken by Cube.

And thus, my friends, this demonstration
Portrays a grievous situation—
A people who would like to find
A way to educate the mind,
And yet, who miss inevitably
A truth we know quite naturally.
And, sometimes, when I contemplate
Their ignorance so desolate
I wonder if *we* fail to see
Some evident reality.
And so, I bring to your attention
The subject of the Fourth Dimension.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about problems to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

2776. Proposed by C. P. SOUSLEY, State College, Penn.

Prove by elementary geometry that the Wallace lines of the extremities of any diameter of the circumscribed circle of a triangle intersect at right angles on the nine-point circle of the triangle.

2777. Proposed by W. D. CAIRNS, Oberlin College.

Prove that the two series

$$1 + \frac{\pi^4}{2^4 \cdot 4!} + \frac{\pi^8}{2^8 \cdot 8!} + \cdots,$$

and

$$\frac{\pi^2}{2^2 \cdot 2!} + \frac{\pi^6}{2^6 \cdot 6!} + \frac{\pi^{10}}{2^{10} \cdot 10!} + \cdots$$

are equal.

2778. Proposed by WARREN WEAVER, University of Wisconsin.

A partition of space is effected by means of five planes, none of which are parallel and no four of which pass through the same point, and six spheres. This divides all space into n regions, some of which are finite and some infinite. Considering it equally probable that a bird be in any one of the n regions show that the probability of its being caught (that is, of its being in one of the finite regions) is equal to or less than 78/99.

2779. Proposed by J. L. RILEY, Junior Agricultural and Mechanics College, Stephenville, Texas.

A parabola is placed with its axis horizontal; find the straight line of shortest descent from the curve to the focus.

406 (Algebra) [March, 1914]. Proposed by S. A. COREY, Albia, Iowa.

Solve the system of equations:

$$(1 - x)(a_1 + a_2y + a_3z) = d, \quad (1 - y)(b_1 + b_2x + b_3z) = g, \quad (1 - z)(c_1 + c_2x + c_3y) = h.$$

411 (Algebra) [April, 1914]. Proposed by V. M. SPUNAR, Chicago, Ill.

Determine $x_1, x_2, x_3, \cdots x_p$, from the equations:

$$\begin{aligned} x_1 + x_2 + x_3 + \cdots + x_p &= a_0, \\ b_1x_1 + b_2x_2 + b_3x_3 + \cdots + b_px_p &= a_1, \\ b_1^2x_1 + b_2^2x_2 + b_3^2x_3 + \cdots + b_p^2x_p &= a_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_1^{p-1}x_1 + b_2^{p-1}x_2 + b_3^{p-1}x_3 + \cdots + b_p^{p-1}x_p &= a_{p-1}. \end{aligned}$$

442 (Geometry) [May, 1914]. Proposed by J. B. SMITH, Hampden-Sidney College.

If any three straight lines AD, BE, CF, be drawn from the corners of the triangle ABC to the opposite sides a, b, c, they will enclose an area. If Δ, Δ'' be the areas of the triangles ABC, DEF, show that

$$\frac{\Delta''}{\Delta} = \frac{(AF \cdot BD \cdot CE - AE \cdot CD \cdot BF)^2}{(ab - CE \cdot CD)(bc - AE \cdot AF)(ca - BF \cdot BD)},$$

where the signs of the factors are to be determined by the following rule: Each segment being measured from one of the corners of the triangle ABC, along one of the sides, is to be regarded as positive or negative according as it is drawn towards or from the other corner in that side.

455 (Geometry) [February, 1915]. Proposed by R. P. BAKER, University of Iowa.

Find the minimum triangle of assigned angles inscribed in a given triangle.

348 (Calculus) [December, 1913]. Proposed by E. L. DODD, University of Texas.

Let $(x_1, x_2, \cdots x_n)$ be a point in n dimensions lying in the "sphere" S defined by

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leqq 1.$$

Let T be that part of S defined by a set of n linear homogeneous inequalities with non-vanishing determinant; thus:

$$a_ix_1 + b_ix_2 + \cdots + k_ix_n \geqq 0, \quad i = 1, 2, \cdots n.$$

Find the value of

$$\frac{\int \cdot \underset{T}{\cdot} \cdot \int dx_1 \cdots dx_n}{\int \cdot \underset{S}{\cdot} \cdot \int dx_1 \cdots dx_n};$$

in other words, find the magnitude of a "solid angle" in n dimensions, with the "sphere" as unit solid angle.

Note. This problem was discussed and left unsolved by Schläfli in the *Quarterly Journal of Mathematics* for 1858, 1860, 1867.—Editor.

349 (Calculus) [December, 1913]. Proposed by C. N. SCHMALL, New York City.

If $y = a \cos (\log x) + b \sin (\log x)$, eliminate the constants a and b and obtain the equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

198 (Number Theory) [November, 1913]. Proposed by the late ARTEMAS MARTIN.

Prove that every even number is the sum of two prime numbers.

Note. This problem has long been known and no proof has ever been given.—Editor.

201 (Number Theory) [December, 1913]. Proposed by E. T. BELL, University of Washington.

Eisenstein proposed (*Crelle*, t. 27, p. 282), as the simplest of several problems: "In the expansion of

$$\frac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1,$$

where p is prime, to show that the coefficients of the various powers of z are all divisible by p."

SOLUTIONS OF PROBLEMS.

231 (Number Theory) [April, 1915]. Proposed by A. J. KEMPNER, University of Illinois.

Is the series whose terms are the reciprocals of all positive integers not containing a given combination of figures, for example not containing the combination 37, convergent or divergent? Numbers such as $\frac{1}{37}$, $\frac{1}{370}$, $\frac{1}{5371}$ shall be omitted, numbers such as $\frac{1}{73}$, $\frac{1}{307}$, $\frac{1}{5317}$ shall be admitted as terms of the series. (Compare AMERICAN MATHEMATICAL MONTHLY, Volume 21, page 123.)

SOLUTION BY FRANK IRWIN, University of California.

Let $f(n)$ be the number of numbers in the class, g_n, of numbers of n digits that do not contain the given combination, for example 37. We can get these numbers by taking any number of g_{n-1} and adding a digit at the end; except that this digit must not be 7 if the number chosen from g_{n-1} ends with a 3. Since the number of these rejected numbers of g_{n-1} is evidently $f(n - 2)$, we have the formula: $f(n) = 10f(n - 1) - f(n - 2)$. In the general case, where the given combination consists of k digits, we should get similarly, for $n > k$,

$$f(n) = 10f(n - 1) - f(n - k). \tag{1}$$

(An exception would arise for such a combination as 37537; here, for instance, the formula $f(8) = 10f(7) - f(3)$ would not hold, for the number 375 belonging to g_3 would have been subtracted, as it should not have been, since the number obtained by writing after it 3753, viz., 3753753, does not belong to g_7, and so has not been counted in $f(7)$. It will be seen that such cases arise when the first l digits of the given combination are the same as the last l, $l < k$. We shall call such combinations "improper" and deal with them separately.)

From (1) we have $f(n - 1) > f(n)/10$. Also since $f(n - 1) = f(n)/10$, $n < k$, and $f(n - 1) \geqq f(n)/10$, $n = k$ (= if the given combination begins with a zero), we have, for $n > k$, the string of relations $f(n - 2) \geqq f(n - 1)/10$, $f(n - 3) \geqq f(n - 2)/10$, $\cdots$. From these last inequalities we derive $f(n - k) \geqq f(n - 1)/10^{k-1}$. Comparing this with (1), we see that $f(n) \leqq 10f(n - 1) - f(n - 1)/10^{k-1}$, or $f(n) \leqq 10rf(n - 1)$, if we put $(10^k - 1)/10^k = r$. Since the reciprocal of any number in g_n is $\leqq 1/10^{n-1}$, it follows that the sum of the reciprocals of the numbers in g_k, g_{k+1}, $\cdots$ are less than the corresponding terms of the series

$$\frac{f(k)}{10^{k-1}}(1 + r + r^2 + \cdots).$$

This series converges, for r is < 1; and hence our given series converges.

The case of "improper" combinations may be rapidly dealt with by the following device. Write the digit a, k times at the end of the given combination, where a is any digit other than the first digit of the combination. This certainly gives us a "proper" combination (of $2k$ digits), so that, as already proved, the series formed from the harmonic series by omitting terms whose denominators contain this combination converges. But this series evidently contains all terms of the given series; this latter, then, converges too.

In like manner we may prove that *the series obtained by omitting terms containing all of a given set of combinations* (*some of which may be repeated*) *converges.* Here we may compare the series with that obtained by omitting terms containing the single combination formed from the given combinations by juxtaposition.

This supplies another proof of the proposition in my paper on "A Curious Convergent Series" in this MONTHLY for May, 1916, that proposition being the special case of the above in which each of the combinations consists of a single digit.

The proposition stated by Professor Moulton in his solution of Algebra 453 in this MONTHLY for Oct., 1916, refers, I suppose—the proof is not given in full—not to combinations of the kind here considered, but to selections, and without any reference to the juxtaposition of the digits of the selection in the omitted terms.

239 (Number Theory) [March, 1916; March, 1919]. Proposed by HAROLD T. DAVIS, Colorado Springs, Colorado.

Give a general method for determining the solution in integers of the equation

$$x^r - 10xy - (n + 1) + y = 0,$$

where r and n are positive integers.

II. SOLUTION BY FRANK IRWIN, University of California.

If $x = 0$, $y = n + 1$; noting this solution, we shall suppose in what follows $x \neq 0$.

The degree in x of the given equation may be reduced by the following device. $y - (n + 1)$ must be divisible by x, say $y - (n + 1) = y_1x$. Then our equation reduces to

$$x^{r-1} - 10xy_1 - 10(n + 1) + y_1 = 0.$$

Putting $y_1 - 10(n + 1) = y_2x$, $y_2 - 10^2(n + 1) = y_3x$, etc., we finally reach the equation

$$1 - 10xy_r - 10^r(n + 1) + y_r = 0, \tag{1}$$

where y and y_r are connected, it will be found, by the relation

$$y = (n + 1)[1 + 10x + 10^2x^2 + \cdots + 10^{r-1}x^{r-1}] + y_rx^r,$$

or

$$y = (n + 1)\frac{10^rx^r - 1}{10x - 1} + y_rx^r. \tag{2}$$

Rewriting (1) as

$$(10x - 1)y_r = 1 - 10^r(n + 1), \tag{3}$$

we see that x and y_r must have opposite signs.

(*i*) Let y_r be negative, $y_r = -y_r'$. Then (3) becomes $(10x - 1)y_r' = 10^r(n + 1) - 1$. Here the number on the right ends with the digit 9, as does also the first factor on the left. Hence we see that y_r' must end with 1. We get, then, a solution by taking for y_r' any factor, ending with 1, of the right side, and determining x and y from (3) and (2).

(*ii*) Let x be negative, $x = -x'$. Then (3) becomes

$$(10x' + 1)y_r = 10^r(n + 1) - 1.$$

We take y_r equal to any factor of the right side that ends with 9.

This gives us all solutions.

Suppose that $10^r(n + 1) - 1$ can be factored as $a \cdot b$, where a ends with 1, b with 9. Then

we may take either $y_r = -a$, whence $x = (b + 1)/10$; or $y_r = b$, $x = (-a + 1)/10$. Thus solutions go in pairs, the x of either solution being obtained by adding 1 to the y_r of the other and dividing by 10. There are, then, always an even number of solutions, and always at least two, viz., those where $y_r = -1$, $x = 10^{r-1}(n + 1)$; $y_r = 10^r(n + 1) - 1$, $x = 0$.

Finally it may be noted that the more general equation

$$ax^r - bxy + y - c = 0$$

may be solved by similar methods. Here we should have to pick out such divisors of $b^rc - a$ as are $\equiv \pm 1 \pmod{b}$. It may be interesting to apply the method to an example. Let us solve

$$x^4 - 10xy - 22 + y = 0.$$

We have to solve

$$(10x - 1)y_4 = -219999 = -3 \times 13 \times 5641$$

(5641 prime). The values of y_4 with the corresponding solutions of our equation are:

$y_4 = -5641$, $x = 4$, $y = 6$; $y_4 = -1$, $x = 22000$ (y a very large number);

$y_4 = 39$, $x = -564$, $y = -17937434$; $y_4 = 219999$, $x = 0$, $y = 22$.

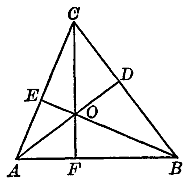
247 (Number Theory) [June, 1916]. Proposed by NORMAN ANNING, Chilliwack, B. C.

To dissect the triangle whose sides are 52, 56, 60 into three Heronian triangles by lines drawn from the vertices to a point within.

The word Heronian is used in the sense of the German Heronische (Wertheim, *Anfangsgründe d. Zahlenlehre*, p. 140) to describe a triangle whose sides and area are integral.

Solution by Frank Irwin, University of California.

The orthocenter, O, may be taken as the required point. Let ABC be the triangle with $a = 60$, $b = 52$, $c = 56$; and let the feet of the perpendiculars from A, B, C on the opposite sides be D, E, F, respectively. Then the various lines in the figure, calculated as indicated below, are: $BD = 168/5$, $DC = 132/5$, $CE = 396/13$, $EA = 280/13$, $AF = 20$, $FB = 36$; $AO = 25$, $BO = 39$, $CO = 33$. Finally, area $BOC = 594$, area $COA = 330$, and area $AOB = 420$; so that the sides and areas of these three triangles are integral, as asserted.



The explanation of these facts depends on the following proposition: If the sides and area of the triangle ABC are rational, the same is true of the triangles BOC, COA, AOB. (Then by multiplying the dimensions of the figure by a suitable integer everything can be made integral.) For the three altitudes are rational, as also the radius r of the inscribed circle (since rs = area). Thus $\tan A/2$ is rational, and so, then, are $\cos^2 A/2$ and $\cos A$. Therefore, $AF = b \cos A$ is rational, and similarly, FB, BD, etc. Then the triangle AOF is rational (that is, has rational sides), since one of its sides, AF, is rational, and it is similar to the rational triangle ABD.

2678 [February, 1918]. Problem proposed by C. F. GUMMER, Queen's University, Canada.

Find necessary and sufficient conditions that the roots of the equation $x^{n+1} + a_1x^n + a_2x^{n-1} + \cdots + a_{n+1} = 0$ may be all real and separated by the roots of $x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0$.

Solution by the Proposer.

Consider the equations

$$f(x) \equiv x^{n+1} + a_1x^n + \cdots = 0, \tag{1}$$

$$g(x) \equiv x^n + b_1x^{n-1} + \cdots = 0, \tag{2}$$

$$R_1(x) \equiv c_0x^{n-p} + c_1x^{n-p-1} + \cdots = 0, \tag{3}$$

$$R_2(x) \equiv d_0x^{n-p-q} + d_1x^{n-p-q-1} + \cdots = 0, \tag{4}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where $R_1(x)$ is the remainder with sign changed on dividing $f(x)$ by $g(x)$, $R_2(x)$ has the same relation to $g(x)$ and $R_1(x)$, etc.

That the roots of (1) may be real and separated by those of (2), it is necessary and sufficient that

(a) the roots of $g(x)$ shall be real and distinct (such as $\beta_1 < \beta_2 < \cdots < \beta_n$) and

(b) $f(-\infty), f(\beta_1), \cdots, f(\beta_n), f(\infty)$ shall have alternate signs.

Now $f(\beta_i) = -R_1(\beta_i)$, and $R_1(x)$ cannot change sign more than $n-1$ times. Hence, (a) and (b) are equivalent to the conditions that c_0 shall be positive, and that the roots of (2) are all real and separated by those of (3) (implying that $p = 1$). These are in turn equivalent to the conditions that c_0, d_0 shall be positive, $p = q = 1$, and the roots of (3) real and separated by those of (4), and so on.

Hence, a set of necessary and sufficient conditions is

$$p = q = \cdots = 1;\ c_0,\ d_0,\ \cdots, \text{ all positive.}$$

It may be shown that these are equivalent to the conditions

$$\begin{vmatrix} 1 & b_1 & b_2 \\ 1 & a_1 & a_2 \\ 0 & 1 & b_1 \end{vmatrix}, \quad \begin{vmatrix} 1 & b_1 & b_2 & b_3 & b_4 \\ 1 & a_1 & a_2 & a_3 & a_4 \\ 0 & 1 & b_1 & b_2 & b_3 \\ 0 & 1 & a_1 & a_2 & a_3 \\ 0 & 0 & 1 & b_1 & b_2 \end{vmatrix}, \quad \cdots$$

all positive. For the calculation of the remainders see papers by E. B. Van Vleck (*Annals of Mathematics*, 1899, pp. 1–13) and A. J. Pell and R. L. Gordon (*Annals of Mathematics*, June, 1917, pp. 188–193).

2701 [May, 1918]. Proposed by E. H. WORTHINGTON, University of Pennsylvania.

Find the sum of the infinite series

$$\frac{1}{5} + \frac{1\cdot 2}{5\cdot 7}r + \frac{1\cdot 2\cdot 3}{5\cdot 7\cdot 9}r^2 + \cdots + \frac{n!}{5\cdot 7\cdot 9\cdots(2n+3)}r^{n-1} + \cdots$$

Verify your result for $r = 0$ and $r = 1$.

SOLUTION BY A. M. HARDING, University of Arkansas.

It can be easily shown that the following equation holds for all integral values of $n \geqq 0$:

$$\int_0^1 t^{1/2}(1-t)^{n+1}dt = \frac{2(n+1)}{2n+5}\int_0^1 t^{1/2}(1-t)^n dt.$$

Multiplying both sides by x^n and substituting $n = 0. 1, 2, \cdots (n-1)$, gives

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)dt = \tfrac{1}{5},$$

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)^2 x dt = \frac{1\cdot 2}{5\cdot 7}2x,$$

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)^3 x^2 dt = \frac{1\cdot 2\cdot 3}{5\cdot 7\cdot 9}(2x)^2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)^n x^{n-1} dt = \frac{n!}{5\cdot 7\cdot 9\cdots(2n+3)}(2x)^{n-1}.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Letting $x = r/2$ and adding, we have

$$\frac{3}{4}\int_0^1 t^{1/2}(1-t)[1 + (1-t)x + \cdots + (1-t)^{n-1}x^{n-1} + \cdots]dt$$
$$= \frac{1}{5} + \frac{1\cdot 2}{5\cdot 7}r + \cdots + \frac{n!}{5\cdot 7\cdot 9\cdots(2n+3)}r^{n-1} + \cdots.$$

The series in the right member of this equation converges if $r < 2$. In this case, the series under the integral converges uniformly to the sum $\dfrac{1}{1-(1-t)r/2}$. Hence, the sum of the given series is given by

$$S = \frac{3}{2}\int_0^1 \frac{t^{1/2}(1 - t)}{2 - (1 - t)r}\,dt.$$

When $r = 0$, we find $S = 1/5$, and when $r = 1$, the integral reduces to $5 - (3\pi/2)$; we do not know of any evaluation of the series independent of the integral calculus.

Also solved by H. F. Gummer and E. H. Clarke.

2709 [June, 1918]. Proposed by E. V. HUNTINGTON, Harvard University.

The following problem was suggested to the proposer by a professor of biology, who has found the result useful in certain problems concerning the equilibrium of chemical reactions. Starting with

$$\mu(c_1 + y - x)(c_2 - x)(c_3 - x)\cdots(c_n - x) = \lambda(b_1 + x)(b_2 + x)(b_3 + x)\cdots(b_m + x),$$

where $\mu c_1c_2c_3\cdots c_n = \lambda b_1b_2b_3\cdots b_m$ (all the letters being positive), find the limit of x/y as y approaches zero; and show that for small values of y, the value of x/y is always *less* than this limit.

I. Solution by the Proposer.

Let $y = f(x) = f(0) + f'(0)\cdot x + \frac{1}{2}f''(0)\cdot x^2 + \cdots$.

First. From the given equations, when $x = 0$, $y = 0$; hence $f(0) = 0$.

Second. Taking logarithms and differentiating, we have

$$\frac{f'(x) - 1}{c_1 + y - x} = \left(\frac{1}{c_2 - x} + \frac{1}{c_3 - x} + \cdots + \frac{1}{c_n - x}\right) + \left(\frac{1}{b_1 + x} + \frac{1}{b_2 + x} + \frac{1}{b_3 + x} + \cdots + \frac{1}{b_m + x}\right),$$

whence

$$\frac{f'(0)}{c_1} = \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \cdots + \frac{1}{c_n}\right) + \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots + \frac{1}{b_m}\right).$$

Third. Differentiating again, we have

$$\frac{(c_1 + y - x)f''(x) - [f'(x) - 1]^2}{(c_1 + y - x)^2} = \left[\frac{1}{(c_2 - x)^2} + \frac{1}{(c_3 - x)^2} + \cdots + \frac{1}{(c_n - x)^2}\right]$$
$$- \left[\frac{1}{(b_1 + x)^2} + \frac{1}{(b_2 + x)^2} + \frac{1}{(b_3 + x)^2} + \cdots + \frac{1}{(b_m + x)^2}\right],$$

whence

$$\frac{f''(0)}{c_1} = \left[\left(\frac{1}{c_2} + \frac{1}{c_3} + \cdots + \frac{1}{c_n}\right) + \left(\frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots + \frac{1}{b_m}\right)\right]^2$$
$$+ \left[\frac{1}{c_2^2} + \frac{1}{c_3^2} + \cdots + \frac{1}{c_n^2}\right] - \left[\frac{1}{b_1^2} + \frac{1}{b_2^2} + \frac{1}{b_3^2} + \cdots + \frac{1}{b_m^2}\right],$$

which is clearly *positive*.

Fourth. With these values of $f(0)$, $f'(0)$, and $f''(0)$, we have

$$\frac{y}{x} = f'(0) + \frac{f''(0)}{2}\cdot x + \cdots, \qquad \text{or} \qquad \frac{x}{y} = \frac{1}{f'(0)} - \frac{f''(0)}{2[f'(0)]^2}\cdot x + \cdots,$$

from which the required solution is obvious.

Note: It is clear that x and y have like signs for small values of x since $f'(0) > 0$.

II. Solution by A. M. Harding, University of Arkansas.

Solving the given equation for y gives

$$y = x - c_1 + \frac{\lambda(b_1 + x)(b_2 + x)\cdots(b_m + x)}{\mu(c_2 - x)(c_3 - x)\cdots(c_n - x)} = x - c_1 + \frac{c_1(1 + x/b_1)(1 + x/b_2)\cdots(1 + x/b_m)}{(1 - x/c_2)(1 - x/c_3)\cdots(1 - x/c_n)}.$$

Let c_k denote the smallest of $c_2, c_3, \cdots c_n$. Then, if $x < c_k$, each of the expressions $\dfrac{1}{1 - x/c_i}$, $(i = 2, 3, \cdots, n)$, may be expanded into a convergent series and we obtain, after reduction,

$$y = Ax + Bx^2 + Cx^3 + Dx^4 + \cdots, \tag{1}$$

where

$$A = 1 + c_1 \sum_{i=1}^{m} \frac{1}{b_i} + c_1 \sum_{i=2}^{n} \frac{1}{c_i} = c_1 \left[\sum_{i=1}^{m} \frac{1}{b_i} + \sum_{i=1}^{n} \frac{1}{c_i} \right]$$

and B, C, D, $\cdots$ are all positive.

Since each of the series formed from $\dfrac{1}{1 - x/c_i}$ converges in the interval $0 \leqq x < c_k$, the power series (1) will converge in this interval. And, since in (1) A is not zero, there is one and only one solution for x in terms of y, and this solution may be obtained as a power series in y convergent for sufficiently small values of y.

Let

$$x = \alpha y + \beta y^2 + \gamma y^3 + \delta y^4 + \cdots. \tag{2}$$

Substitute (2) in (1), equate coefficients of like powers of y, and obtain

$$\alpha = \frac{1}{A}, \qquad \beta = -\frac{B}{A^3}, \qquad \gamma = \frac{2B^2}{A^5} - \frac{C}{A^4}, \cdots.$$

That is

$$x = \frac{1}{A} y - \frac{B}{A^3} y^2 + \left(\frac{2B^2}{A^5} - \frac{C}{A^4} \right) y^3 + \cdots$$

or

$$\frac{1}{A} - \frac{x}{y} = \frac{B}{A^3} y - \left(\frac{2B^2}{A^5} - \frac{C}{A^4} \right) y^2 + \cdots. \tag{3}$$

Thus

$$\lim_{y \to 0} \frac{x}{y} = \frac{1}{A} = \frac{1}{c_1 \left(\sum_{i=1}^{m} \frac{1}{b_i} + \sum_{i=1}^{n} \frac{1}{c_i} \right)}.$$

For sufficiently small values of y the right member of (3) is positive. Hence, for small values of y, the value of x/y is always less than $1/A$.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Dr. G. H. LIGHT has been promoted from assistant professor to associate professor of mathematics at the University of Colorado.

The fellows in mathematics at the University of Chicago for next year are Miss GLADYS GIBBENS, Mrs. MAYME I. LOGSDON, and Mr. FRANK E. WOOD.

Dr. C. C. CAMP, who recently returned from war service in France, has been appointed instructor in mathematics at the University of Illinois.

At Brown University Captain R. W. BURGESS, now of the Bureau of Statistics, Washington, and Dr. R. E. GILMAN have been appointed assistant professors of mathematics, and Mr. C. R. ADAMS instructor.

At Cornell University Assistant Professor F. R. SHARPE has been promoted to a full professorship and Mr. V. G. GROVE, of the University of Kentucky, has been appointed instructor.

Dr. E. A. KIRCHER, now in the employ of the National City Bank, New York City, has been appointed instructor of mathematics at Yale University.

Mr. C. A. NELSON has been appointed instructor of mathematics at the University of Kansas.

Mr. A. D. CAMPBELL, now instructor in mathematics at Cornell University, has been appointed instructor at Yale.

Dr. G. W. MULLINS of Barnard College, Columbia University, has been promoted to an assistant professorship of mathematics.

At Harvard University, Assistant Professor G. D. BIRKHOFF has been promoted to a full professorship, Messrs. B. H. BROWN, C. A. RUPP, and R. S. TUCKER have been appointed instructors in mathematics, and Dr. I. A. BARNETT and Dr. H. C. M. MORSE have been appointed Benjamin Peirce instructors in mathematics. Professor H. N. DAVIS, assistant professor of physics since 1910, has been appointed professor of mechanical engineering.

At the Massachusetts Institute of Technology the following instructors in mathematics have been appointed for next year: Mr. RAYMOND DOUGLASS, of the University of Maine; Dr. J. S. TAYLOR, of the University of California, and Dr. NORBERT WIENER, of Harvard University. Assistant Professor H. C. BRADLEY of the department of drawing and descriptive geometry has been promoted to an associate professorship; in the same department Mr. S. A. BREED has been appointed instructor.

The following announcements come from the mathematics department of the University of Minnesota. Assistant Professor DUNAHM JACKSON, of Harvard, has been appointed to a professorship. Major W. L. HART has been appointed to an assistant professorship. Mr. R. W. BRINK has been promoted to an assistant professorship, and given leave of absence for a year to enable him to accept a lectureship at the University of Edinburgh. Dr. C. H. YEATON has been appointed as an instructor to take Mr. Brink's place for the year 1919–1920. Miss MINNA J. SCHICK has been appointed instructor for the first Quarter of 1919–1920, as a substitute for Professor G. N. BAUER who is absent on leave until January 1, 1920.

Professor F. E. MILLER, head of the department of mathematics in Otterbein College and a charter member of the Association, died March 26, 1919.

On March 28, Major J. L. COOLIDGE, of Harvard University, commenced a course of lectures (twice a week) at the University of Paris on "Geometry in the complex domain."

Of the six three-hour examinations held at Annapolis this month for the

purpose of selecting instructors of mathematics at the Academy, one was on theoretical mechanics as in Miller and Lilly's *Analytic Mechanics,* and another was on strength of materials and hydromechanics as in Smith's *Strength of Material* and Alger's *Hydromechanics.* The other examinations were on differential and integral calculus as in Granville's work, analytic geometry as in Smith and Gale's *New Analytic Geometry,* trigonometry as in Brown's *Trigonometry and Stereographic Projections,* and Wentworth, *College Algebra* (revised), 1902, and Wentworth's *Solid Geometry* (revised), 1909.

Higher Educational Circular No. 14 issued in February, 1919, by the Bureau of Education, Washington, gives an account of advanced educational work at the Bureau of Standards where, during the past ten years, graduate courses in physics, mathematics, and chemistry have been maintained. Seventeen men have used investigations made at the Bureau for their doctoral theses. Eleven of these have received credit for attendance upon the lecture courses at the Bureau. Courses requiring considerable mathematical development, such as advanced optics, and thermodynamics have been given; but the emphasis in these courses was primarily upon the physics. The following courses in pure and applied mathematics have been offered recently: 1917–18—Least squares by Dr. P. W. MERRILL, Bureau of Standards; Electrical oscillations by Dr. LOUIS COHEN, Signal Corps; Introduction to mathematical physics by P. G. AGNEW, Bureau of Standards; Electric waves by Professor W. S. FRANKLIN, Massachusetts Institute of Technology. 1918–19—Advanced differential equations by Major F. R. MOULTON, University of Chicago; Introduction to mathematical physics by Professor J. S. AMES, Johns Hopkins University.

In addition to the reports of summer courses in mathematics given in the March, April and May numbers of the MONTHLY, we have the following:

Leland Stanford University: June 17–August 30. Professor H. F. BLICHFELDT, Algebra (4 units); coördinate geometry (4 units); Reading course.

University of Pennsylvania: July 8–August 16. Professor G. H. HALLETT: Elementary algebra (to quadratics); plane trigonometry; analytic geometry; higher calculus. Professor H. H. MITCHELL: College algebra; differential calculus; integral calculus; mathematical theory of probability. Professor R. L. MOORE: Elementary algebra (quadratics and beyond); plane geometry; solid geometry; introduction to the theory of functions of a complex variable.

University of Michigan, June 29–August 2. Professor W. W. BEMAN: Geometry and Algebra for teachers (2 hours credit); differential equations (2 hours). Professor T. L. MARKLEY: Theory of functions of a complex variable (2 hours credit); algebra (2 hours); modern geometry and analytic geometry of three dimensions (2 hours). Professor L. C. KARPINSKI: Elementary algebra (for entrance); history of mathematics (2 hours credit). Professor C. J. COE: Algebra (2 hours credit); Analytic geometry (4 hours); Professor V. C. POOR: Integral calculus and differential equations (four or five hours credit). Dr. G. H. CRESSE: Plane Geometry (for entrance); plane trigonometry (two hours

credit). Dr. E. S. ALLEN: Solid geometry (for entrance). Dr. R. B. ROBBINS: Mathematical theory of statistics (two hours credit); introduction to the mathematical theory of interest (two hours); introduction to the mathematical theory of life insurance (two hours). Dr. A. L. NELSON: Elementary course in differential and integral calculus (four or five hours credit). Dr. L. J. ROUSE: Analytic geometry (continuation of Professor Coe's course—four hours credit).

In addition to those recorded in our last issue, as having been elected fellows of the National Academy of Sciences at the April meeting, mention should be made of Professor E. J. WILCZYNSKI, University of Chicago.

At the May meeting of the American Academy of Arts and Sciences, Professors JOSEPH LIPKA, G. A. MILLER, F. R. MOULTON, and VIRGIL SNYDER were elected Fellows in Class I, section 1—Mathematics and Astronomy.

To the Division of Physical Sciences of the National Research Council the American Mathematical Society elected Professors E. W. BROWN, L. E. DICKSON, and H. S. WHITE, as representatives; the American Physical Society elected Professor E. B. WILSON. Professor Wilson is a member of the Executive Committee of the Division.

At the seventeenth annual meeting of the Association of Teachers of Mathematics in New England held at Boston University, May 3, 1919, Miss FLORENCE P. LEWIS, exchange professor from Goucher College at Wellesley College, presented a paper entitled "History of the parallel postulate," and Professor C. L. BOUTON, of Harvard University, discussed "Photogrammetry." Professor W. R. RANSOM, of Tufts College, is president of the Association.

At the Educational Congress held under the direction of The University of the State of New York in Albany, May 19th to 28th, the papers read included the following by members of the Association: "Mathematical Requirements" by Professor J. W. YOUNG and President F. C. FERRY; "Mathematics in the Junior High School" by Mr. WILLIAM BETZ; "Mathematics of the Senior High School of the Future" by Professor H. E. HAWKES; "Experiments in Teaching Secondary Mathematics" by Miss VEVIA BLAIR; "Applied Mathematics in High School Courses" by Professor W. E. BRECKENRIDGE; "Projects for Mathematical Research" by Professor D. E. SMITH; "Training of Mathematics Teachers" by Professors R. C. ARCHIBALD and HARRY BIRCHENOUGH.

The Association of Teachers of Mathematics in the Middle States and Maryland held a joint meeting with the Association of Mathematics Teachers of New Jersey at Newark, N. J., on May 3. The program included: "Test of intelligence for admission to college," by Professor A. L. JONES, Columbia University; "Certain undefined elements and tacit assumptions in the first book of Euclid's Elements,"—the presidential address of the New Jersey Association—by Mr.

H. E. WEBB, Central High School, Newark, N. J.; "On Newton's method of approximation," by Dean H. B. FINE, Princeton University; "Some illustrations of statistical methods," by Mr. P. C. H. PAPPS, Newark, N. J.

The two hundred and third regular meeting of the American Mathematical Society at New York City was held on April 26, 1919, at Columbia University. Twenty-three papers of the usual research nature were presented. In addition to these papers, Professor DUNHAM JACKSON and Dr. T. H. GRONWALL gave reports on work in ballistics at Aberdeen and Washington. Attending members of the Society took luncheon and dinner together at the Faculty Club. A full report of the meeting will be found in the *Bulletin of the American Mathematical Society*.

SUMMER MEETING OF THE ASSOCIATION.

The fourth summer meeting of the Association will be held at the University of Michigan, Ann Arbor, Michigan, on Thursday and Friday, September 4–5, 1919. It will immediately follow the meeting of the American Mathematical Society to be held at the same place September 2–4. A joint meeting of the two organizations will be arranged for Thursday afternoon, and a joint dinner for Thursday evening. Detailed programs of the meeting will be mailed to all members of the Association at a later date, but it may be announced at this time that general arrangements have already been made for the accommodation of attending members. For this purpose, both the Helen Newberry Residence and the recently built Michigan Union Building will be available. The rates for rooms in Newberry Residence will be one dollar per day; in the Michigan Union Building, one dollar and fifty cents per day. Meals will be furnished under the auspices of the Michigan Union at reasonable rates. Special provisions will be made for ladies and married couples.

Ann Arbor is on the main line of the Michigan Central Railroad and of the Ann Arbor railroad.

The American Astronomical Association will meet in Ann Arbor probably from Monday to Wednesday of the same week.

The American Mathematical Monthly

OFFICIAL ORGAN OF

The Mathematical Association of America

**Is the Only Journal of Collegiate Grade in
The Mathematical Field in this Country**

This means that its mathematical contributions can be read and understood by those who have not specialized in mathematics beyond the Calculus.

The Historical Papers, which are numerous and of high grade, are based upon original research.

The Questions and Discussions, which are timely and interesting, cover a wide variety of topics.

Surveys of the contents of recent books and periodicals constitute a valuable guide to current mathematical literature.

The Topics in the department of Undergraduate Mathematical Clubs have excited wide interest both in this country and in Great Britain.

The Notes and News cover a wide range of interest and information both in this country and in foreign countries.

The Problems and Solutions hold the attention and activity of a large number of persons who are lovers of mathematics for its own sake.

There are other journals suited to the Secondary field, and there are still others of technical scientific character in the University field: but the MONTHLY is the only journal of Collegiate grade in America suited to the needs of the non-specialist in mathematics.

THE MATHEMATICAL ASSOCIATION OF AMERICA now has over eleven hundred individual and institutional members. There are already nine sections formed, representing twelve different states. The Association has held so far two national meetings per year, one in September and one in December. The sections, for the most part, hold two meetings each year. All meetings, both national and sectional, are reported in the Official Journal, and many of the papers presented at these meetings are published in full.

The slogan of the Association is to include in its membership every teacher of collegiate mathematics in America and to make such membership worth while. Application blanks for membership may be obtained from the Secretary, W. D. Cairns, 27 King Street, Oberlin, Ohio.

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

THIRD ANNUAL MEETING OF THE KENTUCKY SECTION.[1]

The Third Annual Meeting was held at the University of Kentucky, Saturday, May 17, 1919, Professor E. L. Rees presiding. The following papers were read: "A chart of mathematical history" by E. L. Rees, Univ. of Kentucky; "A problem in linkages" by H. P. Pettitt, Univ. of Kentucky (by invitation); "What is a curve?" by C. H. Richardson, Georgetown College (by invitation); "A problem in maxima and minima" by H. H. Downing, Univ. of Kentucky; "Syzygies obtained from the associative law for $n = 2$" by G. W. Smith, Univ. of Kentucky; "The catenoid the only minimal surface of revolution" by W. W. Elliott, Univ. of Kentucky; "Present day tendencies in the analytics course" by P. P. Boyd, Univ. of Kentucky; "Review of Shaw's *Philosophy of Mathematics*" by J. M. Davis, Univ. of Kentucky.

Professor C. G. Crooks, Center College, Danville, Ky., was elected chairman and Dr. G. W. Smith, University of Kentucky, was elected secretary-treasurer. After the meeting the members were the guests of the retiring chairman, Professor Rees, at a very delightful luncheon served in the rooms of the University Cafeteria.

H. H. Downing, *Secretary-Treasurer.*

THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS.

The General Education Board has appropriated the sum of sixteen thousand dollars for the use of the National Committee on Mathematical Requirements. This Committee was appointed some three years ago by the Mathematical Association of America for the purpose of giving national expression to the movement for reform in the teaching of mathematics in secondary schools and colleges, which movement had for many years been actively furthered by various sectional organizations of teachers throughout the country.

According to plans adopted, one college man and one secondary school man are to devote their whole time to the work of the Committee for one year beginning July 1, 1919. Professor J. W. Young, of Dartmouth College, has been selected as the college man in question. He will act as chairman and treasurer of the Committee and will have general charge of the work with headquarters in Hanover, N. H. Mr. J. A. Foberg, of the Crane Technical High School of Chicago, is the representative of the secondary schools who will devote his whole time to work of the Committee. He will act as vice-chairman and secretary of the Committee with headquarters in Chicago. The other members of the Committee are Professors A. R. Crathorne, of the University of Illinois,

[1] This section was formerly the Mathematics Section of the Association of Kentucky Colleges which held eight annual meetings.

E. H. Moore, of the University of Chicago, C. N. Moore, of the University of Cincinnati, D. E. Smith, of Columbia University, and H. W. Tyler, of the Massachusetts Institute of Technology, representing the colleges; and Miss Vevia Blair, of the Horace Mann School, New York, Mr. G. W. Evans, of the Charlestown High School, Boston, and Mr. Raleigh Schorling, of the Lincoln School, New York, representing the secondary schools. Two or three additional members representing secondary schools will be added to the Committee in the near future.

Two specific problems face the Committee: (1) the revision of secondary school and college courses in mathematics; (2) the revision of college entrance requirements in mathematics. The latter problem has been referred to the Committee not only by the Mathematical Association of America but also by the Council of the American Mathematical Society.

Detailed plans of the Committee are in the formative stage. Certain of its functions seem to be clear now, however. It must complete the reports already under way, it must formulate for discussion general principles which are to govern the proposed revisions (1) and (2) referred to above, it must elaborate, also for discussion, the detailed application of such general principles, it must establish intimate contact with all organizations and agencies throughout the country having similar problems in hand, it must do all it can to organize a nation-wide discussion of these problems and seek to coordinate the results of such discussion. Pending the adoption of definite plans for action the chairman or vice-chairman will welcome any suggestions looking toward the most effective methods of procedure.

A THEORY AND GENERALIZATION OF THE CIRCULAR AND HYPERBOLIC FUNCTIONS.

By A. F. FRUMVELLER, Marquette University.

There is a striking lack of symmetry and elegance in the usual treatment of hyperbolic functions; the formulas are one-sided, some admitting i, others not, in a manner apparently arbitrary. An air of unreality is added by the fact that they are made to depend essentially upon

$$e^{ix} = \cos x + i \sin x = \sum_{\infty} \frac{(ix)^n}{n!},$$

whose small geometric content entirely evaporates after a few transformations; many authorities frankly abandon the geometric phase altogether, and treat the whole topic as an exercise in the algebra of complex numbers. Long ago the writer became convinced that the one-sidedness of the formulas with respect to i arose from building the theory on too narrow a foundation; the circle after all is not a primary curve, but merely an ellipse in which the axes have accidentally become equal. If this be so, circle trigonometry is only a minor case of the

trigonometry of the ellipse; and a true insight into the relation between the circular and hyperbolic functions can be attained only by moving the ellipse into the place of honor, and studying the functions that naturally present themselves in connection with its equation.

The results of an investigation into this matter proved to be highly interesting. First of all, the primary or ancestral forms for both circular and hyperbolic functions were brought to light; and secondly, the trigonometry of all the central conics, real or imaginary, presented itself in a systematic manner, and without reference to De Moivre's theorem. In fact, the ancestral type of De Moivre's formula was discovered at once.

A previous attempt at a generalization in this field was made by Prof. Irving Stringham in the *American Journal of Mathematics*, vol. 14, 1892, and in a little book entitled *Uniplanar Algebra* (Univ. of California Press). He lays down two entirely arbitrary definitions:

$$\sin_k w = \frac{k}{2}(e^{w/k} - e^{-w/k}), \qquad \cos_k w = \tfrac{1}{2}(e^{w/k} + e^{-w/k}).$$

These, with four others derived from them ($\tan_k$, $\cot_k$, $\sec_k$, $\csc_k$), he calls "modocyclic" functions; for $k = 1$, we get the hyperbolic functions, for $k = i$ the circular. Stringham's $\sin_k$ and $\cos_k$ are mutually unsymmetric, whereas the circle and hyperbola are symmetric in two ways; his formulas are therefore not the ultimate ancestral types; they are realizable in the curve $x^2 + y^2/k^2 = 1$. though he did not advert to this fact. Later on, Becker and Van Orstrand (*Smithsonian Tables of Hyperbolic Functions*, 1909) used this k-ellipse in developing their theory; but the presence of only one parameter makes it impossible to derive formulas in this way for the general central conic. Our formulas will have to have two parameters corresponding to the two parameters of the locus

1. *The "Protocyclic" functions.*

Consider the hyperbola

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1;$$

the area APM is given by

$$\frac{b}{a}\int_x^a \sqrt{x^2 - a^2}\,dx = \frac{b}{2a}\left(x\sqrt{x^2 - a^2} - a^2 \log \frac{x + \sqrt{x^2 - a^2}}{a}\right)$$

$$= OPM - OAP;$$

hence if OAP is called H, we have

$$\frac{2H}{ab} = \log\left(\frac{x}{a} + \frac{y}{b}\right) \qquad \text{or} \qquad \frac{x}{a} + \frac{y}{b} = e^{2H/ab}.$$

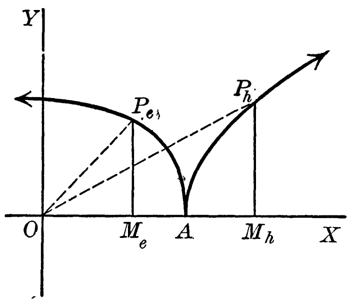
The curve-equation furnishes as the factorial mate to this

$$\frac{x}{a} - \frac{y}{b} = e^{-(2H/ab)};$$

so that

$$x_h = \frac{a}{2}\left(e^{2H/ab} + e^{-(2H/ab)}\right), \qquad y_h = \frac{b}{2}\left(e^{2H/ab} - e^{-(2H/ab)}\right).$$

Consider in the next place, the ellipse



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1;$$

its area APM is

$$\frac{b}{a}\int_a^x \sqrt{a^2 - x^2}\,dx = \frac{b}{ai}\int_a^x \sqrt{x^2 - a^2}\,dx.$$

Hence if OAP is called E, we get as above

$$\frac{2E}{ab}i = \log\left(\frac{x}{a} + \frac{iy}{b}\right),$$

and

$$x_e = \frac{a}{2}\left(e^{(2E/ab)i} + e^{-(2E/ab)i}\right), \qquad y_e = \frac{b}{2i}\left(e^{(2E/ab)i} - e^{-(2E/ab)i}\right).$$

Comparing these E and H formulas, we observe that if we put

$$2H \text{ (or } 2E) = w, \qquad \frac{1}{a} = k_1, \qquad \text{and} \qquad \frac{1}{b}\left(\text{or } \frac{i}{b}\right) = k_2,$$

the expressions for x become identical; likewise those for y; and the equation of both curves becomes $k_1^2x^2 - k_2^2y^2 = 1$. We therefore lay down the following proposition:

Every central conic $q_1^2x^2 - q_2^2y^2 = 1$ has associated with it two functions, which define x and y for any point P on the curve in terms of the area of the vertex-sector $OAP = \Omega/2$. These functions are called the cosine, and the sine, respectively; and are defined by the equations

$$(1) \qquad x = \cos\Omega = \frac{e^{q_1q_2\Omega} + e^{-q_1q_2\Omega}}{2q_1}; \qquad y = \sin\Omega = \frac{e^{q_1q_2\Omega} - e^{-q_1q_2\Omega}}{2q_2}.[1]$$

[1] In the case of the circle and equilateral hyperbola this proposition has been a familiar one of our text books—for example, *An elementary treatise on the differential calculus* by J. M. Rice and W. W. Johnson, New York, 1880, pp. 139–140; according to Montucla (*Histoire des mathématiques*, Vol. 3, 1802, p. 287) the conception dates back to the time of Lambert. The extension of this idea as above to the general conic by means of an imaginary integration for the ellipse, and the consequent development of a train of two-parameter formulas including those for the circle and hyperbola as sub-cases carries this process to its ultimate completion in the most direct manner. A generalization from circle and equilateral hyperbola to any ellipse and any hyperbola was considered by C. A. Laisant, *Essai sur les fonctions hyperboliques*, Paris, 1874, p. 37 f. (also in *Mémoires de la société des sciences de Bordeaux*, Vol. 10). In particular Laisant indicates the relations between v and w (see page 286 of this paper), in which "the figure is unified in a remarkable manner." Professor Frumveller's presentation of his rediscovery of this and other relations, and of his deductions therefrom, is well worthy of perusal, even by those to whom Laisant's scarce volume is accessible. See also S. Günther, *Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunktionen*, 1881, p. 371 f.—EDITOR.

These, with their quotients and inverses, constitute the six functions that we shall call "protocyclic"; the number Ω will be taken to be essentially real, at least for the present, while q_1, q_2, are numbers of the general type $u + iv$. If the reference-curve be $\varphi(q_1, q_2) = 0$, the four simplest classes of these functions are those associated with the curves $\varphi(k_1, k_2)$; $\varphi(k_1, ik_2)$; $\varphi(ik_1, k_2)$; $\varphi(ik_1, ik_2)$; the two first mentioned degenerating into the ordinary hyperbolic and circular functions respectively, when $k_1 = k_2 = 1$ and both are real.

Solving equations (1) for the exponentials, and raising both sides to the nth power, we obtain the most general form of DeMoivre's theorem, valid for all n's;

$$(2) \qquad e^{\pm q_1 q_2 n\Omega} = q_1 \cos n\Omega \pm q_2 \sin n\Omega = (q_1 \cos \Omega \pm q_2 \sin \Omega)^n.$$

When n is negative, we must note that $\cos \Omega = \cos(-\Omega)$, since the cosine is an even function, while the sine is odd. In like manner

$$(3) \qquad \prod_{\lambda=1}^{n} (q_1 \cos \Omega_\lambda \pm q_2 \sin \Omega_\lambda) = q_1 \cos\left(\sum_1^n \Omega_\lambda\right) \pm q_2 \sin\left(\sum_1^n \Omega_\lambda\right),$$

as is evident on changing both sides into exponentials.

The equation of the curve, together with the definitions (1), gives

$$(4) \qquad \begin{aligned} &q_1^2 \cos^2 \Omega - q_2^2 \sin^2 \Omega = 1, \text{ with its corollaries,} \\ &q_1^2 - q_2^2 \tan^2 \Omega = \sec^2 \Omega, \qquad q_1^2 \cot^2 \Omega - q_2^2 = \csc^2 \Omega. \end{aligned}$$

No end of formulas may now be evolved from these data! Let w and w_1 be two special values of Ω; form the four products $\sin w \cos w_1$, $\cos w \sin w_1$, $\sin w \sin w_1$, $\cos w \cos w_1$, using exponentials as in equations (1); then

$$(5) \qquad \begin{aligned} \sin(w \pm w_1) &= q_1(\sin w \cos w_1 \pm \cos w \sin w_1), \\ \cos(w \pm w_1) &= (1/q_1)(q_1^2 \cos w \cos w_1 \pm q_2^2 \sin w \sin w_1). \end{aligned}$$

From these we get tangent and secant formulas, formulas for $w/2$, $2w$, $3w$, etc., in the usual way. On adding these two sine or cosine equations, and writing $\xi = w + w_1$, $\eta = w - w_1$, product-formulas result:

$$(6) \qquad \begin{aligned} \sin \xi \pm \sin \eta &= 2q_1 \sin\left(\frac{\xi \pm \eta}{2}\right) \cos\left(\frac{\xi \mp \eta}{2}\right), \\ \cos \xi + \cos \eta &= 2q_1 \cos\left(\frac{\xi + \eta}{2}\right) \cos\left(\frac{\xi - \eta}{2}\right), \\ \cos \xi - \cos \eta &= 2\frac{q_2^2}{q_1} \sin\left(\frac{\xi + \eta}{2}\right) \sin\left(\frac{\xi - \eta}{2}\right). \end{aligned}$$

To find the general formulas for $\sin^m w$ and $\cos^m w$ in multiples of w, we write $\xi^r = q_1 \cos rw + q_2 \sin rw$, and $1/\xi^r = q_1 \cos rw - q_2 \sin rw$, so that

$$\cos rw = \frac{1}{2q_1}\left(\xi^r + \frac{1}{\xi^r}\right), \qquad \sin rw = \frac{1}{2q_2}\left(\xi^r - \frac{1}{\xi^r}\right).$$

Thus

$$\cos^{2n} w = \left(\frac{1}{2q_1}\right)^{2n}\left(\xi + \frac{1}{\xi}\right)^{2n}$$
$$= \left(\frac{1}{2q_1}\right)^{2n}\left[\left(\xi^{2n} + \frac{1}{\xi^{2n}}\right) + \binom{2n}{1}\left(\xi^{2n-2} + \frac{1}{\xi^{2n-2}}\right) + \cdots\right].$$

A similar expansion is found for $\sin^m w$, the results being

$$\cos^{2n} w = \left(\frac{1}{2q_1}\right)^{2n-1}\left[\cos 2nw + \binom{2n}{1}\cos(2n-2)w + \binom{2n}{2}\cos(2n-4)w + \cdots + \frac{1}{2q_1}\binom{2n}{n}.\right]$$

$$\cos^{2n+1} w = \left(\frac{1}{2q_1}\right)^{2n}\left[\cos(2n+1)w + \binom{2n+1}{1}\cos(2n-1)w + \cdots + \binom{2n+1}{n}\cos w\right]$$

$$(7)\qquad \sin^{2n} w = 2q_1\left(\frac{1}{2q_2}\right)^{2n}\left[\cos 2nw - \binom{2n}{1}\cos(2n-2)w + \binom{2n}{2}\cos(2n-4)w - \cdots + (-1)^n\frac{1}{2q_1}\binom{2n}{n}\right]$$

$$\sin^{2n+1} w = \left(\frac{1}{2q_2}\right)^{2n}\left[\sin(2n+1)w - \binom{2n+1}{1}\sin(2n-1)w + \cdots + (-1)^n\binom{2n+1}{n}\sin w\right].$$

These will suffice to give an idea of what the new formulas look like; $q_1 = 1$, $q_2 = i$, reduces them all to the case of ordinary trigonometry.

Since our fundamental Ω-functions are exponential, they are *periodic:* their period is $2\pi i/q_1q_2$, so that

$$f(q_1q_2\Omega) = f[q_1q_2(\Omega \pm 2n\pi i/q_1q_2)]; \qquad f(q_1q_2\Omega/i) = f[q_1q_2(\Omega/i \pm 2n\pi/q_1q_2)].$$

To establish this, we observe that when $q_1q_2\Omega$ becomes $i\Omega$ by putting $q_1 = 1, q_2 = i$, the x and y of formulas (1) become the ordinary trigonometric functions $\cos_c$, $\sin_c$, associated with $x^2 + y^2 = 1$; thus

$$y_c = \sin_c \Omega = (1/2i)(e^{i\Omega} - e^{-i\Omega}) = \sin_c(\Omega \pm 2n\pi) = (1/2i)(e^{i(\Omega\pm 2n\pi)} - e^{-i(\Omega\pm 2n\pi)});$$

in general therefore, we get $\sin_c$, $\cos_c$, by placing $q_1q_2\Omega = i(\Omega \pm 2n\pi)$, *i. e.*, by placing

$$q_1q_2\left(\Omega \pm \frac{2n\pi i}{q_1q_2}\right) = i\Omega.$$

It is now an easy matter to follow the variation of our general functions sin, cos, tan, etc., through quarter-periods;

$$
(8)\quad
\begin{aligned}
&\text{if } q_1q_2\Omega = \frac{\pi}{2}i, && \left(\text{or } \Omega = \frac{\pi i}{2q_1q_2}\right), && \sin\Omega = \frac{i}{q_2}\sin_c\frac{\pi}{2} = \frac{i}{q_2};\\
&\text{if } q_1q_2\Omega = \frac{2\pi}{2}i, && \left(\text{or } \Omega = \frac{2\pi i}{2q_1q_2}\right), && \sin\Omega = \frac{i}{q_2}\sin_c\frac{2\pi}{2} = 0;\\
&\text{if } q_1q_2\Omega = \frac{3\pi}{2}i, && \left(\text{or } \Omega = \frac{3\pi i}{2q_1q_2}\right), && \sin\Omega = \frac{i}{q_2}\sin_c\frac{3\pi}{2} = -\frac{i}{q_2};\\
&\text{if } q_1q_2\Omega = \frac{4\pi}{2}i, && \left(\text{or } \Omega = \frac{4\pi i}{2q_1q_2}\right), && \sin\Omega = \frac{i}{q_2}\sin_c\frac{4\pi}{2} = 0.
\end{aligned}
$$

The corresponding cycle for $\cos\Omega$ is $(0, -1/q_1, 0, 1/q_1)$; from these the tangent and secant follow at once. The tangent has a period $\pi i/q_1q_2$.

The combined use of formulas (5) and (8) now gives us the generalized forms of the trigonometric expressions for $\sin_c(90 \pm A)$, $\cos_c(180 \pm A)$, etc., *i.e.*, for *complements* and *supplements* of Ω; the following are specimens:

$$
(9)\quad
\begin{aligned}
&\sin\left(\Omega \pm \frac{\pi i}{2q_1q_2}\right) = \pm\frac{iq_1}{q_2}\cos\Omega; && \sin\left(\Omega \pm \frac{2\pi i}{2q_1q_2}\right) = -\sin\Omega,\\
&\cos\left(\Omega \pm \frac{\pi i}{2q_1q_2}\right) = \pm\frac{iq_2}{q_1}\sin\Omega; && \cos\left(\Omega \pm \frac{2\pi i}{2q_1q_2}\right) = -\cos\Omega.
\end{aligned}
$$

We might mention in passing that if w is a value of Ω,

$$\frac{d}{dw}\sin w = q_1^2\cos w,$$

and

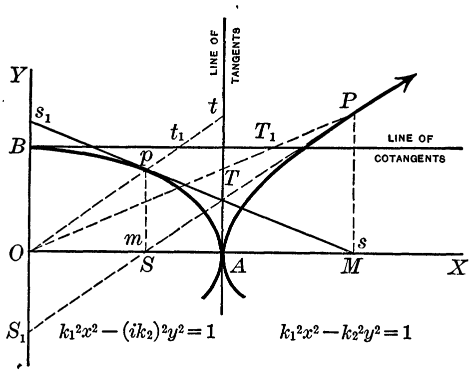
$$\frac{d}{dw}\cos w = q_2^2\sin w,$$

so that our sin and cos functions satisfy the differential equation $D_w^2z - q_1^2q_2^2z = 0$. As a final rounding out of our theory, we give the series-expansions of these two functions:

$$(10)\qquad \sin w = \frac{1}{q_2}\sum_0^\infty \frac{(q_1q_2w)^{2n+1}}{\underline{|2n+1}}; \qquad \cos w = \frac{1}{q_1}\sum_0^\infty \frac{(q_1q_2w)^{2n}}{\underline{|2n}}.$$

2. *Geometry of the protocyclic functions.*

If k_1, k_2, are real numbers, the reference-curve $q_1^2x^2 - q_2^2y^2 = 1$ has four fundamental types, namely $(q_1, q_2) = (k_1, k_2)$, (k_1, ik_2), (ik_1, k_2), or (ik_1, ik_2); the first and second form a pair (hyperbola and ellipse), likewise the third and fourth. Each of these types has a geometrical representation for the six functions; and the geometry of each pair has cross-relationships of a very interesting kind.



We shall use the subscripts e, h, c, for the functions connected with the ellipse, hyperbola, and circle.

(*a*) *Types* (k_1, k_2) *and* (k_1, ik_2). In the adjacent hyperbola let $w = 2(OAP)$; AT, PS, are tangents to H; $OA = 1/k_1$; $OB = 1/k_2$; $PS \equiv k_1^2 xx_h - k_2^2 yy_h - 1 = 0$; $OP \equiv y - (\sin_h w/\cos_h w)x = 0$. The various intercepts of these lines give at once the following table of line-values for the protocyclic functions of a point P on H:

$$PM = \sin_h w, \qquad OS_1 = \frac{-1}{k_2^2}\csc_h w,$$

$$OM = \cos_h w, \qquad OS = \frac{1}{k_1^2}\sec_h w,$$

$$AT = \frac{1}{k_1}\tan_h w, \qquad BT_1 = \frac{1}{k_2}\cot_h w.$$

In the same way precisely, the ellipse E has the following line-values at the point p;

$$pm = \sin_e v, \qquad Os_1 = \frac{1}{k_2^2}\csc_e v,$$

$$Om = \cos_e v, \qquad Os = \frac{1}{k_1^2}\sec_e v,$$

$$At = \frac{1}{k_1}\tan_e v, \qquad Bt_1 = \frac{1}{k_2}\cot_e v.$$

When p and P are chosen at random, there is no relation between v and w; but if ordinate pm be erected at the point S, the entire figure is unified in a remarkable manner.[1] We have first of all, $OS = \cos_e v = (1/k_1^2)\sec_h w$; ps, the tangent at p, has for its x-intercept, $1/(k_1^2 x_e) = (1/k_1^2)\sec_e v = \cos_h w$, so that it passes through M. Using formulas (4) of § 1 on the two values of OS, we have

$$(1/k_1^2)(1 - k_2^2 \sin_e^2 v) = (1/k_1^4)(k_1^2 - k_2^2 \tan_h^2 w)$$

whence

$$\sin_e v = \left(\frac{1}{k_1}\right)\tan_h w;$$

applying

$$\cos_e v = \left(\frac{1}{k_1^2}\right)\sec_h w$$

to this, we get $\tan_e v = k_1 \sinh_h w$; which shows that the points t and P, T and p,

[1] Cf. footnote on page 282.

are respectively at the same level above OX. The relation thus existing between v and w is expressed by the functional symbol $v_e = \text{am}\, w_h$, or $v_e = \text{gd}\, w_h$ (read, "v is the *amplitude* or *gudermannian*[1] of w").

The operation "*gd*" is carried out as follows, by means of formula 2, § 1;

$$e^{k_1k_2w} = k_1 \cos_h w + k_2 \sin_h w = k_1^{-1} \sec_e v + (k_2/k_1) \tan_e v = V,$$

whence

$$w = \frac{1}{k_1k_2} \log V.$$

If there were in existence a table of the various functions of v_e, we could consequently compute w_h and its functions from that table; ordinary hyperbolic functions for $k_1 = k_2 = 1$ are thus easily found from the corresponding tables of circular trigonometry.[2]

(*b*) *Types* (ik_1, k_2) *and* (ik_1, ik_2). The fundamental loci now become

$$\begin{cases} H_{-1}; \quad (ik_1)^2x^2 - (ik_2)^2y^2 = 1, \quad \text{or} \quad -k_1^2x^2 + k_2^2y^2 = 1. \\ E_{-1}; \quad (ik_1)^2x^2 - k_2^2y_2 = 1, \quad \text{or} \quad -k_1^2x^2 - k_2^2y^2 = 1. \end{cases}$$

The ellipse E_{-1} is the imaginary form of our previous E; hence H_{-1} must be regarded as the imaginary form of H. All the formulas of § 1 were predicated on the supposition that Ω was *real*; but we may generalize our point of view as regards x and y in the following manner: "*the theory as well as the geometry of the protocyclic functions remain valid, even when x or y is a pure imaginary number.*"

Graphing under such circumstances has been fully worked out by the writer in a former paper[3] to which the reader must be referred for details.

The equation H_{-1}, or $-k_1^2x^2 + k_2^2y^2 = 1$, when $x = u + iv$, and y is real, becomes $k_2^2y^2 = 1 + k^2{}_1(u^2 - v^2)$, with the condition $uv = 0$; if $v = 0$,

$$k_2^2y^2 - k_1^2u^2 = 1$$

in the plane $\overline{uoy}$; if $u = 0$, $k_2^2y^2 + k_1^2v^2 = 1$ in plane $\overline{voy}$. The other equation E_{-1}, or $k_1^2x^2 + k_2^2y^2 = -1$, gives $1 = -k_1^2u^2 - k_2^2y^2$ in $\overline{uoy}$, and $1 = k_1^2v^2 - k_2^2y^2$ in $\overline{voy}$; in the complex plane $\overline{voy}$, we therefore find the geometrical locus of case (*a*) reproduced exactly!

When the k's become complex numbers, we can still write

$$(H) \qquad y^2 = \frac{1}{k_2^2} - \left(\frac{k_1^2}{k_2^2}\right)x^2 = (r_1 + is_1) - (r_2 + is_2)(u + iv)^2$$

$$= (r_1 - r_2u^2 + r_2v^2 + 2s_2uv) + i(s_1 - s_2u^2 + s_2v^2 - 2r_2uv),$$

[1] The term "Gudermannian" was introduced by Cayley in a paper of 1862 (see *Collected Mathematical Papers*, Volume 5, p. 86); if $u = \log \tan [(\pi/4) + (\phi/2)]$, then $\phi = 1/i \log \tan [(\pi/4) + \frac{1}{2}ui]$ = "gd u (Gudermannian of u)." The name was given in honor of Gudermann who called the function the "longitude of u" (Crelle's *Journal*, Vol. 6, 1830, p. 165).—EDITOR.

[2] The *Smithsonian Tables* referred to above contain an extensive table of the gudermannian (and another of the antigudermannian); a brief one may be found in B. O. Peirce, *A Short Table of Integrals*, revised edition.—EDITOR.

[3] "The graph of $f(x)$ for complex numbers," AMERICAN MATH. MONTHLY, Vol. 24, November, 1917.

with a similar equation for (E), but there is no simple geometrical method of attacking this wilderness of ordinates. The y's stand at every point in the x-plane, but they have various slants towards the fourth direction of space; all we notice is, that those of the same slant or tilt have their feet on an hyperbola around O as center. Methods of function-theory must here be used.

Conclusion.—One primary advantage of treating hyperbolic functions from this general standpoint lies, in the writer's judgment, in the relegation of i to its proper subordinate position; protocyclic functions are real, and represent geometric entities from the very start; the generalization that occurs in their definition is seen to be geometrically appropriate. The theoretic interest is mainly in the fact that circle-trigonometry and hyperbola-trigonometry can now be comprehended in the wider topic of the trigonometry of the central conics.

A PROOF OF A THEOREM OF COMPOUND PROBABILITIES.

By UGO BROGGI,[1] Universidad De La Plata.

1. The definition of a geometric probability is well known. We call the probability, $\phi(a, b)$, that a real number, x, belongs to the interval (a, b) a real function, ϕ, of a and b which satisfies the conditions

$$0 \leqq \phi(a, b) \leqq 1,$$

and, if $a \leqq c \leqq b$,

$$\phi(a, b) = \phi(a, c) + \phi(c, b).$$

It would be an easy matter, but beside our purpose, to generalize the definition, and apply it not only to intervals but also to more general one- or more-dimensional sets of points.

Since the definition of our new probability is based upon the theorem of total probabilities, it is evident that it must agree with that theorem.

We cannot without proof say the same of the theorem of compound probabilities, which asserts that if p is the probability that x belongs to (a, b), and π the probability that y belongs to (α, β) and there is no relation between x and y, the probability P that the two-dimensional number (x, y) belongs to the region

$$a \leqq x \leqq b, \qquad \alpha \leqq y \leqq \beta,$$

is expressed by $p\pi$.

It is a surprising fact that nowhere do we find the demonstration of this theorem, one which is evidently fundamental and without which the theory of errors and the kinetic theory of gases would come to nothing.

We propose to give here the missing demonstration.

[1] Professori Broggi is a native of Italy, received his doctorate at Göttingen (1907) and has long been professor of mathematics at the University of La Plata in Buenos Aires. His dissertation was entitled *Die Axiome der Wahrscheinlichkeitsrechnung* and his work *Matematica Attuariale* (Milano, 1906) was issued in French form in 1907, and in German in 1911.—Editor.

2. Suppose a and α to be constant, and put

$$b = a + h, \qquad \beta = \alpha + k; \qquad p = p(a, h), \qquad \pi = \pi(\alpha, k); \qquad P = P(a, \alpha; h, k).$$

It is evident that

$$p \geqq P, \qquad \pi \geqq P,$$

and, when $h_2 > h_1 > 0$, $k_2 > k_1 > 0$,

$$p(a, h_2) - p(a, h_1) \geqq P(a, \alpha; h_2, k) - P(a, \alpha; h_1, k) \geqq 0$$

$$\pi(\alpha, k_2) - \pi(\alpha, k_1) \geqq P(a, \alpha; h, k_2) - P(a, \alpha; h, k_1) \geqq 0.$$

P cannot be an increasing function of h or of k unless p and π are increasing functions of their arguments. If p and π are increasing functions of h and of k, there is only one value of h or of k corresponding to a given value of p or of π, respectively. Hence in

$$P = P(a, \alpha; h, k)$$

we can express h and k through p and π and have

$$P = F(p, \pi).$$

If

$$\pi_1 = \pi(\alpha, k_1), \qquad \pi_2 = \pi - \pi_1 \qquad (k_1 \leqq k)$$

we have also, from the theorem of total probabilities

$$P = F(p, \pi_1) + F(p, \pi_2)$$

and therefore[1]

$$P = p\pi.$$

3. Let us now suppose that the inequality

$$p(a, h_1) < p(a, h_2) \qquad (h_1 < h_2)$$

holds true only for $h \leqq h_1$, and that two values h_1 and h_2 exist, such that

$$p(a, h_1) = p(a, h_2).$$

Then for every h belonging to the interval (h_1, h_2) we must have

$$p(a, h) = p(a, h_1).$$

$$0 = p(a, h) - p(a, h_1) \geqq P(a, \alpha; h, k) - P(a, \alpha; h_1, k) \geqq 0$$

$$P(a, \alpha; h, k) = P(a, \alpha; h_1, k) = p(a, h)\pi(\alpha, k).$$

The theorem must still hold true when we have identically

$$p(a, h) = 0$$

[1] Broggi, "Il teorema della probabilità composta," in *Rendiconti del Circolo Matematico di Palermo*, Vol. 28 (1909), 245–247.

for $h \leq h_1$. In this case

$$p \geqq P \geqq 0, \qquad P = p\pi = 0.$$

4. We can summarize the results here obtained by saying that the theorem of compound probabilities holds true when p and π are increasing functions of their arguments, or increase in certain intervals and are constant elsewhere. But this condition is always realized, as we cannot have

$$p(a, h_1) = p(a, h_2), \qquad \pi(\alpha, k_1) = \pi(\alpha, k_2)$$

unless the functions p and π are constant in the intervals (h_1, h_2) and (k_1, k_2) respectively.

BITS OF HISTORY ABOUT TWO COMMON MATHEMATICAL TERMS.

By G. A. Miller, University of Illinois.

In 1841 A. L. Cauchy defined the term *indicator* (indicateur) corresponding to a given modulus n as the exponent to which a positive integer m relatively prime to n belongs mod n, and gave tables for the determination of the maximum indicator I corresponding to a given modulus n, where n may be replaced successively by a series of positive integers. For instance,

$n = 2$	3	4	5	6	7	8	9	10	11	12	13
$I = 1$	2	2	4	2	6	2	6	4	10	2	12

About four years later E. Prouhet published a note in volume 5 of the *Nouvelles Annales de Mathêmatiques*, page 75, in which he defined the term indicator of n as the number $\phi(n)$ of the positive integers less than n and prime to n. He added incorrectly that this term had not been employed previously in mathematics. It seems somewhat singular that this later definition came into common use, especially since Cauchy was a more noted mathematician than Prouhet and this definition was due to ignorance on the part of its author.

In the language of group theory Cauchy's definition of indicator is equivalent to that of the order of the totitives of n, while that of Prouhet is equivalent to that of the order of the group formed by these totitives mod n. The I of n is therefore always a divisor of $\varphi(n)$ and a necessary and sufficient condition that I of n is equal to $\phi(n)$ is that n has a primitive root, or that the group of totitives of n is cyclic. In general, the value of I is equivalent to the order of the largest cyclic subgroup contained in this group.

As an instance where a later definition of a common term had an entirely different fate we may refer to the definition of *simple group* found on page 65, volume 20, *Proceedings of the London Mathematical Society*. According to this definition a group is called *simple* when it is both cyclic and its order is a power of a prime number, while the common definition of simple group as a group which

does not involve any invariant subgroup besides the identity had been used by C. Jordan about twenty years earlier.

No one else seems to have adopted this later definition of simple group. Its implicit use in L. E. Dickson's *History of the Theory of Numbers,* volume 1, page 131, is doubtless due to an oversight. Curiously H. W. L. Tanner continued the use of his unfortunate definition in an article published about seven years later in volume 27 of the same journal, page 331.

The mathematical literature contains numerous definitions due to a lack of knowledge on the part of their authors. The different fates of the two cited above may serve to illustrate how difficult it is to deal with some of these definitions. The fact that they are potential sources of error, especially when they appear in periodicals which are frequently used, makes it undesirable to ignore even those which are not adopted by others.

QUESTIONS AND DISCUSSIONS.

Edited by W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

36. A number of Discussions have been published in this department relating to cubic and biquadratic equations (cf. Vol. XXXV, p. 29, pp. 268–269 and 343–347; Vol. XXIV, pp. 136–137 and 436–439; Vol. XXIII, pp. 314–315). Below are given a number of questions sent in by Professor Harris Hancock of the University of Cincinnati which relate to the cubic and biquadratic and might, perhaps, more properly be proposed as problems were it not for the advantage to be gained, if possible, by treating them all, or at least several of them, in one discussion.

1. For what values of n can $\cos 2\pi/n$ be expressed in the form $(a + \sqrt{b})/c$ where a, b and c are integers?

2. Write the biquadratic in the form $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$. Show that its reducing cubic may be expressed by means of a determinant of the third order which when expanded is

$$4y^3 - g_2y - g_3 = 0,$$

where $g_2 = ae - 4bd + 3c^2$, and

$$g_3 = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = ace - ad^2 - eb^2 - c^3 + 2bcd.$$

3. For the same biquadratic show that

$$a^3(x_0 + x_1 - x_2 - x_3)(x_0 + x_2 - x_1 - x_3)(x_0 + x_3 - x_1 - x_2) = 32(3abc - a^2d - 2b^3)$$

without making any use of symmetric functions, where $x_0, \cdots, x_3$ are the roots of the biquadratic equation.

4. If x_0, x_1, x_2 are the roots of a cubic and D its discriminant, show that x_1 is a rational function of x_0 and $\sqrt{D}$. Derive a much simpler relation than that given in Serret, *Cours d'Algèbre Supérieure,* 5th ed., No. 511.

5. If x_0, x_1, x_2, x_3 are the roots of a biquadratic, D its discriminant, e_1, e_2, e_3 the roots of the reducing cubic, show that x_1 is a rational function of x_0, e_1, e_2, e_3 and consequently also of x_0, e_1, and $\sqrt{D}$.

6. If the biquadratic

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0 \tag{1}$$

has a double root, show that the reducing cubic (cf., for example, Burnside and Panton's *Theory of Equations*)

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0 \tag{2}$$

has a root in common with the cubic

$$8t^3 + 12Ht^2 + G^2 = 0, \qquad (G \neq 0); \tag{3}$$

and conversely, if (2) and (3) have a common root, then (1) has a double root.

REPLY BY HARRIS HANCOCK, University of Cincinnati.

The following reply deals with parts 2–6 of Question 36.

2. With Descartes write the biquadratic $f(x) = ax^4 + 4bx^3 + 6cx^2 + 4dx + e$ in the form $(px^2 + 2qx + r)(p'x^2 + 2q'x + r')$, where

$$pp' = a, \qquad pq' + qp' = 2b, \qquad pr' + rp' + 4qq' = 6c, \qquad qr' + rq' = 2d, \qquad rr' = e.$$

The third of these expressions becomes $pr' + rp' = 2c - 4s$, if we put $qq' = c + s$. Note that the determinant

$$\begin{vmatrix} 2pp', & pq' + qp', & pr' + rp' \\ qp' + q'p, & 2qq', & qr' + rq' \\ rp' + r'p, & rq' + r'q, & 2rr' \end{vmatrix}$$

is the product

$$\begin{vmatrix} p, & p', & 0 \\ q, & q', & 0 \\ r, & r', & 0 \end{vmatrix} \begin{vmatrix} p', & p, & 0 \\ q', & q, & 0 \\ r', & r, & 0 \end{vmatrix} = 0.$$

Substituting the above values in the first determinant, we have

$$\begin{vmatrix} a, & b, & c - 2s \\ b, & c + s, & d \\ c - 2s, & d, & e \end{vmatrix} = 0;$$

or,

$$4s^3 - g_2 s - g_3 = 0,$$

where

$$g_2 = ae - 4bd + 3c^2,$$

and

$$g_3 = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}.$$

This form of the reducing cubic is well known. It is very important in the Theory of Elliptic Integrals and should be given in every text-book on the subject.

3. Let x_0, x_1, x_2, x_3 be the roots of the above quartic. It follows that

$$\frac{f(x) - f(x_0)}{x - x_0} = a(x - x_1)(x - x_2)(x - x_3).$$

Writing

$$x = -x_0 - \frac{2b}{a} = \tfrac{1}{2}(x_1 + x_2 + x_3 - x_0),$$

this expression becomes

$$\frac{f\left(-x_0 - \dfrac{2b}{a}\right) - f(x_0)}{2x_0 + \dfrac{2b}{a}} = \frac{a}{8}P,$$

where

$$P = (x_0 + x_1 - x_2 - x_3)(x_0 + x_2 - x_1 - x_3)(x_0 + x_3 - x_1 - x_2).$$

Note that P remains unchanged when we interchange x_0 with x_1, or with x_2, or with x_3. It follows

that the equation of the third degree

$$\frac{f\left(-x-\frac{2b}{a}\right)-f(x)}{ax+b}=\tfrac{1}{4}P,$$

holds for four values of x, and consequently for every value of x. Hence, writing $x = 0$, it is seen that

$$\frac{8}{a^3}(3abc - a^2d - 2b^3) = \tfrac{1}{4}P.$$

In a similar manner it may be proved that

$$f'(x_0) + f'(x_1) + f'(x_2) + f'(x_3) = P.$$

4. Write the cubic in the form

$$K(x) = x^3 - p_1x^2 + p_2x + p_3 \equiv (x - x_0)(x - x_1)(x - x_2) = 0,$$
$$K'(x_0) = (x_0 - x_1)(x_0 - x_2),$$
$$\sqrt{D} = (x_1 - x_0)(x_2 - x_0)(x_2 - x_1).$$

It follows that

$$x_2 - x_1 = \frac{\sqrt{D}}{K'(x_0)};$$

and since $x_2 + x_1 = p_1 - x_0$, we have

$$2x_2 = p_1 - x_0 + \frac{\sqrt{D}}{K'(x_0)}, \qquad 2x_1 = p_1 - x_0 - \frac{\sqrt{D}}{K'(x_0)}.$$

Similarly[1] it is seen that

$$2x_2 = p_1 - x_1 - \frac{\sqrt{D}}{K'(x_1)}, \qquad 2x_0 = p_1 - x_1 + \frac{\sqrt{D}}{K'(x_1)};$$

$$2x_0 = p_1 - x_2 - \frac{\sqrt{D}}{K'(x_2)}, \qquad 2x_1 = p_1 - x_2 + \frac{\sqrt{D}}{K'(x_2)}.$$

5. In Burnside and Panton, *Theory of Equations*, fourth edition, p. 124, it is shown that the roots of the above biquadratic may be written in the form

$$ax_0 + b = \sqrt{e_1} + \sqrt{e_2} + \sqrt{e_3} = z_0, \qquad ax_1 + b = \sqrt{e_1} - \sqrt{e_2} - \sqrt{e_3} = z_1,$$
$$ax_2 + b = \sqrt{e_2} - \sqrt{e_1} - \sqrt{e_3} = z_2, \qquad ax_3 + b = \sqrt{e_3} - \sqrt{e_1} - \sqrt{e_2} = z_3,$$

where e_1, e_2, e_3 are the roots of Euler's equation

$$t^3 + 3Ht^2 + \left(3H^2 - \frac{a^2I}{4}\right)t - \frac{G^2}{4} = 0.$$

Those that are acquainted with the theory of the Galois realms of rationality will observe that if the algebraic number x_0 is adjoined to the realm R of the coefficients, and if the norm of the resulting realm $R(x_0)$ is taken, a realm, say $G = R(x_0)\cdot R(x_1)\cdot R(x_2)\cdot R(x_3) = R(x_0, x_1, x_2, x_3) = R(x_0, x_1, x_2)$ is formed, which is of degree 4!. This realm contains all rational functions of $\sqrt{e_1}$, $\sqrt{e_2}$, $\sqrt{e_3}$ and consequently also all rational functions of e_1, e_2, e_3. Hence G contains as a divisor the realm A, where A is the norm of the conjugate realms $R(e_1)$, $R(e_2)$, $R(e_3)$, a realm which is of degree 3!. It follows that the realm G taken with respect to the realm A is of the fourth degree, and that x_0 is a primitive quantity in G/A. Proofs of these statements are given below without using the Galois theory.

The following results may be established at once: Write $\sigma = z_0 = \sqrt{e_1} + \sqrt{e_2} + \sqrt{e_3}$ and $\tau = z_1 = \sqrt{e_1} - \sqrt{e_2} - \sqrt{e_3}$. It is seen that $\sigma^2 = e_1 + e_2 + e_3 + 2(\sqrt{e_1}\sqrt{e_2} + \sqrt{e_1}\sqrt{e_3} + \sqrt{e_2}\sqrt{e_3})$ with corresponding values for σ^3, σ^4, $\sigma^3\tau$, etc. Noting that $e_1 + e_2 + e_3 = -3H$, and $\sqrt{e_1}\sqrt{e_2}\sqrt{e_3} = -G/2$, we have the *rational relations*

$$\tau^4 + \tau^3\sigma + 3H(\tau^2 + \tau\sigma + 6e_1) + G(3\sigma + 5\tau) - 2e_1(\tau^2 + 2\tau\sigma - 6e_1) = 0,$$
$$\sigma^4 + \sigma^3\tau + 3H(\sigma^2 + \sigma\tau + 6e_1) + G(3\tau + 5\sigma) - 2e_1(\sigma^2 + 2\sigma\tau - 6e_1) = 0.$$

[1] These formulæ exhibit the rational expressions desired unless $D = 0$; if $D = 0$, they may be modified without difficulty.

If in this pair of relations we put $\tau = z_2$, then $\sigma = z_3$. Change e_1 to e_2, then $\tau = z_1$ when $\sigma = z_3$, and further $\tau = z_2$ when $\sigma = z_0$; change e_1 to e_3, then $\tau = z_1$ when $\sigma = z_2$ and $\tau = z_0$ when $\sigma = z_3$. The above relations may be written in the form

$$-\sigma = \frac{\tau^4 + \tau^2(3H - 2e_1) + 5G\tau + 18He_1 + 12e_1^2}{\tau^3 + \tau(3H - 4e_1) + 3G},$$

or

$$\frac{-\sigma - \tau}{2} = \frac{e_1\tau^2 + G\tau + 9He_1 + 6e_1^2}{\tau^3 + \tau(3H - 4e_1) + 3G} = A_0\tau^3 + A_1\tau^2 + A_2\tau + A_3.$$

To determine the coefficients A_0, A_1, A_2, A_3, interchange σ and τ, and we have

$$\frac{-\tau - \sigma}{2} = A_0\sigma^3 + A_1\sigma^2 + A_2\sigma + A_3.$$

Through subtraction, it follows that $0 = A_0(\sigma^3 - \tau^3) + A_1(\sigma^2 - \tau^2) + A_2(\sigma - \tau)$, or

$$0 = A_0(\sigma^2 + \sigma\tau + \tau^2) + A_1(\sigma + \tau) + A_2.$$

Writing for σ and τ their values it is seen that $0 = A_0(-3H + 2e_1 + 2\sqrt{e_2}\sqrt{e_3}) + A_1 2\sqrt{e_1} + A_2$. From this it follows that

$$A_1e_1 = A_0\frac{G}{2} \qquad \text{and} \qquad A_2 = A_0(3H - 2e_1).$$

These values substituted in the expression above give

$$-\sqrt{e_1} = A_0\left(\sigma^3 + \frac{G}{2e_1}\sigma^2 + 3H\sigma - 2e_1\sigma\right) + A_3.$$

Writing for σ its value, it follows, if we put $D = 8e_1^3 + 12He_1^2 + G^2$, that

$$A_0 = \frac{2e_1^2}{D}, \qquad A_1 = \frac{Ge_1}{D}, \qquad A_2 = \frac{6He_1^2 - 4e_1^3}{D}, \qquad A_3 = \frac{3Ge_1(2e_1 + H)}{D}.$$

It is thus seen that

$$\frac{-\sigma - \tau}{2} = \frac{e_1\tau^2 + G\tau + 9He_1 + 6e_1^2}{\tau^3 + \tau(3H - 4e_1) + 3G}$$
$$= \frac{2e_1^2\tau^3 + Ge_1\tau^2 + (6He_1^2 - 4e_1^3)\tau + 3Ge_1(2e_1 + H)}{8e_1^3 + 12He_1^2 + G^2}.$$

We thus have the pair of *integral* relations

$$0 = \sigma(8e_1^3 + 12He_1^2 + G^2) + 4e_1^2\tau^3 + 2Ge_1\tau^2 + (24He_1^2 + G^2)\tau + 6Ge_1(2e_1 + H),$$
$$0 = \tau(8e_1^3 + 12He_1^2 + G^2) + 4e_1^2\sigma^3 + 2Ge_1\sigma^2 + (24He_1^2 + G^2)\sigma + 6Ge_1(2e_1 + H),$$

with five other pairs of relations which correspond to those which have already been indicated for the rational relations.

The completion of the reply to Part 5 will be postponed until after the consideration of Part 6.

6. Writing $x = 2t$, $\lambda = 12H^2 - a^2I$, the reducing cubic is $F(x) = x^3 + 6Hx^2 + \lambda x - 2G^2$. Further put $F_1(x) = x^3 + 3Hx^2 + G^2$. By the algorithm of the greatest common divisor, the remainder, say $F_2(x)$ which is had by dividing $F(x)$ by $F_1(x)$ is $F_2(x) = 3Hx^2 + \lambda x - 3G^2$. The remainder derived by dividing $F_1(x)$ by $F_2(x)$ is, say, $F_3(x) = Ax + B$, where

$$A = \frac{9HG^2 - 9\lambda H^2 + \lambda^2}{9H^2} = \frac{9HG^2 + \lambda(3H^2 - a^2I)}{9H^2},$$

and

$$B = \frac{12G^2H^2 - \lambda G^2}{3H^2} = \frac{G^2a^2I}{3H^2}.$$

The remainder obtained on dividing $F_2(x)$ by $F_3(x)$, expressed also in terms of $F(x)$ and $F_1(x)$ is

$$9H^2[3G^2A^2 + (\lambda A - 3BH)B] = F_1(x)[9H^2A^2 + (3HAx + \lambda A - 3BH)(3Hx + 18H^2 - \lambda)] + F(x)\Phi(x),$$

where as seen below it is unnecessary to write down the value of the polynomial $\Phi(x)$. The term

on the left-hand side of this equation is $G^2a^6(27J^2 - I^3) = G^2a^6\Delta$, where Δ is the discriminant of the biquadratic under discussion. This proves the theorem proposed. It is of course understood that G^2 is *different* from zero.

To continue the discussion further under the preceding Question, write the right-hand side of the above expression in the form

$$F_1(x)[C_0x^2 + C_1x + C_2] + F(x)\Phi(x).$$

It may be shown that

$$\begin{aligned}
C_0 &= 9HG^2 - 9\lambda H^2 + \lambda^2 = 9HG^2 + (3H^2 - a^2I)\lambda = (a^2I)^2 - 6H^2a^2I - 9Ha^3J,\\
C_1 &= 3[3a^3Ja^2I + 2G^2a^2I - H(a^2I)^2 - 18H^2a^3J]\\
&= 3[6H^2G^2 + \lambda(G^2 + 2H\lambda - 18H^3)]\\
&= 3[a^3Ja^2I + H(a^2I)^2 - 8H^3a^2I - 18H^2a^3J],\\
C_2 &= 9(G^4 - \lambda^2H^2) + \lambda^3 + 12G^2\lambda H - 18HG^2a^2I\\
&= a^6\Delta + 6G^2(3a^3J + Ha^2I).
\end{aligned}$$

Writing $x = 2e_1$ in the expression above and noting that $F(2e_1) = 0$, it is seen that

$$\frac{1}{8e_1^3 + 12He_1^2 + G^2} = \frac{4C_0e_1^2 + 2C_1e_1 + C_2}{G^2a^6\Delta}.$$

Returning to the integral relations in the preceding Question it is seen that

$$\begin{aligned}
G^2a^6\Delta\tau &= B_0\sigma^3 + B_1\sigma^2 + B_2\sigma + B_3,\\
G^2a^6\Delta\sigma &= B_0\tau^3 + B_1\tau^2 + B_2\tau + B_3,
\end{aligned}$$

where the B's are integral functions of the second degree in e_1. The coefficients of these latter functions are integral in H, G^2 and I. Writing for σ, τ the values z_0, z_1, z_2, z_3 and interchanging e_1 with e_2 and e_3, we have here six pairs of integral relations corresponding to the rational relations indicated in 5.

The last part of 5 follows from the first part by use of 4.

NOTE. To Question 36, proposed by Professor Hancock, his own reply, printed above, is the only one as yet received. A reply to part 1 of the question will be welcome.—EDITOR.

DISCUSSIONS.

We present this month two discussions, both related more closely to secondary than to collegiate mathematics, but of considerable interest also to teachers of the latter. Professor Johnson, in advocating, with much reason, the early use of the complex number and its representation by a point of the plane, has made several comments on which there may well be differences of opinion. It is hoped that expressions on such questions will be forthcoming from readers.

Professor McClenon states without argument certain modes of approaching the teaching of logarithms. Many, perhaps most, teachers will be able to express a preference regarding these diverse methods, without hesitation.

I. THE COMPLEX QUANTITY IN ELEMENTARY ALGEBRA.

BY W. WOOLSEY JOHNSON, U. S. Naval Academy.

It is the purpose of this note to advocate an earlier introduction, to the student of elementary algebra, of the geometrical interpretation of the complex quantity $a + bi$ (Argand's diagram) than seems usually to be made.

Immediately after the solution of quadratic equations, let it be pointed out that the case of so-called impossible or imaginary roots is but a fresh instance of the occurrence of apparent "anomalies" in the results of mathematical operations, of which several are already familiar to the student. These anomalies always deny the existence of an answer to certain questions in the field of quantity to which they apply. Thus the subtraction indicated by 7–10 denotes the impossibility of an answer in certain number-systems; but in others, notably in connection with the location of a point on a line (a zero point and a unit of length being assumed), such a result presents no difficulty, but gives rise to the so-called negative quantities, of which the very name embodies the earlier view *denying* their existence.

So too we speak of a number (that is, an *integer*) as one that can or cannot be divided by a given smaller number. But when dealing with a straight line all lengths are divisible, and accordingly our number system is enlarged to include fractions.

Again, the indirect operation of extracting a root gave rise to an extension of the field of number, and the name *surd* attached to such a result as $\sqrt{2}$, for instance, reminds us that the attempt to represent it in the field of rational quantity, at which we had hitherto arrived, resulted in a "reductio ad absurdum" proving its impossibility. Yet, by a geometrical construction, the square root of two can be determined with as much exactness as that of an "exact" square.

The new anomaly at which we are supposing the student to have now arrived merits, to be sure, a name implying the impossibility of an answer far more than does, for instance, the negative quantity, which, as a result, may indicate a loss where a gain was expected, or a position on the left where one on the right was assumed. The impossible or imaginary quantity of quadratic algebra on the other hand generally indicates the non-existence of that which was sought, as when we seek to divide a number into parts which shall have a given product. On generalizing the constants, we here find occasion for an interesting class of questions which concern limits of existence and non-existence, taking the form of problems of maxima and minima.

But the question remains, can we find a field of quantity in which questions resulting in quadratic equations can always find an answer?

Having denoted the roots of $x^2 + 1 = 0$ (the simplest quadratic which presents the anomaly) by $\pm i$, we find the roots of every quadratic to be expressible in one of the forms $a \pm b$ or $a \pm bi$, in which a and b, being quantities of the kind hitherto known, may be called "scalars," because measurable on a scale or straight line. But, whereas in the first case the parts may be merged into single scalars, they are in the second case *heterogeneous* and cannot be so merged. Space of two dimensions now presents us the opportunity to form another scale, just as it does when we seek to represent a function of a scalar associated with its independent variable. But whereas, in that case, we were free to choose the direction of the new scale, consideration of i as a turning factor fixes the direction of the scale of imaginaries. It is, in my opinion, not necessary to say that we have

invented the new numbers $a + bi$ and annexed them to complete our field of number; rather let us say that, as in the former case of the straight line for scalars, we have found, ready to hand, a field of quantity of the desired kind, namely, one[1] in which the roots of a quadratic are always possible.

The mode of performing geometrically the fundamental operations of algebra with complex quantities; the mode in which the idea of absolute magnitude is attached to them; the construction of the roots of unity with their introduction to the many-valued function; and later their application to the power series; can hardly fail to interest and stimulate the student.[2]

Finally we observe that, at this point, the student will have an intelligent appreciation of the meaning of the fact, which may now be stated (without proof, for which he is, of course, not yet prepared), that every equation of the nth degree has n roots; no new anomaly appearing in the solution of those of higher degree, although new incommensurables do occur; thus the system of complex quantities is a complete set reproducing itself through all the operations of algebra, direct and indirect.

II. On the Teaching of Logarithms.

By R. B. McClenon, Grinnell College.

How should the subject of logarithms be presented to the beginner? The usual method has been, to start with the laws of exponents, define logarithms as powers of the base, and thus infer the fundamental laws of operation. This approach, however, is extremely likely to prove confusing to the student, because it does not bring out clearly enough the functional relation involved; and this disadvantage is accentuated by the impossibility of giving a satisfactory discussion of the meaning of an irrational exponent until a later stage of the mathematics course.

Two other methods have been proposed as a substitute for this one. One suggestion is, to use the historical approach, as employed by Napier in his invention of logarithms.[3] This would begin with the fundamental idea of the relation between an arithmetic and a geometric progression, the notion of "base" being at first entirely ignored. With the help of graphical methods, and an abundance of concrete problems chosen from physics, economics, and finance, it is said to be possible to bring out clearly and vividly the relation between the exponential and logarithmic functions. The other suggestion is, to abandon frankly all attempt to *explain* logarithms at the outset, but rather to focus the attention upon the actual use of logarithmic tables, until the mechanical rules for computa-

[1] A former student of Kronecker once told me that Kronecker would never admit the existence of a complex quantity. He said "there was not any complex variable $x + iy$. There were two variables, x and y." Whatever we say to this verbal contention, we shall continue to say that it takes two quantities (*i. e.*, scalars) to determine the position of a point in a given plane.

[2] In this connection the writer may refer to a construction of the roots of the quadratic with complex coefficients given by him to the New York Mathematical Society in 1893 and, in abstract, in the *Bulletin*, Vol. 2, p. 171.

[3] See, for example, Cajori, "History of the Exponential and Logarithmic Concepts," Am. Math. Monthly, Vol. 20 (1913), pp. 5–14.

tion have become thoroughly familiar. Then, it is urged, the student will find it easier to appreciate the theoretical discussion which comes later, as this will merely take the line of explaining the reasons for facts of whose truth he is already convinced.

Both of these suggestions seem worthy of being tried out, and they may very likely have been already. If so, it is to be wished that some testimony as to the results might be presented in this department of the MONTHLY; as well as any other suggestions that have been found helpful in the teaching of this subject.

The last plan mentioned above, the postponement of the theoretical part of the work until the practical use of logarithms has been mastered, would prove most satisfactory if this practical part of the work could be introduced into the secondary school mathematics course, as would certainly seem desirable on many grounds. When this is done, the problem of the college teacher will be much simplified, and, what is of course of incomparably greater importance, the secondary school student will have the added knowledge and power that come from acquaintance with the practical use of so valuable a mathematical tool.

RECENT PUBLICATIONS.

REVIEWS.

HINDU SCIENCE.

Hindu Achievements in Exact Science, A Study in the History of Scientific Development. By B. K. SARKAR. Longmans, New York, 1918. 13 + 82 pp. Price, $1.00.

"The Astronomical Observatories of Jai Singh." By G. R. KAYE. *Archeological Survey of India,* New Imperial Series, Vol. 40. Calcutta, 1918, 8 + 154 pages + 26 plates and frontispiece photograph of Jai Singh. Price 15 rupees.

These two works are illustrative of two well-defined tendencies which have been existent for fifty years or more, in the discussion of Hindu science. The work of Sarkar is that of the enthusiast for all things Hindu; here we have the acceptance of practically all claims for the Hindu origin of different scientific ideas, with the general denial of foreign influence. In the work of Kaye we have the insistence upon the absolute dominance of foreign ideas in Hindu science, with the practical denial of any indigenous contribution.

Unfortunately *Hindu Achievements* is written by one who does not treat historical material critically; thus we have works of real value and works of no value cited as authorities, with a preponderance in favor of obsolete and even worthless works relating in one way and another to Hindu science. On the other hand Mr. Kaye is familiar with the modern authorities in the field of the history of science. This work of Jai Singh is a work of real merit but even it exhibits the tendency to depreciate the value of Hindu contributions which has characterized much of Mr. Kaye's work.

The denial of Hindu originality in science is paralleled by the denial on the part of certain writers of Babylonian, Egyptian, and even Arabic originality. This denial rests upon a fundamental misconception of the nature of science and scientific progress; it is of the same naïve nature as the common German view (before the war) that all discoveries in science of any value were German in origin.

To deny to Babylon, to Egypt, and to India, their part in the development of science and scientific thinking is to defy the testimony of the ancients,[1] supported by the discoveries of the modern authorities.[2] The efforts which have been made to ascribe to Greek influence the science of Egypt, of later Babylon, of India, and that of the Arabs do not add to the glory that was Greece. How could the Babylonians of the golden age of Greece or the Hindus, a little later, have taken over the developments of Greek astronomy? This would only have been possible if they had arrived at a state of development in astronomy which would have enabled them properly to estimate and appreciate the work which was to be absorbed. People in one stage of civilization do not borrow the science of another people in a higher state of civilization. There has never been any question concerning the nature and origin of such feeble beginnings of science as are found among the American Indians. With the Hindus and the Babylonians and the civilization of Europe in the time of Alexander the Great and up to 600 A. D., the problem is entirely different. Here we have peoples who had reached approximately the same stage of development. The admission that the Greek astronomy immediately affected the astronomical theories of India carries with it the implication that this science had attained somewhat the same level in India as in Greece. Without serious questioning we may assume that a fundamental part of the science of Babylon and Egypt and India, developed during the times which we think of as Greek, was indigenous science. Nor do we thereby detract from the real greatness of Greece. The Hellenic civilization remains as an integral and vital part of all civilization, and not as something apart.

Recently Mr. Kaye has published an article[3] in which the denial of Hindu originality has reached the ultimate limit, including not only all Hindu astronomy, but even the sine function, the Hindu numerals, the value for π of 3.1416 and the solutions of indeterminate equations. To justify this total repudiation it is necessary here to postulate the nature of the contents of Greek works which are lost. This is a new method in history, and not one to be commended. Not a scintilla of real historical evidence is advanced to support the contentions. Those who lack training in the weighing of historical evidence may take a new *theory* for *proof*, but there is a profound difference.

[1] Hipparchus and Ptolemy, in Ptolemy's *Almagest*; Theon of Smyrna, and Diodorus Siculus; Herodotus, II, 109; Berossus, as given in fragments; Clemens of Alexandria; Pliny, *Hist. nat.*, VI, 122 and VII, 193; and numerous others.

[2] Heath, *Aristarchos of Samos* (Oxford, 1913); Berthelot, *Les origines de l'alchimie* (Paris, 1885), Chap. III; Boll, *Sphaera* (Leipzig, 1903), *p.* 461; Cumont, *Astrology and Religion among the Greeks and Romans*; Kugler, *Die babylonische Mondrechnung* (Freiburg, 1900), pp. 50–51, 200–211; Epping, *Astronomisches aus Babylon* (Freiburg, 1889), pp. 183–190.

[3] G. R. Kaye, "Influence grecque dans le développement des mathématiques hindoues," *Scientia*, Vol. 25 (Jan., 1919), pp. 1–14.

The work of Mr. Sarkar is in no sense scholarly. As a popular exposition it may be of interest particularly to Hindus, but it is distinctly unfortunate that so much is claimed which is easy to refute. This only results in the non-acceptance even of those claims which are justified.

Kaye's work shows the truly magnificent equipment of the observatories established by Jai Singh (1686–1743). Only among the Arabs would we find any buildings which could be compared with the Delhi, Jaipur, Ujjain, Benares, and Mathura structures. At Delhi quadrants, in the plane of the equator, with a radius of 49.5 feet, are graduated to minutes; the high masonry gnomon, 7.5 feet in width and 113.5 feet in length, rises above these quadrants to a height of about 68 feet, and its shadow on the quadrants gives the apparent solar time. Two complementary concave hemispheres of masonry work, diameter 27 feet, 5 inches, formed a part of the equipment; so also did two circular buildings with inside diameter of 49.1 feet, with 30 openings each 6° in width. Other masonry instruments and metal astrolabes and armillary spheres, in large part of Persian and Arabic workmanship, still survive from the instruments which were used in these observatories. A set of astronomical tables was prepared by Jai Singh, dedicated to the Emperor Muhammad Shah, being called Zij Muhammad Shahi; these tables follow closely the methods of Ulugh Beg. However, "polar longitudes" are given, as in the Surya Siddhanta, and it is more reasonable to suppose a Hindu source than a Moslem one, as Kaye does.

The wonderful ruins of these observatories of Jai Singh are a tribute to Hindu science, revealing an unusual power of appreciation of pure science whether or not they bear direct testimony to other positive achievements of the Hindus themselves.

LOUIS C. KARPINSKI.

The Italian Universities and their Opportunities for Foreign Students. By KENNETH MCKENZIE. Roma, Tipografia Nazionale Bertero, 1919. 8vo. 16 pp.

This timely and very interesting pamphlet was written by Dr. McKenzie, professor of romance languages in the University of Illinois, but now Director of the Italian branch of the American University Union in Europe. The headings of the thirteen sections are as follows—'The Universities of Italy: their history; The universities of Italy: their present condition; Primary and secondary schools; Organization of the universities;—["The university year extends nominally from October 16 to July 31; but the instruction begins in November or even later, and ends by June 15. Examinations are as a rule given in two periods, beginning respectively October 16 and June 16"]—Registration and fees; University courses and degrees;—["The University courses are of four years in the case of the Faculties of Letters and Philosophy, Sciences and Law"]—Foreign students;—["Foreigners who have received a degree from a University of approved standing, and who can prove that they have studied all the subjects required for one of the regular degrees in Italy, are not obliged to study in residence or to take the examinations in the separate subjects, but may present themselves directly for the final examination for the degree which they desire. This does not apply, however, to the new degree mentioned below. The character of the final examination is . . . written dissertation and oral discussions."]—Requirements for the regular degrees; The new doctor's degree;—["On October 28, 1917, a new doctorate degree was established, equivalent in grade and in the amount of work required to the old four years' degrees, but not demanding their rigid selection of courses. This new degree is not professional in the sense of giving legal rights in connection with the practice of the professions in Italy, but is intended to encourage scientific attainment for the purpose of advancing knowledge and increasing

the personal culture of the individual; for this reason it is sometimes referred to as a scientific degree (as opposed to professional). It was destined at first exclusively for foreigners, who would not wish to conform to the requirements of the professional curricula. In March, 1918, however, it was opened to Italians as well."]—Degrees for foreigners;—["At present no degree of higher grade than the regular laurea or the new doctorate is conferred in Italy; and from what precedes, it is obvious that the requirements for these degrees are less in amount than the requirements for the doctorate in Graduate Schools of good standing in America. It must be borne in mind, however, that for Americans the successful following of a course of university study in Italy implies residence abroad and mastery of the Italian language"]—The language; Choosing a University; —["Concise descriptions of all the Universities and other higher institutions in Italy, including libraries and learned societies, with lists of the professors and other officials, will be found in the *Annuario degli Istituti Scientifici Italiani*, compiled by Professor Silvio Pivano for the Associazione Italiana per l'Intesa Intellettuale fra i Paesi Alleati ed Amici (Rome, 1918; price 10 lire)."]—Special provisions 1919–1920.

Number Stories of Long Ago. By D. E. SMITH. Boston, Ginn, 1919. 12mo. 7 + 136 pp. + 8 plates in color. Price 48 cents.

Extract from "Preface number two for the grown-ups, and not worth reading"—

". . . This book is intended for supplementary reading in the elementary school. It is written in nontechnical language, and the effort has been made to connect with the history enough of the human element to make it more interesting than any mere recital of facts. With it there is also joined something of the history of writing materials, this being connected naturally with the story of our numbers. Chapters I–VIII can easily be read aloud, and the Question Box at the end of each chapter can be used as a basis for conversation or for written work.

"The facts stated in the book are as nearly exact as the circumstances permit. It is not to be expected, however, that changes in the form of various numerals will be considered. Such changes are of no moment in a work of this nature and do not contradict the statement that the historical facts are presented with substantial accuracy.

"It is the author's hope that this little series of human incidents will create a new interest not merely in the study of arithmetic but in the story of the development of our civilization."

The solutions of the problems in chapters IX and X of *Number Stories* have been given by D. E. SMITH in a fourteen page pamphlet entitled: *Number Puzzles before the Log Fire*.

Essentials of Algebra and Geometry. By F. M. MORGAN. New York, Association Press, 1919. 12mo. 58 pp.

This is the second of the series, published under the direction of Professor J. W. Young for the National War Work Council of Young Men's Christian Associations, to which reference has been made already in the MONTHLY (March, 1919). It contains six lessons preparatory to the study of trigonometry. The requirement has been cut down to a minimum and it is intended that everything given shall be of importance. There are numerous "Exercises," "Oral review exercises" and "Review exercises."

NOTES.

The *Harvard Alumni Bulletin* for April 24, 1919, contains the report by a Faculty Committee (of which Professor G. D. BIRKHOFF is a member) on General Final Examinations for Degrees. Such examinations are not to be used in the Divisions of Mathematics and the Natural Sciences.

The twenty-page *List of Members, 1919*, of the Indian Mathematical Society contains 195 names. Of these one name is that of a "Patron," one of an "Honorary Member," and nine of "Life Members." Of the 185 "Ordinary Members" only one resides outside of India.

The thoroughly unsound foundation for J. M. Child's contention that "Isaac Barrow was the first inventor of the infinitesimal calculus" was set forth clearly by Professor Cajori in this MONTHLY for January, 1919. It seems unfortunate, therefore, that the contention is whole-heartedly accepted by such an influential journal as *Science Progress.* (Cf. the issue for April, 1919.)

Announcements of the Cambridge University Press: Sir Thomas L. Heath's *Euclid in Greek* (Book 1) will include the Greek text of Euclid's first book together with an English translation and notes; the author holding the view that neither Euclid nor Greek can ever be more than apparently in abeyance—New and completely revised editions of Love's *Elasticity* and Lamb's *Infinitesimal Calculus* are in the press—Professor J. H. Jeans has written a new work entitled *Problems of Cosmogony and Stellar Dynamics,* which it is hoped to publish this autumn.

Revista de la Sociedad Matemática Española ceased publication with the completion of año 6 in July, 1917. In January, 1919, a new mathematical periodical, entitled *Revista matematica Hispano-Americana,* appeared at Madrid under the editorship of J. Rey Pastor. It is the Sociedad's official organ. At this writing five numbers have been received in America. Their contents include: "Notas sobre la teoria de grupos"[1] by G. A. Miller (pages 148–152), a sketch and fine portrait of Don Eduardo Torroja, 1847–1918, (pages 1–13), and a portrait and sketch of Hadamard and his scientific work (pages 65–80, 105–112). In the latter is given a list of 135 papers published by Hadamard 1888–1914.

We have also received Tome 1, no. 1 of Professor Z. G. de Galdeano's new periodical: *Supplemento a la Revista matemática Hispano-Americana, boletin, de critica, pedagogia, historia y bibliografia* (32 pages). Professor Galdeano has also written or edited four other periodicals: (1) *El Progreso Matematico,* 7 vols., 1891–95, 1899–1900; (2) *Boletin de critica, ensenanza, y bibliografia,* 2 nos., 1907–08; (3) *Supplemento a la revista de la Sociedod Matematica Española,* 3 nos., 1917; (4) *El progreso scientifico, revista semestral,* 1 no., July 1918 (See this MONTHLY, March, 1919, p. 118).

L. Huxley has recently published his *Life and Letters of Sir Joseph Dalton Hooker,*[2] the distinguished botanist and traveller who died in 1911 at the age of 94. The first volume contains interesting particulars (pages 538–546) of the famous x Club[3] of which there were 240 meetings from the first in 1864 to the last in 1892. No additions were ever made to the original membership of nine: Hooker, T. H. Huxley, John Tyndall, Herbert Spencer, Edward Frankland, John Lubbock, George Busk, and the mathematicians William Spottiswoode and T. A. Hirst. The first break in the circle was caused by the death of Spottiswoode in 1883.

[1] In part, a translation of "Remarks on the bearing of the theory of groups" in *Tôhoku Mathematical Journal,* vol. 11, 73–78, December, 1914.

[2] New York, Appleton, 1918.

[3] See also *Life and Letters of T. H. Huxley,* Vol. 1, 368 *seq.*; *Sketches from the Life of Edward Frankland,* page 148 *seq.*; and Huxley's reminiscences of John Tyndall in the *Nineteenth Century,* Jan., 1894.

There was only one meeting of the Club after Hirst's death in February, 1892.

The second volume of the *Life* contains the following extracts (pages 126, 336) culled from Hooker's letters:

(1) To *Charles Darwin*, August 5, 1871: "I have been reading W. Thomson's[1] address [at the Edinburgh meeting of the British Association for the Advancement of Science in 1871], and am anxious to hear your opinion of it. What a belly-full it is, and how Scotchy! It seems to be very able indeed, and what a good notion it gives of the gigantic achievements of mathematicians and physicists—it really makes one giddy to read of them. I do not think that Huxley will thank him for his reference to him as a positive unbeliever in spontaneous generation—these mathematicians do not seem to me to distinguish between un-belief and a-belief—I know no other name for the state of mind that is traduced under the term scepticism. I had no idea before that pure mathematics had achieved such wonders in practical science, and I wonder how far Thomson's statements will be contested. The total absence of any allusion to Tyndall's labors, even when comets are his theme, seems strange to me."

(2) To *Rev. J. D. La Touche*, Dec. 24, 1893: "What you say of A, B, and C does not surprise me. They are 'ne plus ultra' mathematicians, have not a conception of biological science, and in fact are only *half intellects* (I suppose I deserve to be burned), but so it is, that I have often found such men to be impervious to reasoning out of their own circle, in matters of natural science. With biologists, who have to found everything, beyond pure observation, on circumstantial evidence, the case is quite different. For hundreds of biologists who are good mathematicians, you will not find ten vice versa."

ARTICLES IN CURRENT PERIODICALS.

Bulletin of the American Mathematical Society, volume 25, no. 7, April, 1919: "Mathematics in war perspective" by L. E. Dickson, 289–311 [Presidential address delivered before the American Mathematical Society, December 27, 1918]; "A partial isomorph of trigonometry" by E. T. Bell, 311–321; "The trains for the 36 groupless triad systems on 15 elements" by Louise D. Cummings, 321–324; "A theorem on areas" by T. Hayashi, 324–325; "Concerning the definition of a simple continuous arc" by G. H. Hallett, Jr., 325–326; "The transformation of a regular group into its conjoint" by J. E. McAtee, 326–329; "Notes" and "New publications," 329–336.—No. 8, May: "The life and services of Maxime Bôcher" by W. F. Osgood, 337–350; "A theorem on linear point sets" by H. Blumberg, 350–353; "A general form of Green's Theorem," 353-357; "Rotating cylinders and rectilinear vortices" by H. Bateman, 358–374; Review by A. Dresden of Shaw's *Lectures on Philosophy of Mathematics* (Chicago, 1918), 374–377; Review by R. D. Carmichael of Macrobert's *Functions of a complex variable* (London, 1917), 377–378; Review by A. Emch of Montessus de Ballore's *Lecons sur les fonctions elliptiques en vue de leurs applications* (Paris, 1917), 378-379; "Notes" and "New Publications," 379-384—No. 9, June: "The March meeting of the American Mathematical Society at Chicago," 385-392; "The April meeting of the San Francisco section, 393–397; "On a certain generation of rational circular and isotropic curves" by Arnold Emch, 397-404; "The self-dual plane rational quintic" by L. E. Wear, 405–408; "Groups containing a relatively large number of operators of order two" by G. A. Miller, 408-413; "The derivative of a functional" by P. J. Daniell, 414–416; Review by L. W. Dowling of *Scritti matematici offerti ad Enrico D'Ovidio*, etc. (Torino, 1918), 417-422; "Shorter Notices," 422-424; "Notes," 424-429; "New Publications," 429-432.

Journal of Accountancy, New York, volume 27, 1919, January: "Practical interpolation" by A. S. Little, 48–60—April: "Rapid calculation of compound interest processes" by F. C. Belser, 241-248; "Mathematics of credit extension" by F. Thulin, 259-267.

Journal of Educational Psychology, Hershey, Pa., volume 10, no. 2, February, 1919: "English and mathematical abilities of a group of college students" by E. C. Tolman, 95-103. [Based on observations made on an introductory class in psychology at the University of California made up of sophomores, juniors and seniors (men and women)].

[1] William Thomson (1824–1907), afterwards Lord Kelvin, was the son of James Thomson who became professor of mathematics at the University of Glasgow where William matriculated when he was a little more than ten years of age, and was afterwards to be for fifty-three years a "professor of Natural Philosophy."

Journal of Philosophy, Psychology and Scientific Methods, volume 15, no. 10, May 9, 1918: "Doctrinal functions" by C. J. Keyser, 262–266. [Synopsis in *Mind*: Starts from Russell's notion of a propositional function which is neither true nor false until values have been assigned to its variables, and points out that values may always be given which make nonsense of the function and hence are to be called *inadmissible constants*. Admissible constants are divided into *verifiers* and *falsifiers*: the former "*satisfy* it and are called the *values* of its *variables*. Thus the values of a given function are the true propositions that are derivable from it by replacing its variables by admissible constants." Applying these distinctions to "the postulational method of founding and constructing mathematical sciences," it appears that as "any postulate-system contains one or more undefined terms and at least one of them denotes an *element*," which gives it the appearance of having a definite subject-matter, the system will require interpretation. In this process "the *rôle* of the undefined terms is the *rôle* of variables"; hence "a postulate system is not a system of propositions, as it is commonly said to be, but it is a system of propositional functions." It should be called therefore 'a doctrinal function,' and it is shown that "the number of values of any doctrinal function is equal to any given transfinite cardinal number. It is a corollary that "Hilbert's *Foundations of Geometry* is not a geometry at all, nor is it any other doctrine; it is a doctrinal function having an infinitude of values, some of them geometric, some of them algebraic, some of them neither one nor the other."]—No. 11, May 23: "The definition of infinity" by R. H. Dotterer, 294–301. [Synopsis in *Mind*: "Criticises the new 'infinity' of Dedekind and Cantor as doubly ambiguous. (1) Two infinite series do not stand *merely* in a one-to-one correspondence, but also in an infinity of others. But *unless* they do, the new definitions of 'similarity' and 'equality' break down. (2) The 'new infinite' is only the old in disguise, for that also involved an inexhaustible series and the possibility of a one-to-one correspondence (or of any other). Hence it retains also the old difficulties. Only they are hidden away in its definition. Thus the infinite series of cardinal numbers cannot be called a 'system' or a 'totality' without assuming a realised infinite. If 'totality' is defined to mean *determinable* only, the 'new infinite' cannot claim existence any more than the old. Hence it does not help in the solution of any of the problems of philosophy or theology."]—No. 14, July 4: "Mechanism and causality in physics" by M. P. Cohen, 365–386—No. 21, October 10: "A list of articles, mostly book reviews, contributed by Charles S. Peirce to *The Nation* to which is appended some additions to the bibliography of his published writings in this *Journal*, December 21, 1916," 578–584.

Journal of the Indian Mathematical Society, volume 11, February, 1919: Frontispiece, Indian Mathematical Society, 2d Conference, January, 1919, Bombay; "Second Conference," 3–30; "Introductory Notes to differential geometry" by E. H. Neville, 30–33; "On the factorization of two large numbers" by N. B. Mitra, 34; Questions and Solutions, 37–40.

Mathematical Gazette, volume 9, March, 1919: "The Annual Meeting of the Mathematical Association—Report of the Council for 1918," 309–311; "Mathematics and the pivotal industries" by W. P. Milne, 312–316; "The teaching of geometry to first-year pupils" by B. A. Howard, 317–321; "Cubic graphs of the form $y = ax^3 + bx^2 + cx + d$" by A. Lodge, 322–326; Review by H. Hilton of Frost's *An Elementary Treatise on Curve Tracing* (London, 1918); [Quotation: "The reviser's hope that 'this edition will be found to be comparatively free from inaccuracies' unfortunately proves fallacious on a casual glance at the diagrams. Some errors, e. g. plate V., fig. 24, and VI. 17, are doubtless due to the printer. But III. 27; IV. 25, 26, 29 represent curves cut by a line in points whose number is greater than the curve's degree; while IV. 28 is also rather rough and ready. There is a tendency to draw cusps as though the tangents thereat were distinct, to exaggerate the rate of approach of a curve to its asymptote, and to misplace inflexions and their tangents. This is the more to be regretted, in that these are just the mistakes against which a beginner most needs warning"]. Review by W. P. Milne of Carey's *Infinitesimal Calculus* (London, 1917–18), 327–328; Review of Whitehead's *The Organization of Thought* (London, 1917), and Hermite's *Oeuvres*, tome 4, (Paris, 1917); "The Teacher's Library, "Section II, by W. P. Milne, i–v.

Memoirs of the National Academy of Sciences, Washington, D. C., Volume 14, second memoir, 1919: "Complete classification of the triad systems on fifteen elements" by H. S. White, F. N. Cole, and Louise D. Cummings, 1–89: [Part 1: "Triad systems on 15 elements whose group is of order higher than unity" by H. S. White, 5–25; Part 2: "Trains for triad systems on 15 elements whose group is of order higher than unity" by L. D. Cummings, 27–68 (216 figures); Part 3: "Groupless triad systems on 15 elements" by H. S. White and L. D. Cummings, 69–72; Part 4: "Structure as defined by interlacings, heads, and semiheads; a complete census of triad systems in fifteen elements" by F. N. Cole, 73–80; Part 5: "Sequences and indices for all groupless triad systems on fifteen elements" by L. D. Cummings, 81–89.]

MONIST, volume 29, no. 2, April, 1919: "Pandiagonal magics of orders 6 and 10 with minimal numbers" by C. Planck, 307–316; "The publication of *Isis*" by G. Sarton, 318–319—[Reprinted from *Science*, February 14, 1919]; Review of *The philosophy of Mr. B*rtr*nd R*ss*ll, with an appendix of leading passages from certain other works.* Edited by P. E. B. Jourdain (London, Allen and Unwin, 1918), 319–320. [Quotation: ". . . No amount of quotation (finite by reason of editorial control) could exhaust the transfinite number of good things in this amusing volume. For the proof that this is not mere hyperbole I must refer the reader to the chapter on 'The Hierarchy of Jokes.' And when one comes delightedly across such gems of delicate irony as the logical analysis of Mr. Chesterton's method of disguising platitudes as paradoxes (p. 41) one can only hope that Mr. Jourdain will discover among the papers of the late Mr. R*ss*ll more, and still more, pin-prickings of popular bombastics."]

NIEUW ARCHIEF VOOR WISKUNDE, Amsterdam, series 2, volume 12, no. 4, 1918: "Nieuwe behandeling van een bekend kansvraagstuk" by F. Schuh, 361–373; "Remarque sur la singularité que présente la courbe nodale d'une surface développable quand l'arête de rebroussement possède une tangente double" by W. A. Versluys, 374–378; "Onderzoek van de functie die voor $R(z) > 0$ wordt voorgesteld door de reeks

$$\sum_{n=1}^{\infty} \left(\frac{n}{a}\right) \frac{e^{-nz}}{n^s}$$

by N. G. W. H. Beeger, 379–392; "Ueber grözsten gemeinsamen Teiler und kleinstes gemeinsames Vielfaches" by I. G. van der Corput, 393–404; "Constructie van de assen der ellips, die de centrale projectie is van een cirkel" by F. Schuh, 405–406; "Beweging van een materieel punt op den bodem eener draaiende vaas onder den invloed der zwaartekracht" by L. E. J. Brouwer, 407–419; "Elementair bewijs der intgebreide wederkeerigheidswet van Legendre" by A. L. Bartelds and F. Schuh, 420–438; "Addenda en corrigenda over de grondslagen der wiskunde" by L. E. J. Brouwer, 439–445; "On cylindrical functions" by L. Crijns, 446–449; "Over het differentieeren van bepaalde integralen" by J. Wolff, 450–453; "Eenige stellingen over limieten van integralen" by H. Bremekamp, 454–460; "Notes on some differential equations in physics and differential geometry" by T. Hayashi, 461–468; Review (in Dutch), 468–480.

NOTES AND QUERIES, London, 12th series, volume 5, February, 1919: "EULER ON THE END OF THE WORLD.—Euler the mathematician (1707–83) is said to have predicted that the end of the world would take place in a certain year. It is likely that some reference to the statements would be found in the letters of Catherine II (1729–96) to F. M. Grimm (1723–1807).

"Could a reader give some precise information?

R. G. H."

NOUVELLES ANNALES DE MATHEMATIQUES, volume 77, December, 1918: "Sur les troisième et quatrième centres de courbure des courbes de Cesàro" by R. Goormaghtigh, 441–445; "Sur un problème concernant des groupes de points sur l'hyperbole équilatère" by R. Goormaghtigh, 445–448; "Sur les nombres complexes de deuxième et de troisème espèce" by L. G. Du Pasquier, 448–461; "Propriété de certaines courbes et surfaces enveloppes" by M. Weill, 462–464; "Sur les cercles bitangents a la parabole" by J. Bouchary, 464–471; "Sur les courbes définies par une relation entre les distances de chacune de leurs tangentes à des points fixes" by M. d'Ocagne, 471–472.

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, series 2, volume 17, part 5, March, 1919: "Professor Olaus Henrici" by M. J. M. Hill, xlii–xlix. [Quotations: "Olaus Magnus Friedrich Erdmann Henrici was born in the year 1840, in Meldorf, on the West Coast of Holstein, where his father held a post in the Danish Civil Service. His father had studied Science and Engineering at the Gewerbschule in Berlin, which afterwards became the College at Charlottenburg, and he was one of the three who prepared the plans for the canal from Kiel to the mouth of the Elbe . . . Henrici will be remembered chiefly as a great teacher. During his years of struggle [1865–1869; he went to London in 1865] when he was obliged to spend much time in giving lessons in elementary mathematics to school boys, he had learned to probe the working of the minds of his pupils. This was of great value to him in his subsequent career as a teacher and a writer, for he acquired the faculty of expressing himself with great clearness in both capacities. He published nothing until he felt completely satisfied as to its form. But for this characteristic more of his methods and ideas would have been preserved. I understand that he has left a large amount of manuscript, and it is much to be hoped that some one will be found to go through it with extreme care."]

School Science and Mathematics, volume 19, no. 5, May, 1919: "Construction work in solid geometry" by E. W. Schreiber, 407–413; "Historical notes in the mathematical text-books" by G. A. Miller, 414–416; "Graphical algebra" by F. E. Nipher, 417–420; "History of mathematics at University of California" by O. Schmiedel, 462; [Quotation: "Three courses are given the first term of the present session; a largely attended course on the History of Mathematics and Physics, a course on The Teaching of Mathematics in Secondary Schools, and a most interesting seminar course on The History of Fundamental Concepts of the Calculus and of Fluxions."] "Useful benefits from study of mathematical history" by O. Schmiedel, 463; Problems and solutions, 468–473.

Science Progress, volume 13, No. 3, January, 1919: Mr. Oscar S. Adams has called attention to the fact that, in quoting from R. Ross's article on isosceles trigonometry in our April issue, two misprints occur: (1) *for* "$(\text{bas } s\theta + i \text{ bas } \theta)^n = \text{bas } sn\theta + i \text{ bas } n\theta$" *read*

$$\text{"}(\text{bas } s\theta + i \text{ bas } \theta)^n = 2^{n-1}(\text{bas } sn\theta + i \text{ bas } n\theta)\text{"};$$

(2) *for* "between θ and π" *read* "between 0 and π." The first of these slips was due to an error in the original. Mr. Adams remarks further: "De Moivre's theorem can of course apply only to complex numbers that lie on the unit circle with the origin as center or, in other words, to those with modulus equal to unity. bas $s\theta + i$ bas θ lies on the circle with radius equal to two"—No. 4, April: "The discovery of the Calculus" by R. R., 634–635; "Mathematics in an encyclopædia" by P. E. B. Jourdain, 648–651; [First paragraph: "A discussion of the nature and extent of articles in a proposed mathematical dictionary has lately excited a good deal of interest in America. The Mathematical Association of America has appointed a committee to investigate the subject, and Dr. G. A. Miller (Amer. Math. Monthly, 1918, *25*, 383–7; cf. 428) has given an example of a proposed article dealing with the theory of groups. This article has the conventional characteristics of an article for a dictionary. This is not meant to imply that the article is not a good one of its kind: it is, in fact, a very thorough and exhaustive treatment of the things about which it means to give information. But it seems to me that what it and most other dictionary articles discuss is exactly what nobody really wants who is not bent on acquiring merely the sort of knowledge that is required by examinations. Nearly all of Dr. Miller's specimen article is devoted to definitions of "groups" in the general technical meaning of the word, and particular qualities of groups. It is no uncommon thing to read in an examination paper such a question as: 'Define the terms *isomorphism* and *transitivity*;' but such information, at least in this form, is not required outside the examination room. Surely an intelligent being who consults an article on a science in an encyclopædia does so from a wish to know what such-and-such a department of knowledge is about and what has been done in it, whether he does so for cultivating his own mind, or as a preliminary for cultivating the minds of others, or for his own work of discovery. He does not want simply to fill his memory with what certain words mean in the technical language of the present: his aim would be to get a firm grasp of the principles underlying that particular subject, and to find out why certain notions were so important as to be fixed by a name—such as "isomorphism," for instance. It is not the formal *definitions* that we must seek in the first place; it is those more or less vague ideas which have lost part of their vagueness so as to become apparently definable. Of course we can never be quite certain that the definitions we may arrive at in some science at a particular time are quite free from vagueness: a mathematician who is interested in the principles of his subject can find many instances of 'definitions' which were seriously given only a few years ago and which can now be seen not to define at all. The theory that definitions should form the subject-matter of articles in a dictionary or encyclopædia, and a great part of the subject matter of text-books, is a theory held by those schoolmasters who think that examinations are the goal of knowledge; and we must always remember that professors are only a better class of schoolmasters." In the remainder of the article Mr. Jourdain deals chiefly with the mathematical articles in the *Encyclopædia Britannica*.] "The general theory of relativity and Einstein's theory of gravitation" by G. W. Tunzelman, 652–657—[Reprinted in *Scientific American Supplement*, May 31, 1919, Vol 87, pp. 346-347]; Reviews by P. E. B. Jourdain of Bateman's *Differential equations* (London, 1918), Veblen and Young's *Projective Geometry* (Boston, 1910–1918), Vollenhoven's *De Wijsbegeerte der Wiskunde van Thëistisch Standpunt* (Amsterdam, 1918) and Tuttle's *The Theory of Measurements* (Philadelphia, 1916), 666–672; Review by A. E. Heath of Jourdain's *The Philosophy of Mr. B*rtr*nd R*ss*ll with an appendix of leading passages from certain other works* (London. 1918), 670–671; Review by H. S. J[ones] of Kaye's *The Astronomical Observatories of Jai Singh*

(Calcutta, 1918), 672–674; Review by E. H. B. of C. V. Raman's *On the mechanical theory of the vibrations of bowed strings and of musical instruments of the violin family, with experimental verification of the results*, Part I (Calcutta, 1918), 683. [Quotations: "Helmholtz on an experimental basis was able to construct a partial theory of the bowed string. F. Krigar Menzel and A. Raps photographed, upon a revolving drum carrying a film, various points of bowed strings so as to exhibit their displacement-time graphs. E. H. Barton and his pupils took simultaneous photographs of the behaviour of the strings and either bridge, belly or air of a monochord or violin. But in none of the foregoing cases was a direct mechanical theory of the string, bridge, etc., attempted. This is now done by C. V. Raman.

"The equations of motion of the string are written and solved for the case of a periodic force applied transversely by the bow at any given position. The equations of motion of the bridge are next written and dealt with. The *modus operandi* of the bow is afterwards examined and a simplified kinematical theory of the bowed string is based upon it. . . . Another interesting subject here treated is that of the effect of the *mute*, which, by loading the bridge, enfeebles and veils the tone of the instrument. . . . This work contains twenty-eight figures in the text and twenty-six excellent full-page photographic reproductions, and well deserves the careful attention of those interested in such a notable contribution to an important subject."]

SCIENTIA, volume 84, April, 1919: "Synthèses et visions d'histoire de la science" by A. Mieli, 310–313; [Discussion of topic with special reference to three works: S. Günther, *Geschichte der Naturwissenschaften*, 2 little volumes, Leipzig, Reclam; F. Dannemann, *Die Naturwissenschaften in ihrer Entwickelung und ihrem Zusammenhang*, 4 volumes, 1910–1913; and W. Libby, *An Introduction to the History of Science*, New York, 1918.] Review by G. Scorza of Borel's *Introduction géométrique à quelques théories physiques* (Paris, 1914) and Borel's *Leçons sur les fonctions monogènes uniformes d'une variable complexe* (Paris, 1917), 314–316; Review by R. Marcolongo of Whittaker's *A Treatise on the analytical dynamics of particles and rigid bodies* (Cambridge, 1917).

SCIENTIFIC AMERICAN SUPPLEMENT, volume 87, April 5, 1919: "A scientific coin problem. How some mysterious numerical effects can be produced" by T. L. De Land, 220.

TEXAS MATHEMATICS TEACHER'S BULLETIN, volume 4, no. 3, May, 1919: "Summer school courses in mathematics," 3–6; "Number systems and numerals" by Phillis Henry, 7–10; "Saving time and gaining efficiency in teaching trigonometry" by Goldie P. Horton, 11–14; "A few facts about mathematics" by P. M. Batchelder, 15–17; "Bibliography of the teaching of mathematics" by C. D. Rice, 18–20; "Methods *vs.* results and mathematical induction" by A. A. Bennett, 21–28; "A trigonometric solution of the irreducible case of the cubic equation with applications" by C. E. Brand, 29–34; "Checking the solution of an equation" and "Familiar tricks based on literal arithmetic," 35–38—[Reprinted from W. F. White's *A Scrap Book of Elementary Mathematics*]; "The Pentagram during 1918–1919" by Goldie P. Horton, 39–40; "The straight edge," 41.

TÔHOKU MATHEMATICAL JOURNAL, volume 15, nos. 1–2, March, 1919: "On a determinate system of non-independent trials" by M. Watanabe, 1–134; "Sur l'équation $x_1^5 + x_2^5 + x_3^5 + \cdots + x_k^5 = 0$" by A. Filippov, 135–145; "On closed curves described by a spherical pendulum" by A. Emch, 146–165; "On a transformation theorem relating to spheroidal harmonics" by B. Datta, 166–171; "On certain mean curves defined by the series of orthogonal functions" by K. Ogura, 172–180; "A remark on the dynamical system with two degrees of freedom" by K. Ogura, 181–183; "A contribution to the question of linear dependence in linear integral equations" by L. J. Rouse, 184–216.

ZEITSCHRIFT FUR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT ALLER SCHULGATTUNGEN, volume 49, no. 12, December, 1918: "Die Entwicklung des Satzes vom vollständigen Vierseit und Viereck zu einem Grundpfeiler des natürlichen Systems der Geometrie" by W. Dieck, 329–341; "Über die Anwendung beweglicher Figuren im geometrischen Unterricht" by Elisabeth K. Staiger, 341–348; Reviews of Neuendorff's *Praktische Mathematik*, 2. Teil (Leipzig, 1918), and of Seyffarth's *Elementarmathematik zum Gebrauch an Lehrerbildungsanstalten*, 1. Teil: Allgemeine Arithmetik und Algebra, 5. Auflage (Dresden–Blasewitz, 1917).

AMERICAN DOCTORAL DISSERTATIONS.

GILLIE A. LAREW, *Necessary conditions in the problems of Mayer in the calculus of variations*. Pp. 1–22. [Reprinted from *Transactions of the American Mathematical Society*, 1919.] (Chicago, 1916.)

A. S. MERRILL, *An isoperimetric problem with variable end-points*. Pp. 60–78. [Reprinted from *American Journal of Mathematics*, 1919.] (Chicago, 1916.)

PAULINE SPERRY, *Properties of a certain projectively defined two-parameter family of curves on a general surface.* Pp. 213-214. [Reprinted from *American Journal of Mathematics*, 1918.] (Chicago, 1916.)

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

CLUB ACTIVITIES.

BARNARD COLLEGE MATHEMATICS CLUB, Columbia University, New York, N. Y. [*1918*, 226–227].

The officers of the club for 1918–19 are: Honorary president, Professor Edward Kasner; president, Janet Meneely '19; vice-president, Veronica Jentz '20; secretary, Georgia Schaaf '19; treasurer, Evelyn Baldwin; program committee: Gretchen Torek '19, Catherine Piersall '20, Marian Haskell '21.

The following are programs of meetings for 1918–19:

November 19, 1918: "Theory of ballistics" by Professor Edward Kasner.
December 16: "Descartes" by Alice Johnson '21 and Marian Haskell '21; "Leibniz" by Helen Clarke '20; "Newton" by Mimosa Pfaltz '19.
January 14, 1919: "Occupations open to women taking mathematics" by Ellen Leut '18.
February 18: "Cycloids" by George W. Mullins, instructor in mathematics.
March 11: "Physics and mathematics" by Elaine Kennard '20; "Relation of curves to physics" by Catherine Piersall '29.
April 16: "Accounting" by Professor Wildman, New York University.
May: Business meeting.

DENISON MATHEMATICS CLUB, Denison University, Granville, Ohio. [*1918*, 403–04].

The officers of the club for the year 1918–19 are: President, Edgar King '19; vice-president, Esther Weaver '20; secretary-treasurer, Fern Whitney '21.

The programs for the first part of the year 1919 were as follows:

January 21, 1919: "The significance of mathematics" (excerpts from a paper by Professor Cassius J. Keyser[1]) by Professor Anna B. Peckham.
February 4: "Parallel coördinates" by Professor Forbes B. Wiley.
February 18: "Mathematics in artillery" by Lieut. R. Thompson '21.
March 4: "Fundamentals in mathematics" by Professor Charles C. Morris, Ohio State University.
March 18: "Parallel coördinates" (continued) by Professor Wiley.
April 15: "Centrodes" by Esther Weaver '20; "Magic squares" by Alva Shumaker '22.
April 29: "Applications of mathematics to astronomy" by Professor Paul Biefeld, department of astronomy, Denison University; election of officers.

[1] "The Human Significance of Mathematics," *Science*, new ser., Vol. 42, No. 1089 (Nov. 12, 1915), pp. 663–680.

May 13: First annual banquet.

The club has about ten dollars in its treasury which it plans to use in purchasing portraits of noted mathematicians for the walls of the mathematics rooms.

The officers elected for the year 1919–20 are: President, Richard Howe '20; vice-president, Esther Weaver '20; secretary, Evangeline Nellis '22; treasurer, Alva Shumaker '22.

UNDERGRADUATE MATHEMATICS CLUB, University of Illinois, Urbana, Ill. [*1918*, 404–405].

Owing to the war conditions no meetings of the club were held during 1918–19 until February 5, 1919, when the following officers were elected: President, Irene Doyle '19; vice-president, Laura Stoll '19; secretary-treasurer, Beulah Prante '19.

Membership is limited to juniors and seniors majoring in mathematics and to those students who have attended three consecutive meetings. The average attendance is about twelve. The following programs have been given since February fifth.

February 19, 1919: "The training value of mathematics" by Beulah Prante '19.
March 5: "Mathematical puzzles" by Margaret Walker '19.
March 19: "Women mathematicians" by Irene Doyle '19.
April 9: "Maxima and minima" by Agnes Nelson '19.

THE WHITE MATHEMATICS CLUB AT THE UNIVERSITY OF KENTUCKY, Lexington, Ky. [*1918*, 90, 451–345].

The following is the list of programs for the first part of the year 1919.

January 13, 1919: "Geometric determination of π" by Professor Elijah L. Rees.
January 20: "Analytic determination of π" by Professor Harold H. Downing.
January 27: "Transcendence of e" by Professor Harold H. Downing.
February 10: "Euclid's Elements" by H. P. Pettit Gr.
February 26: "Transcendence of π" by Guy W. Smith, instructor in mathematics.
March 3: "A non-euclidean world" by W. W. Elliott Gr.; "Discussion of problems" by H. P. Pettit Gr.
March 10: "Introduction to linear associative algebras" by Dr. Guy W. Smith; "Solution of problems" by Professor Downing and H. P. Pettit Gr.
March 17: "History of the parallel postulate" [1] by Mary E. Beall '19; "Discussion of problems" by H. P. Pettit Gr.
March 26: "Some applications of vector analysis to the theory of ruled surfaces" by Professor Rees.
March 31: "Logical significance of definitions, axioms and postulates" by Edna Berkele '19.
April 7: "The net of Moebius" by Professor Paul P. Boyd.

[1] The subjects discussed by undergraduates are selected from J. W. Young's *Fundamental Concepts of Algebra and Geometry* (Macmillan, 1911).

April 14: "Consistency, independence and categoricalness of a set of assumptions" by Frank W. Tuttle '20.

April 21: "A problem in partial differential equations" by Professor J. Morton Davis.

April 28: "Class, correspondence, number" by Frances Kimbrough '20.

CLUB NOTES.

A number of inquiries have been received asking if the editor of this department can suggest some mathematical games for the use of clubs at social meetings. Can not some of our readers supply us with descriptions of mathematical games or plays which they have found successful? The editor gives below several forms of entertainment, more or less mathematical, which have been used in social meetings of the club of which he is a member.[1]

Mathematical contests. Two captains choose sides as in an old-fashioned spelling match. The captain of each side designates a contestant to represent his side and the two contestants stand with their backs to the blackboard[2] while a member of the faculty places on the board back of one contestant a mathematical exercise and an assistant copies the same exercise on the board back of the other contestant. At a given signal the contestants turn and attack the exercise. The one first obtaining the correct answer is declared the winner and must defend his position against the next contestant designated by the captain of the losing side. The losing contestant drops out of the game. The play is continued until all of the contestants on one team have been defeated and the side whose representative is still undefeated is declared the winner of the contest. By continuing the play among the undefeated members of the winning team the champion of the club can be determined.

Where membership in the club is limited to upperclassmen, there is considerable latitude in the choice of exercises, but the use of simple exercises in arithmetic and participation in the contest by all members of the faculty as well as by all of the students are likely to produce a merrier time.

Telling fortunes by mathematics. The Ancient Science of Numbers, by Luo Clement (New York, Rogers Bros., 1909) is a book which is not likely to be taken very seriously by many of our readers; but it contains a system, taken seriously by its author, for telling something of the "apparently mysterious operation of the Science of Numbers, and its effect upon the health, happiness and success of the individual" which, in the hands of a clever operator, who has

[1] The *Teachers College Record,* New York, for November, 1912 (Vol. 13, pp. 385–495), is devoted to articles on "Number Games and Number Rhymes" by D. E. Smith, C. W. Hunt, F. J. Flynn, C. C. Eaton, R. K. Atwell, and F. B. Selkin. Chapter 3 (pp. 413–422) on "Rithmomachia, the great medieval number game" by D. E. Smith and C. C. Eaton, is reprinted, with a few modifications from this MONTHLY, April, 1911; a bibliography is given on page 495.

References to the literature of "Nim, a Game with a Complete Mathematical Theory" were given, in this MONTHLY, in Topics for Club Programs—No. 7, March, 1918. "Probabilities in the game of 'Shooting Craps'" will be discussed in the next number of the MONTHLY.—Editor-in-Chief.

[2] The meetings attended by the editor were held in a private home and a portable blackboard was brought in for the evening.

imagination and the ability to act a part gravely, may be made the basis of a pleasant evening's entertainment. The writer has seen it used on more than one occasion, but most successfully as the main feature of a Halloween party with a member of the faculty taking the part of the fortune-teller. The house was decorated with jack-o'-lanterns, paper witches, black cats, etc., and an appropriate booth in a corner of a room was provided for the fortune-teller. The "fortunes" told were based on the "Ancient Science of Numbers" but obviously supplemented with much other real and imaginary information more or less surreptitiously obtained by the amateur revealer of mysteries.

An evening at chess. Chess is essentially a mathematical game, as is evidenced by the fact that the Royal Society Index lists no less than thirty mathematical papers on the "Knight's Move" alone. Mathematical students generally are fond of the game or enjoy learning it when not already players.

For an evening at chess the first requisite is a sufficient number of boards and sets of men so that every member can play. Before the play is begun a member of the club presents a brief sketch of the history of the game and the mathematical problems connected with it. The club is then divided into players and beginners. If convenient, the players may go into one room and the beginners into another. The players are seated at small tables and the play proceeds as in a progressive card game except that the time for play is limited (say to 10 or 15 minutes) and one of the best players, who has been chosen to act as judge and timekeeper, awards all unfinished games to the player who, in his judgment, holds the more advantageous position.

The beginners include all who have not yet learned the game. Two or three experienced players explain the game to them, illustrating by means of simple games based on easy checkmates such as the "fool's mate" and the "scholar's mate." After sufficient instruction has been given, the beginners divide into pairs and spend the evening playing, their instructors remaining in attendance to answer questions or give other assistance.

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel and Otto Dunkel.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

2780. Proposed by ELMER LATSHAW, West Philadelphia, Pa.

A polygon whose sides are a, $2a$, $3a$, $4a$ is inscribed in a circle. Find the radius of the circle.

2781. Proposed by J. L. RILEY, Stephensville, Texas.

Show that the asymptotic lines on a pseudospherical surface are curves of constant torsion.

2782. Proposed by WARREN WEAVER, University of Wisconsin.

A great number N of jackstraws are jumbled up in such a way that any one is as likely to have one direction as another. Show that the probable number that make an angle lying between θ_1 and θ_2 as measured from any given direction is equal to $\dfrac{N(\cos\theta_1 - \cos\theta_2)}{2}$.

2783. Proposed by C. C. BRAMBLE, U. S. Naval Academy.

Two players A and B take turns throwing a single die, A leading. The one first making a score of 3 aces is to be the winner. Find the probability that A will win.

416 (Algebra) [May, 1914]. Proposed by C. E. FLANAGAN, Wheeling, Va.

The sides of a given rectangle are a and b in which a rectangle is to be inscribed one of whose sides is c. Find the other side, using Euler's rule for quartics.

353 (Calculus) [February, 1914]. Proposed by R. P. LOCHNER, Philadelphia, Pa.

The center of a sphere, radius $R = 5$ inches, is $a = 10$ inches above the surface of a sphere, radius $12\frac{1}{2}$ inches. There is a point of light at $b = 1$ inch horizontally from a point $c = 10$ inches vertically above the surface of the first sphere. What is the area of the shadow which the upper sphere casts on the lower one?

287 (Mechanics) [February, 1914]. Proposed by W. H. DRANE, Lebanon, Tenn.

While sitting in an empaled enclosure, I noticed that the spokes of the wheels of passing automobiles, when viewed through the pickets of the fence, appeared to revolve more slowly than they really did, and in some instances even appeared to be revolving in a direction opposite to that in which they were really turning. Explain this optical illusion.

202 (Number Theory) [December, 1913]. Proposed by A. R. SCHWEITZER, Chicago, Ill.

There exists an infinitude of systems of dyads $\{\alpha\beta\}$ in 7, 9, 11, etc., elements such that each system has the following properties: (1) if $\alpha\beta$ is in the set, then $\beta\alpha$ is not in the set; (2) for each dyad $\alpha\beta$ in the set there exists an element ξ such that $\xi\beta$ and $\alpha\xi$ are also in the set. For example, such a system is,

12,	23,	34,	45,	56,	67,	78,	89,	91
13,	24,	35,	46,	57,	68,	79,	81,	92
14,	25,	36,	47,	58,	69,	71,	82,	93
51,	62,	73,	84,	95,	16,	27,	38,	49.

Investigate the existence of

I. A finite set of triads $\{\alpha\beta\gamma\}$ such that (1) if $\alpha\beta\gamma$ is in the set, then $\beta\gamma\alpha$, $\gamma\alpha\beta$ are also in the set but $\beta\alpha\gamma$ is not in the set; (2) for each triad $\alpha\beta\gamma$ in the set there exists an element ξ such that $\xi\beta\gamma$, $\alpha\xi\gamma$, $\alpha\beta\xi$ are also in the set.

II. A finite set of tetrads $\{\alpha\beta\gamma\delta\}$ such that (1) if $\alpha\beta\gamma\delta$ is in the set, then $\beta\gamma\alpha\delta$, $\gamma\alpha\beta\delta$, $\gamma\delta\alpha\beta$ are also in the set but $\beta\alpha\gamma\delta$ is not in the set; (2) for each tetrad $\alpha\beta\gamma\delta$ in the set there exists an element ξ such that $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ are also in the set.

The problem for alternating n-ads for $n > 4$ is obvious.

SOLUTIONS OF PROBLEMS.

2699, 2710 [May, June, 1918; April, 1919]. Proposed by the late R. E. MOORE, University of Wisconsin.

If $a_k^{(r)}$ denotes the kth term of an arithmetic progression of order r, and c_k denotes the kth binomial coefficient in the expansion of $(a - b)^n$ (n being a positive integer), show that

$$s \equiv \sum_{k=1}^{n+1} c_k a_k^{(r)} = 0, \text{ if } n > r.$$

II. Solution by A. Pelletier, Montreal, Canada.

The terms of the arithmetic progression may be expressed in the usual form as follows:

$$a, \quad a + \Delta_1, \quad a + 2\Delta_1 + \Delta_2, \quad a + 3\Delta_1 + 3\Delta_2 + \Delta_3, \quad \cdots.$$

We have to prove that

$$c_0 a + c_1(a + \Delta_1) + c_2(a + 2\Delta_1 + \Delta_2) + c_3(a + 3\Delta_1 + 3\Delta_2 + \Delta_3)$$
$$+ \cdots + c_n\left(a + n\Delta_1 + \frac{n(n-1)}{1\cdot 2}\Delta_2 + \cdots\right) = 0,$$

or that

$$\begin{aligned}&a[c_0 + c_1 + c_2 + c_3 + \cdots + c_n]\\&+ \Delta_1[c_1 + c_2\cdot 2 + c_3\cdot 3 + \cdots + c_n\cdot n]\\&+ \Delta_2\left[c_2 + c_3\cdot 3 + c_4\cdot 6 + \cdots + c_n\frac{n(n-1)}{2!}\right]\\&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\&+ \Delta_k[c_k + c_{k+1}\cdot c_1{}^{k+1} + c_{k+2}\cdot c_2{}^{k+2} + \cdots + c_n\cdot c_{n-k}{}^n] = 0,\end{aligned}$$

where k is used for r, and $c_{n-k}{}^n = \binom{n}{n-k}$. Now, each of the quantities in the brackets is equal to zero; for, the general form of the bracketed expressions is $c_k + c_{k+1}\cdot c_1{}^{k+1} + c_{k+2}\cdot c_2{}^{k+2} + \cdots + c_n{}^n\cdot c_{n-k}$ and this is equal to $c_k[1 - c_1{}^{n-k} + c_2{}^{n-k} - \cdots (-1)^{n-k}\, c_{n-k}{}^{n-k}] = c_k{}^n(1-1)^{n-k} = 0$, for $n > k$.

Also solved by A. M. Harding, E. E. Witmer, and the Proposer.

Note by the Editor: The solution of Mr. Pelletier does not differ greatly from solution I by Mr. E. H. Clarke. Problem 481 (Algebra) [May, 1917], also proposed by Mr. Moore, is really the same as problem 2699 (or 2710), and has been solved by Professor F. H. Safford.

2702 [May, 1918]. Proposed by N. P. PANDYA, Sojitra, India.

A conic of variable eccentricity has a focus and corresponding directrix fixed. The latus rectum cuts a fixed circle in A' and B'. If A be the vertex of the conic, find the locus of the centroid of the triangle $AA'B'$.

Solution by H. L. Olson, Chicago, Illinois.

If the directrix be taken as the y axis and the perpendicular from the focus to the directrix as the x axis, the equation of the conic may be written,

$$\{(1 - e^2)x - k\}^2 + (1 - e^2)y^2 = k^2e^2,$$

where k is the (fixed) perpendicular distance from the focus to the directrix. Let the equation of the fixed circle be $(x - p)^2 + (y - q)^2 = r^2$. The latus rectum, $x = k$, of the conic cuts this circle in the points for which $x = k$, $y = q \pm \sqrt{r^2 - (k - p)^2}$. The coördinates of the vertex A of the conic are $x = k/(1 + e)$, $y = 0$. Hence, the coördinates of the centroid of the triangle $AA'B'$ are

$$x = \tfrac{2}{3}k + \frac{k}{3(1+e)}, \qquad y = \tfrac{2}{3}q,$$

and its locus is the straight line $y = \frac{2}{3}q$.

Also solved by A. M. Harding.

2703 [May, 1918]. Proposed by S. A. COREY, Albia, Iowa.

Let A_1, A_2, A_3, A_4 and $-(A_1 + A_2 + A_3 + A_4)$ be the vector sides of a pentagon, plane or gauche. Let B_1, B_2, B_3, B_4 and $-(B_1 + B_2 + B_3 + B_4)$ be the sides of a second pentagon where

$$B_1 = (c_1 + c_2)A_1 + (c_1 - c_2)A_3, \qquad B_2 = (c_4 + c_3)A_4 + (c_4 - c_3)A_2,$$
$$B_3 = (c_2 + c_1)A_2 + (c_2 - c_1)A_4, \qquad B_4 = (c_3 + c_4)A_3 + (c_3 - c_4)A_1,$$

c_1, c_2, c_3, and c_4 being ordinary scalars.

Then, if a_r and b_s be the lengths of the sides A_r and B_s respectively and if $\cos A_rA_s$ denotes the cosine of the angle included between the sides A_r and A_s and if $\cos B_rB_s$ denotes the cosine of the angle included between the sides B_r and B_s prove that

$$2(c_1c_4 + c_2c_3)(a_1a_4 \cos A_1A_4 + a_2a_3 \cos A_2A_3) = b_1b_2 \cos B_1B_2 + b_3b_4 \cos B_3B_4.$$

Solution by Louis Weisner, Student, College of the City of New York.

From the given equations, we have

$$B_1 \cdot B_2 = (c_1c_4 + c_1c_3 + c_2c_4 + c_2c_3)A_1 \cdot A_4 + (c_1c_4 - c_1c_3 + c_2c_4 - c_2c_3)A_1 \cdot A_2$$
$$+ (c_1c_4 + c_1c_3 - c_2c_4 - c_2c_3)A_3 \cdot A_4 + (c_1c_4 - c_1c_3 - c_2c_4 + c_2c_3)A_3 \cdot A_2$$
$$B_3 \cdot B_4 = (c_2c_3 + c_2c_4 + c_1c_3 + c_1c_4)A_2 \cdot A_3 + (c_2c_3 - c_2c_4 + c_1c_3 - c_1c_4)A_2 \cdot A_1$$
$$+ (c_2c_3 + c_2c_4 - c_1c_3 - c_1c_4)A_4 \cdot A_3 + (c_2c_3 - c_2c_4 - c_1c_3 + c_1c_4)A_4 \cdot A_1.$$

Adding,

$$B_1 \cdot B_2 + B_3 \cdot B_4 = (2c_1c_4 + 2c_2c_3)A_1 \cdot A_4 + (2c_1c_4 + 2c_2c_3)A_2 \cdot A_3 = 2(c_1c_4 + c_2c_3)(A_1 \cdot A_4 + A_2 \cdot A_3).$$

By definition $M \cdot N = mn \cos MN$, where m and n are the lengths of M and N, respectively, and $\cos MN$ is the cosine of the angle between the vectors M and N.

Hence,

$$b_1b_2 \cos B_1B_2 + b_3b_4 \cos B_3B_4 = 2(c_1c_4 + c_2c_3)(a_1a_4 \cos A_1A_4 + a_2a_3 \cos A_2A_3).$$

Also solved by the Proposer.

2704 [May, 1918]. Proposed by WILLIAM HOOVER, Columbus, Ohio.

A particle moves from rest under gravity down the arc of a parabola with the axis vertical and concavity upward; express the time to the vertex in terms of an elliptic integral of the second kind.

Solution by the Proposer.

Taking the y-axis along the axis of the curve, the equation of the parabola is

$$x^2 = 4ay. \tag{1}$$

In the usual notation, the equation of motion of the particle is

$$mv\frac{dv}{ds} = -mg\frac{dy}{ds}.$$

Multiplying by $2ds$ and integrating,

$$v^2 = -2gy + C. \tag{2}$$

When $v = \dot{s} = 0$, let $y = h$; then $C = 2gh$, and (2) is

$$\dot{s}^2 = 2g(h - y). \tag{3}$$

From (1),

$$\dot{s} = \sqrt{\frac{a+y}{y}}\,\dot{y}$$

and (3) becomes

$$dt = -\frac{1}{\sqrt{2g}}\sqrt{\frac{a+y}{y(h-y)}}\,dy. \tag{4}$$

Now change the variable by setting $y = h - hy$; then

$$dt = \frac{1}{\sqrt{2g}}\sqrt{\frac{(a+h) - hy}{y(1-y)}}\,dy. \tag{5}$$

The upper and lower limits in (4) are $y = h$, $y = 0$, and so (5) gives

$$t = \frac{\sqrt{h}}{\sqrt{2g}}\int_0^1 \sqrt{\frac{\frac{a+h}{h} - y}{y - y^2}}\,dy = \sqrt{\frac{2(a+h)}{g}}\,E\left(\sqrt{\frac{h}{a+h}}\right).$$

Also solved by R. C. Colwell and A. M. Harding.

2705 [May, 1918]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Find the area of a loop of the trochoid

$$x = \tfrac{1}{3}a(3\phi - \pi \sin \phi), \qquad y = \tfrac{1}{3}a(3 - \pi \cos \phi).$$

Solution by L. E. Mensenkamp, Freeport, Illinois.

The curve of the type here given is generated by causing a circle of radius a to roll on a straight line. Then, a point on a fixed radius of this circle, and at a distance $\pi a/3$ from the center, will describe the given trochoid. The axis of X is the line on which the circle rolls, and the axis of Y is taken as the position of the radius containing the generating point when it is pointing vertically downward.

The desired area may be found by evaluating $\int y\, dx$ between the proper limits of integration. It is convenient to express this integral in terms of the parameter ϕ, as

$$\int \frac{a^2}{9}(3 - \pi \cos \phi)^2 d\phi.$$

From the way in which the curve is defined above, it is symmetrical with respect to the Y-axis; therefore, it will suffice to find that portion of the loop to the left of the Y-axis and multiply it by 2. As ϕ starts from zero and increases in value, the point which generates the curve starts from its position on the negative portion of the Y-axis and traverses the left half of the loop in a clock-wise direction about the origin, finally returning to the Y-axis again, this time in the upper half of the coördinate plane. The limits of integration in ϕ must, therefore, correspond to the two points where the loop cuts the Y-axis. Examination of the equations of the curve shows these to be 0 and $\pi/6$.

The area of half the curve is therefore given by

$$\frac{a^2}{9}\int_0^{\pi/6}(9 - 6\pi \cos \phi + \pi^2 \cos^2 \phi)d\phi,$$

or

$$\frac{a^2}{9}\left(9\phi - 6\pi \sin \phi + \frac{\pi^2\phi}{2} + \frac{\pi^2}{4}\sin 2\phi\right)_0^{\pi/6}.$$

The area of the entire loop will then be double this, or

$$\frac{a^2\pi}{108}(-36 + 2\pi^2 + 3\pi\sqrt{3}).$$

Also solved by J. B. Reynolds and the Proposer.

2707 [May, 1918]. Proposed by S. A. COREY, Albia, Iowa.

Let A, B, and C be the vector sides of a triangle. Construct another triangle with vector sides R, S, and T, where

$$R = mA - dnB, \qquad S = nA + (m + en)B.$$

Then prove that

$$(m^2 + emn + dn^2)(a^2 + eab \cos (AB) + db^2) = r^2 + ers \cos (RS) + ds^2,$$

where d, e, m and n are any scalar quantities; a, b, r and s are the tensors, or lengths, of the sides A, B, R and S, respectively; and cos (RS) is the cosine of the angle between R and S when placed coinitially.

Solution by the Proposer.

We have the algebraic identity

$$(m^2 + emn + dn^2)(a^2 + eab + db^2) = r^2 + ers + ds^2, \tag{1}$$

where $r = ma - dnb$, and $s = na + (m + en)b$.

As (1) is a quadratic identity in a, b, r, and s, it holds if these quantities be vectors (or quaternions), provided we consider the scalar part only of the vector products which we would have in (1) under such an interpretation. But such an interpretation gives us exactly the equation of the given problem, which is therefore true.

NOTE.—I have used capitals A, B, C, R, S, and T in some cases where small letters are used in the problem as published in the May MONTHLY. This substitution of capitals for small letters is done for the sake of clearness in distinguishing vectors from their tensors.

2708 [May, 1918]. Proposed by WILLIAM HOOVER, Columbus, Ohio.

A uniform plank of length $2a$ and thickness $2h$ rests in equilibrium on a fixed rough horizontal cylinder of radius c, so that a vertical plane containing the dimension $2a$ and the center of gravity of the plank is at right angles to the axis of the cylinder; find the period of a complete small oscillation of the plank.

SOLUTION BY THE PROPOSER.

Let G be the center of gravity of the plank; G_0 the initial position of G; O the center of gravity of the cylinder and vertically beneath G_0; φ = the angular rotation of the plank after any time t from the beginning of motion, OX, OY the horizontal and vertical coördinate axes $OA = x$, $GA = y$, the coördinates of G, $k = \sqrt{(a^2 + h^2)/3}$ = the radius of gyration of the plank about an axis parallel to the axis of the cylinder. We obtain

$$x = (c + h)\sin\varphi - c\varphi\cdot\cos\varphi, \tag{1}$$

$$y = (c + h)\cos\varphi + c\varphi\cdot\sin\varphi. \tag{2}$$

By *vis viva*,

$$\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + k^2\dot{\varphi}^2) = -mgy + C. \tag{3}$$

From (1),

$$\dot{x} = (h\cos\varphi + c\varphi\sin\varphi)\dot{\varphi}, \tag{4}$$

$$\dot{y} = (-h\sin\varphi + c\varphi\cos\varphi)\dot{\varphi}; \tag{5}$$

(3) then is

$$\frac{m}{2}(h^2 + k^2 + c^2\varphi^2)\dot{\varphi}^2 = C - mg\{(c + h)\cos\varphi + c\varphi\sin\varphi\}. \tag{6}$$

Differentiating both sides of (6) with respect to t using the value of k,

$$3c^2\varphi\cdot\dot{\varphi}^2 + \{(a^2 + 4h^2) + c^2\varphi^2\}\ddot{\varphi} = -3g\{-h\sin\varphi + c\varphi\cos\varphi\}. \tag{7}$$

Let φ be so small that we may put $\sin\varphi \doteq \varphi$, $\cos\varphi = 1$, and omit φ^2, $\dot{\varphi}^2$, etc., because of the nature of the oscillation; we have, then,

$$(a^2 + 4h^2)\ddot{\varphi} = -3g(c - h)\cdot\varphi, \tag{8}$$

an harmonic equation in φ if $c > h$, giving the period required,

$$T = 2\pi\sqrt{\frac{a^2 + 4h^2}{3g(c - h)}}. \tag{9}$$

It may be instructive to derive (8) by another method given by Holditch in the eighth volume of the *Cambridge Transactions* and quoted by Routh, *Dynamics of a System of Rigid Bodies, Elementary Part*, fourth edition, 1882, pages 341–342.

Let the motion of a body in space of two dimensions be given by the coördinates x, y of its center of gravity, and the angle φ which any fixed line in the body makes with a line fixed in space; α = the equilibrium value of φ; x', x'', etc., denoting $dx/d\varphi$, etc.; x_0', etc., the values of x', etc., when $\varphi = \alpha$, and k = the principal radius of gyration; then

$$(x_0'^2 + k^2)\ddot{\varphi} = -gy_0''\varphi. \tag{10}$$

From (4) and (5) we have, with $\varphi_0 = 0$,

$$x_0' = h, \qquad y_0'' = c - h \tag{11}$$

(11) in (10) gives (8).

Also solved by R. C. COLWELL and J. B. REYNOLDS.

2711 [June, 1918]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that the curves (*a*) $a^8y_1^2 = x^4(a^2 - x^2)^3$, (*b*) $a^8y_2^2 = x^8(a^2 - x^2)$ bound ten areas, of which two are each $(a^2/4)(\frac{1}{4}\pi - \frac{1}{3})$ and the remaining eight are each $a^2/24$.

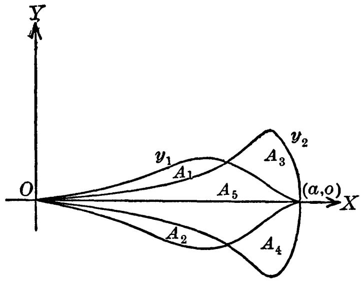
SOLUTION BY A. M. HARDING, University of Arkansas.

Since each of the curves is symmetrical with respect to both axes we shall consider only those areas which lie on the positive side of the x-axis.

$$A_1 = A_2 = \int_0^{a\sqrt{1/2}} (y_1 - y_2)dx = \frac{1}{a^4}\int_0^{a\sqrt{1/2}}$$

$$[x^2(a^2 - x^2)^{3/2} - x^4(a^2 - x^2)^{1/2}]dx$$

$$= \frac{1}{a^4}\int_0^{a\sqrt{1/2}} x^2\sqrt{a^2 - x^2}(a^2 - 2x^2)dx.$$



Let

$$x = a\sin\frac{\theta}{2}.$$

Then

$$A_1 = A_2 = \frac{a^2}{8}\int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta = \frac{a^2}{24}.$$

$$A_3 = A_4 = \int_{a\sqrt{1/2}}^{a} (y_2 - y_1)dx = \frac{1}{a^4}\int_{a\sqrt{1/2}}^{a} x^2\sqrt{a^2 - x^2}(2x^2 - a^2)dx.$$

Let $x^2 = a^2 - u^2$. Then

$$A_3 = A_4 = \frac{1}{a^4}\int_0^{a\sqrt{1/2}} u^2\sqrt{a^2 - u^2}(a^2 - 2u^2)du = \frac{a^2}{24},$$

as above.

$$\tfrac{1}{2}A_5 = \int_0^{a\sqrt{1/2}} y_2dx + \int_{a\sqrt{1/2}}^{a} y_1dx = \frac{1}{a^4}\int_0^{a\sqrt{1/2}} x^4\sqrt{a^2 - x^2}dx + \frac{1}{a^4}\int_{a\sqrt{1/2}}^{a} x^2(a^2 - x^2)^{3/2}dx.$$

The last integral reduces to

$$\frac{1}{a^4}\int_0^{a\sqrt{1/2}} u^4\sqrt{a^2 - u^2}du$$

on setting $x^2 = a^2 - u^2$. Hence,

$$\tfrac{1}{2}A_5 = \frac{2}{a^4}\int_0^{a\sqrt{1/2}} x^4\sqrt{a^2 - x^2}dx = \frac{a^2}{8}\left(\frac{\pi}{4} - \frac{1}{3}\right),$$

or

$$A_5 = \frac{a^2}{4}\left(\frac{\pi}{4} - \frac{1}{3}\right).$$

Also solved by A. R. NAUER, H. L. OLSON, and the Proposer.

2712 [June, 1918]. Proposed by WILLIAM HOOVER, Columbus, Ohio.

Given the conic $ax^2 + 2hxy + by^2 - 2x = 0$. Find the locus on which lie the four points of intersection of pairs of tangents to the conic from a pair of points on the x-axis equidistant from the origin.

SOLUTION BY A. H. WILSON, Haverford College.

The pair of tangents to the conic $C \equiv ax^2 + 2hxy + by^2 - 2x = 0$ from the point $(\alpha, 0)$ are represented by the equation $\lambda C + l^2 = 0$, where, $l \equiv (a\alpha - 1)x + h\alpha y - \alpha = 0$, is the polar of $(\alpha, 0)$, and $\lambda = \alpha(2 - a\alpha)$. Similarly, if $m \equiv (a\alpha + 1)x + h\alpha y - \alpha = 0$, $\mu = -\alpha(2 + a\alpha)$, $\mu C + m^2 = 0$, represents the tangents from $(-\alpha, 0)$.

The elimination of α from the equations $\lambda C + l^2 = 0$ and $\mu C + m^2 = 0$ is effected at once by subtraction and gives for the required locus the conic $hxy + by^2 - x = 0$.

As a does not occur in this equation, it is the locus described for any one of the conics of the one-parameter family obtained by holding h and b fixed and allowing a to vary.

Also solved by A. M. HARDING and the Proposer.

2714 [June, 1918]. Proposed by H. R. HOWARD, University of St. Francis Xavier's College, Nova Scotia.

A shuffled pack of $2(p+q)$ cards contains $2p$ honors. Show that the chance of securing exactly half the honors in taking half the pack is $[F(p, q)]^2 \div F(2p, 2q)$, where $F(p, q)$ denotes the number of different sets of p cards which can be selected from $(p+q)$ cards.

Show also that if one honor is removed from the pack, the chance is not thereby affected. Is this true for the chance of getting any other assigned number of honors?

Solution by the Proposer.

We have to find first the number of ways in which we can get exactly p honors in $(p+q)$ cards.

We can obviously choose our "p" honors in $(2p)!/(p!\,p!)$ ways and the q other cards in $(2q)!/(q!\,q!)$ ways.

Hence, we can effect the required division in $[(2p)!\,(2q)!]/[(p!)^2\,(q!)^2]$ ways. Now the number of ways of taking $(p+q)$ cards from the full pack is $[2(p+q)]!/(p+q)!^2$. Thus the chance is

$$\frac{(2p)!\,(2q)!}{(p!)^2(q!)^2} \div \frac{[2(p+q)]!}{(p+q)!^2}, \tag{1}$$

i.e.,

$$\frac{(p+q)!^2}{(p!)^2\,(q!)^2} \div \frac{(2p+q)!}{(2p)!\,(2q)!},$$

i.e.,

$$[F(p, q)]^2 \div F(2p, 2q).$$

Now suppose one honor removed from the pack. Then the number of ways of taking $(p+q)$ cards from the remainder and obtaining exactly p honors is

$$\frac{(2p-1)!}{(p-1)!\,(p)!} \cdot \frac{(2q)!}{(q!)^2},$$

and the chance is

$$\frac{(2p-1)!}{(p-1)!\,(p)!}\,\frac{(2q)!}{(q!)^2} \div \frac{(2p+2q-1)!}{(p+q)!\,(p+q-1)!}. \tag{2}$$

Dividing (1) by (2) we obtain unity for the quotient and this proves their equality. Let x be the assigned number of honors. We shall show that the condition that the chances be equal can only be satisfied by $x = p$. With the full pack the chance will now be

$$\frac{(2p)!}{x!\,(2p-x)!} \cdot \frac{(2q)!}{(p+q-x)!\,(q-p+x)!} \div \frac{(2p+2q)!}{(p+q)!^2}. \tag{3}$$

With one honor removed the chance will be

$$\frac{(2p-1)!}{x!\,(2p-x-1)!}\,\frac{(2q)!}{(p+q-x)!\,(q-p+x)!} \div \frac{(2p+2q-1)!}{(p+q)!\,(p+q-1)!}. \tag{4}$$

Dividing (3) by (4) we see that these can only be equal if

$$\frac{p}{2p-x} = 1$$

or $x = p$, which proves our statement.

Also solved by H. L. Olson, A. Pelletier, and E. E. Witmer.

2715 [June, 1918]. Proposed by H. R. KINGSTON, University of Manitoba.

A', B', C' are points on the sides BC, CA, AB, respectively, of the triangle ABC, and AA', BB', CC' are concurrent in O. X, Y, Z are the three collinear points in which, by Desargues' theorem, the corresponding sides of the triangles ABC and $A'B'C'$ intersect. If A'', B'', C'' are the vertices of the triangle formed by the lines AX, BY, CZ, show that AA'', BB'', CC'' are concurrent.

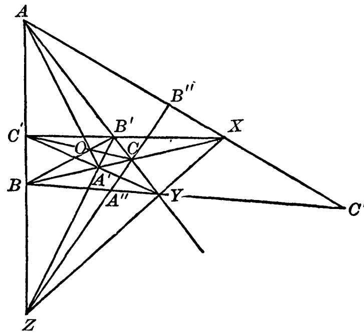
I. Solution by William Hoover, Columbus, Ohio.

Take ABC as the triangle of reference, and the trilinear coördinates of the vertices A, B, C to be $(\alpha_2, 0, 0)$; $(0, \beta_2, 0)$; $(0, 0, \gamma_2)$, and those of O, $(\alpha_1, \beta_1, \gamma_1)$; then the equation of AO is $\gamma_1\beta - \beta_1\gamma = 0$.

The coördinates of A' will be proportional to $(0, \beta_1, \gamma_1)$, and, by symmetry, those of B' to $(\alpha_1, 0, \gamma_1)$, and of C' to $(\alpha_1, \beta_1, 0)$.

The equation of $A'B'$ is $\beta_1\gamma_1\alpha + \alpha_1\gamma_1\beta - \alpha_1\beta_1\gamma = 0$.

The point of intersection of AB and $A'B'$, or Z, has coördinates proportional to $(-\alpha_1, \beta_1, 0)$; those of the intersection BC and $B'C'$, or X, to $(0, -\beta_1, \gamma_1)$; and of CA and $C'A'$, or Y, to $(\alpha_1, 0, -\gamma_1)$; and X, Y, Z are collinear, since



$$\begin{vmatrix} 0, & -\beta_1, & \gamma_1 \\ \alpha_1, & 0, & -\gamma_1 \\ -\alpha_1, & \beta_1, & 0 \end{vmatrix} = \alpha_1\beta_1\gamma_1 \begin{vmatrix} 0, & -1, & 1 \\ 1, & 0, & -1 \\ -1, & 1, & 0 \end{vmatrix}$$

$$= \alpha_1\beta_1\gamma_1 \begin{vmatrix} 0, & 0, & 1 \\ 1, & -1, & -1 \\ -1, & 1, & 0 \end{vmatrix} = \alpha_1\beta_1\gamma_1 \begin{vmatrix} 1, & -1 \\ & \\ -1, & 1 \end{vmatrix} = 0.$$

The equation of AX is $\gamma_1\beta + \beta_1\gamma = 0$, and those of BY and CZ respectively $\gamma_1\alpha + \alpha_1\gamma = 0$, and $\beta_1\alpha + \alpha_1\beta = 0$.

The coördinates of C'', the point of intersection of AX and BY, are proportional to $(\alpha_1, \beta_1, \gamma_1)$; of BY and CZ, to $(\alpha_1, \beta_1, \gamma_1)$; and of CZ and AX to $(\alpha_1, \beta_1, \gamma_1)$, the latter two points being A'', B'' respectively.

It is evident now that the equations to AA'', BB'', CC'' are the same as those of AO, BO, CO, in order, the first three lines then passing through O.

II. Solution by H. L. Olson, Chicago, Illinois.

I shall amplify this theorem by proving that the lines AA'', BB'', CC'' are identical with the lines AA', BB', CC' respectively, and hence intersect in the point O. In the triangles $BC'Y$ and $CB'Z$, the lines BC, $C'B'$, and YZ, joining corresponding vertices, meet at X, and hence the points A, A', A'', in which corresponding sides meet, are collinear. Similarly B, B', B'' are collinear; also C, C', C''. Hence the lines AA'', BB'', CC'' meet at O.

Also solved by A. Pelletier and the Proposer.

2716 [June, 1918]. Proposed by Clifford N. Mills, Brookings, South Dakota.

To a passenger in a train moving at the rate of 40 miles per hour, the rain appears to be rushing downward and towards him at an angle of 20 degrees with the horizontal. If the rain is actually falling in a vertical direction, show that the velocity of the raindrops in feet per second is 21.35.

Solution by E. H. Worthington, Elkins Park, Pa.

A velocity of 40 miles per hour is the same as $58\frac{2}{3}$ feet per second. If v is the velocity of the raindrop, we have $v = 58\frac{2}{3} \tan 20° = 58\frac{2}{3} \times 0.364$ feet per sec. $= 21.35$ feet per second.

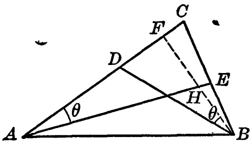
Also solved by H. E. Carleton, A. M. Harding, H. L. Olson, A. Pelletier and J. B. Reynolds.

2735 [December, 1918]. Proposed by H. B. Phillips, Massachusetts Institute of Technology.

If two lines AE and BD, drawn from the vertices A and B of a triangle to the opposite sides, divide the angles A and B so that the parts of A are respectively less than the corresponding parts of B, then AE is greater than BD.

SOLUTION BY THE PROPOSER.

Let C be the third vertex. By hypothesis $CAE < CBD$ and $BAE < ABD$. A point F can then be found on CD such that $DBF = CAE$. Let BF cut AE in H. AHF and BDF are similar, having equal angles at A and B and the same angle at F. Therefore,



$$AH : BD = AF : BF. \tag{1}$$

Also, since $ABD > BAE$, $ABF > BAF$. Consequently,

$$AF > BF. \tag{2}$$

From (1) and (2), it follows that $AH > BD$. Therefore, $AE > AH > BD$, which was to be proved.

COROLLARY: *If AE and BD are equal and divide their angles in the same ratio, the triangle is isosceles.*

For, if the angles A and B were not equal the parts of one would be respectively less than the corresponding parts of the other and AE and BD would be unequal, which is contrary to hypothesis.

In particular, if the bisectors of two angles of a triangle are equal, the triangle is isosceles.

2736 [December, 1918]. Proposed by M. COHEN, Freshman, Johns Hopkins University.

Prove by elementary geometry that the orthocenter, the centroid, and the circumcenter of a triangle lie on a line (the Euler line), and that the centroid lies between the other two and is twice as far from the orthocenter as from the circumcenter.

SOLUTION BY J. L. RILEY, Stephenville, Texas.

Let ABC be the triangle under consideration; O and G the circumcenter and centroid, BE and CF perpendicular, respectively, to AC and AB. Let mid-point of AC be B'.

Produce OG to meet the altitude BE at K. The triangles OGB' and KGB are similar, for OB' is parallel to BK, since each is perpendicular to AC. Then $OG : GK = B'G : GB = 1 : 2$ and hence, $GK = 2\,OG$.

If OG is produced to meet the altitude CF at K', it follows in the same way that $GK' = 2\,OG$. Therefore, $GK' = GK$ and K' coincides with K. Hence BE and CF meet at K and K is the orthocenter. Hence, circumcenter, centroid, and orthocenter lie on the same line.

Also solved by H. L. OLSON, C. P. SOUSLEY, and the Proposer.

2737 [January, 1919]. Proposed by C. N. SCHMALL, New York City.

Employing Maclaurin's theorem, or otherwise, expand the following three functions (1) $e^{\tan^{-1}x}$ as far as x^6; (2) $e^{\sin x}$ as far as x^8; and (3) $\tan x$ as far as x^9.

SOLUTION BY ELMER LATSHAW, West Philadelphia, Pennsylvania.

The successive differentiation required by Maclaurin's theorem in the development of the given functions is long and laborious, but the required developments may be obtained by comparing the derivative of the function with the function itself.

Assume

$$e^{\tan^{-1}x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_6x^6 + \cdots. \tag{1}$$

Differentiating both sides of (1),

$$\begin{aligned} e^{\tan^{-1}x}\frac{1}{1+x^2} &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + 6a_6x^5 + \cdots \\ &= (a_0 + a_1x + a_2x^2 + \cdots a_6x^6 + \cdots)(1 - x^2 + x^4 - x^6 + \cdots). \end{aligned}$$

Equating coefficients of like powers of x,

$$a_1 = a_0, \quad 2a_2 = a_1, \quad 3a_3 = a_2 - a_0, \quad 4a_4 = a_3 - a_1, \quad 5a_5 = a_4 - a_2 + a_0, \quad 6a_6 = a_5 - a_3 + a_1.$$

Equation (1) by making $x = 0$ gives $a_0 = 1$ and the preceding equations give

$$a_1 = 1, \quad a_2 = \tfrac{1}{2}, \quad a_3 = -\tfrac{1}{6}, \quad a_4 = -\tfrac{7}{24}, \quad a_5 = \tfrac{1}{24}, \quad a_6 = \tfrac{29}{144}.$$

Hence,

$$e^{\tan^{-1}x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{7x^4}{24} + \frac{x^5}{24} + \frac{29x^6}{144} - \cdots.$$

$e^{\sin x}$ may be similarly developed.

$$e^{\sin x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_8x^8 + \cdots. \tag{2}$$

Differentiating,

$$\begin{aligned} e^{\sin x}\cos x &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + 8a_8x^7 + \cdots \\ &= (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_8x^8 + \cdots)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right). \end{aligned}$$

Equating coefficients,

$$a_1 = a_0, \quad 2a_2 = a_1, \quad 3a_3 = a_2 - \frac{a_0}{2!}, \quad 4a_4 = a_3 - \frac{a_1}{2!}, \quad 5a_5 = a_4 - \frac{a_2}{2!} + \frac{a_0}{4!},$$

$$6a_6 = a_5 - \frac{a_3}{2!} + \frac{a_1}{4!}, \quad 7a_7 = a_6 - \frac{a_4}{2!} + \frac{a_2}{4!} - \frac{a_0}{6!}, \quad 8a_8 = a_7 - \frac{a_5}{2!} + \frac{a_3}{4!} - \frac{a_1}{6!}.$$

Making $x = 0$ in equation (2) gives $a_0 = 1$ and the preceding equations give

$$a_1 = 1, \quad a_2 = \tfrac{1}{2}, \quad a_3 = 0, \quad a_4 = -\tfrac{1}{8}, \quad a_5 = -\tfrac{1}{15}, \quad a_6 = -\tfrac{1}{240}, \quad a_7 = \tfrac{1}{90}, \quad a_8 = \tfrac{31}{5760}.$$

Hence,
$$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} - \frac{x^6}{240} + \frac{x^7}{90} + \frac{31x^8}{5760} + \cdots.$$

To develop $\tan x$, we notice that $\tan(-x) = -\tan x$. Hence, the expansion will contain only odd powers of x.

Assume
$$\tan x = a_1x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + \cdots.$$

Differentiating,

$$\begin{aligned} \sec^2 x = a_1 + 3a_3x^2 + 5a_5x^4 + 7a_7x^6 + 9a_9x^8 + \cdots &= 1 + \tan^2 x \\ &= 1 + (a_1x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + \cdots)^2. \end{aligned}$$

Equating coefficients of like powers of x,

$$a_1 = 1, \quad 3a_3 = a_1^2, \quad 5a_5 = 2a_1a_3, \quad 7a_7 = 2a_1a_5 + a_3^2, \quad 9a_9 = 2a_1a_7 + 2a_3a_5.$$

From these, we obtain

$$a_1 = 1, \quad a_3 = \tfrac{1}{3}, \quad a_5 = \tfrac{2}{15}, \quad a_7 = \tfrac{17}{315}, \quad a_9 = \tfrac{62}{2835}.$$

Hence,
$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots.$$

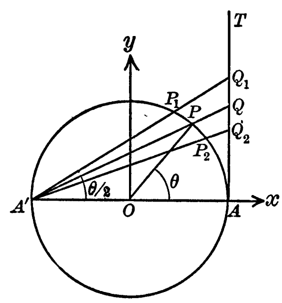
Also solved by E. D. GRANT, H. L. OLSON, J. L. RILEY, and the Proposer.

2738 [January, 1919]. Proposed by W. D. CAIRNS, Oberlin College.

Prove that between any two points on a unit circle with its center at the origin there is a point whose coördinates are rational.

SOLUTION BY P. J. DANIELL, Rice Institute.

Let $A'OA$ be the diameter on the x-axis, and let P_1, P_2 be the two given points. Through A draw AT perpendicular to $A'OA$. Let $A'P_1$, $A'P_2$ intersect AT in Q_1, Q_2. By the theory of irrational numbers between the points Q_1, Q_2 on AT there is a point Q such that AQ is rational and indeed equal to $2m/n$, where m, n are integers. Let $A'Q$ intersect the circle in P. Then P is the required point. It is assumed, and this involves no loss in generality, that P_1, P_2 lie on the same side of the x-axis. Then P lies between P_1 and P_2.



$$x = \cos\theta = \frac{1 - \tan^2\dfrac{\theta}{2}}{1 + \tan^2\dfrac{\theta}{2}} = \frac{1 - \dfrac{AQ^2}{A'A^2}}{1 + \dfrac{AQ^2}{A'A^2}} = \frac{1 - \dfrac{m^2}{n^2}}{1 + \dfrac{m^2}{n^2}} = \frac{n^2 - m^2}{n^2 + m^2}$$

is rational. Similarly $y = \dfrac{2mn}{n^2 + m^2}$ is rational.

Also solved by R. A. JOHNSON, H. L. OLSON, A. PELLETIER, W. R. RANSOM, J. ROSENBAUM, E. SWIFT, and the Proposer.

NOTES AND NEWS.

Edited by E. J. Moulton, Northwestern University, Evanston, Ill.

Mr. I. Roman, now studying at the University of Chicago, will return to Northwestern University as instructor in mathematics this fall.

Mr. P. W. Hill has returned from overseas' service and will again be instructor in mathematics at Wabash College.

Professor E. G. Bill has recently been advanced to the office of Director, Military Service Branch, by the Canadian Government. He expects to return to Dartmouth College this fall.

Dr. M. G. Smith has been appointed professor of mathematics at Greenville College.

Mr. H. L. Smith, of Cornell University, has been appointed instructor in mathematics and astronomy at Trinity College.

Dr. C. A. Fischer, of Columbia University, has been appointed professor of mathematics and astronomy at Trinity College.

Dr. J. R. Kline, instructor in mathematics during the past year has been appointed an "associate" in mathematics at the University of Illinois.

Professor E. B. Van Vleck is to be a lecturer at Harvard University during the second semester of 1919–20.

Mr. H. R. Phalen has been promoted to the position of assistant professor of mathematics at the Armour Institute of Technology.

Assistant Professor H. L. Slobin, of the University of Minnesota, has been appointed head of the department of mathematics in New Hampshire College.

At Lehigh University Mr. M. S. Knebelman has been promoted to be assistant professor of mathematics.

Professor Ida Barney, of Lake Erie College, Painesville, O., has been appointed in charge of the department of mathematics at Meredith College, Raleigh, N. C.

Professor E. E. Moots, of Cornell College, Iowa, has been appointed head of the department of mathematics and engineering.

Professor H. M. Showmann, of Colorado School of Mines, and lecturer in physics at Harvard during the past year, has been appointed assistant professor of mathematics at the Case School of Applied Science.

In accordance with a general policy for encouraging research, initiated in the latter part of June at the Massachusetts Institute of Technology, three members of the department of mathematics (not above the rank of assistant professor) are to be relieved of one-third of their teaching assignments next year and are expected to devote the time released to research. In order to make this possible Dr. S. D. ZELDIN, who has been instructor in the College of Hawaii and professor at Olivet College, has been appointed instructor in mathematics at the Institute.

Science announces that Dr. E. D. ROE, Jr., professor of mathematics at Syracuse University, has been elected director of the observatory. His position in the department of mathematics remains unchanged.

Professor ROBERT W. WILLSON has just become professor emeritus at Harvard University. He was assistant professor of astronomy 1899–1903, and has been professor of astronomy since 1903.

Professor N. E. NÖRLUND, of the University of Lund, has been appointed professor of mathematics at the University of Copenhagen.

The eightieth birthdays of the following mathematicians have occurred recently: LEO KOENIGSBERGER (November 15, 1917), THEODOR REYE (June 20, 1918), and HIERONYMUS ZEUTHEN (February 15, 1919). (*L'Enseignement mathématique* states that Zeuthen died on his birthday.) If MORITZ CANTOR was alive August 23, 1919 he would be ninety years old.

JOHN WILLIAM STRUTT, Lord RAYLEIGH, died June 30, 1919, aged 76.

Don EDUARDO TORROJA, professor of applied geometry at the University of Madrid died September 14, 1918 in the seventy-second year of his age.

PAUL MANSION, professor of mathematics at the University of Ghent, died April 18, 1919, aged 75. He was the author of scores of books and articles dealing with topics in higher algebra, theory of numbers, foundations of geometry and history of mathematics. For forty years he was co-editor of *Nouvelle Correspondance Mathématique* and its successor *Mathesis*.

Professor CHRISTOFORO ALASIA, whose biography and portrait appeared in this MONTHLY, August–September, 1902, died November 19, 1918, aged 49 years. He is probably known in this country chiefly as the author of *La recente geometria del triangolo* (Cità di Castello, 1900) and of *566 Relazioni metriche e trigonometriche fra gli elementi d'un triangolo piano* (ibid., 1900), and as editor of *Le Matematiche* (2 vols., 1901–1903).

Professor O. D. KELLOGG has been made a fellow of the American Association for the Advancement of Science, and elected vice-president and chairman of Section A.

In the proposed organization of an International Astronomical Union the following members of the ASSOCIATION are members of committees in the organization of the American section: E. P. ADAMS, L. A. BAUER, P. P. BOYD, E. W. BROWN, W. J. HUSSEY, FRANK MORLEY, and F. R. MOULTON.

A Norwegian Mathematical Society was founded at Christiania on November 2, 1918 when about one hundred and fifty mathematicians came together from all parts of Norway. This society will meet in larger cities, and will publish a periodical, of which Professor HEEGAARD (recent editor of *Nyt Tidsskrift for Matematik*) and Oberlehrer ALEXANDER have been appointed editors.

From *Nature* we learn of a project for the creation and endowment of a geophysical institute at Cambridge, England, which is being promoted by a committee of prominent British scientists. It is felt that such an institute could render great assistance in connection with geodetic work which forms the basis of all the state surveys, and with the practical problems associated both with leveling, mean sea-level, and vertical movement of the earth's crust, and with a more thorough study of the tides and tide prediction than has ever been made.

Dr. VITO VOLTERRA, professor of mathematical physics in the University of Rome, will deliver a series of six lectures on the Hitchcock Foundation at the University of California in August or September.

At the Colloquium of the American Mathematical Society to be held at the University of Chicago in September, 1920, Professor G. D. BIRKHOFF is to lecture on "Dynamical Systems," and Professor F. R. MOULTON on "Certain topics in functions of infinitely many variables."

In conferring the degree of Doctor of Science on Professor STRATTON at the recent Yale Commencement the public orator spoke as follows: Samuel Wesley Stratton; Mathematician, physicist, professor in the Universities of Illinois and Chicago, a naval officer in the Spanish war, since 1891 Director of the National Bureau of Standards in weights and measures. Dr. Stratton's work in this bureau has been conspicuous and constructive, recognized beyond our own limits, vitally important in war and war research. A man weighed in the balance and not found wanting.

The American Mathematical Monthly

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The Maryland-Virginia-District of Columbia Section of the Mathematical Association of America met at the U. S. Naval Academy, Annapolis, Maryland, May 3, 1919. Among those in attendance were the following members: R. N. Ashmun, International Boundary Commission; H. G. Avers, U. S. Coast and Geodetic Survey; Clara L. Bacon, Goucher College; C. C. Bramble, U. S. N. A.; J. A. Bullard, U. S. N. A.; Paul Capron, U. S. N. A.; G. R. Clements, U. S. N. A.; A. Cohen, Johns Hopkins University; G. H. Cresse, U. S. N. A.; L. S. Dederick, U. S. N. A.; Alexander Dillingham, U. S. N. A.; J. B. Eppes, U. S. N. A.; J. N. Galloway, U. S. N. A.; H. C. Gossard, U. S. N. A.; Angelo Hall, U. S. N. A.; W. M. Hamilton, U. S. Nautical Almanac Office; H. L. Hodgkins, George Washington University; L. S. Hulburt, Johns Hopkins University; W. W. Johnson, U. S. N. A.; A. E. Landry, Catholic University; J. J. Luck, University of Virginia; E. S. Mayer, U. S. N. A.; F. D. Murnaghan, Johns Hopkins University; J. R. Musselman, U. S. Food Commission; C. H. Rawlins, Jr., U. S. N. A.; M. R. Richardson, U. S. N. A.; R. E. Root, U. S. N. A.; W. F. Shenton, U. S. N. A.; Clara E. Smith, Goucher College; H. Ivah Thomsen, Johns Hopkins University; John Tyler, U. S. N. A.

The program consisted of a forenoon and an afternoon session, and contained the following papers:

Forenoon Session—Symposium on "Mathematics for Engineering Students."

(1) Paper, Prof. Abraham Cohen, Johns Hopkins University (Mathematics);
(2) Discussion, Prof. Alexander Dillingham, U. S. N. A. (Mathematics);
(3) Paper, Prof. T. J. MacKavanagh, Catholic University (Engineering), by invitation;
(4) Discussion, Prof. L. A. Doggett, U. S. N. A. (Engineering), by invitation;
(5) General Discussion.

Afternoon Session.

(6) "Circular parts, the general case."—Prof. W. W. Johnson, U. S. N. A.
(7) "A study of map projections in general."—Mr. Oscar S. Adams, U. S. Coast and Geodetic Survey.

The visitors in attendance were the guests of the members of the Naval Academy Department of Mathematics at a delightful luncheon served at one o'clock at The Sign of the Goat Inn.

At the beginning of the afternoon session the following officers were elected for the ensuing year:
Chairman, Professor R. E. Root, U. S. Naval Academy.

Secretary-Treasurer, OSCAR S. ADAMS, U. S. Coast and Geodetic Survey.
Member of the Executive Committee, Prof. CLARA L. BACON, Goucher College.

After the program the visitors were shown about the beautiful grounds and buildings or were taken to see some of the various athletic events that were in progress.

OSCAR S. ADAMS, *Secretary.*

THE GROWTH OF THE SOLAR SYSTEM.

By WILLIAM DUNCAN MACMILLAN, University of Chicago.

Since the sun with its attendant planets is journeying through space with a speed of about 12 miles per second and since there are large regions of space which are visibly nebulous it follows that at rare intervals the sun must pass through such nebulosity and add to its mass. This nebulosity varies from a relatively great density such as occurs in Andromeda and regions in Orion to a density so low that it can be detected only by photographic means, as for example, the region of the Pleiades. The direct evidence ends, perhaps, with the photographs of these regions of low visibility, but the apparently great variation in nebular density compels us to admit the existence of nebulosity below the bounds of visibility even photographically; and from the physical nature of these objects we cannot doubt that regions of very low density are much more extensive than regions of higher density.

In addition to nebulosity which is self-luminous the phenomenon of dark regions with apparently luminous back grounds is interpreted by Barnard and others as indicating the existence of nebulosity which is not self-luminous. How extensive these regions of dark nebulosity are we have but little means of knowing for it would be only rarely that they would be projected upon luminous backgrounds, but such evidence as there is indicates that they are rather extensive.

In addition to matter in a distinctly nebulous form one can scarcely escape postulating the existence of isolated solid fragments, for it is easy to see how such solid fragments can occur. In the course of time—very long, perhaps—some star will pay us a visit, with the result of considerable disturbance to our present orderly system. How serious this disturbance will be will depend, naturally, upon the particular circumstances under which it occurs, and the character of the system which does the visiting. A not impossible result would be to throw one or more of the planets out into space, and it seems almost certain that some of our comets and planetoids and some of our very numerous meteors would receive that kind of treatment. Our sun would doubtless reciprocate upon the other system, so that one of the final results of such a visit would be some loose fragments in space.

Indeed it is not easy to avoid the idea that comets and their resultant meteors

are fragments which the sun has picked up and attached to our system at some epoch in the sun's very ancient history. They are certainly fragments from a once larger body—thereby testifying to the fact that large solid bodies are sometimes broken up and their remains scattered. It seems certain that there must be more or less of this material in space.

How much of this material will the sun and planets pick up as they move along? It will simplify matters to suppose that this material is initially stationary and that the particles move only under the sun's gravitation. This hypothesis is doubtless contrary to the facts but inasmuch as their motions will be at random with respect to the sun this hypothesis will give at least a first approximation to the actual facts. The particles, then, will describe hyperbolas about the sun, and every particle whose perihelion distance is less than the radius of the sun will fall into the sun and add its mass to the sun's mass. All of the others will pass by and be of no consequence.

Imagine the sun moving along a straight line with a speed s, and consider a particle at a distance D from this line while the sun is still very remote. Since the particle is initially at rest it has a speed s with respect to the sun along a line parallel to the sun's motion. If k^2 is the gravitational constant, M the sun's mass and a the semi-axis of the particles' orbit about the sun, then the velocity of the particle with respect to the sun at a distance r is given by the formula

$$v^2 = k^2M\left(\frac{2}{r}+\frac{1}{a}\right).$$

When $r = \infty$ then $v = s$, from which it follows that

$$a = \frac{k^2M}{s^2}. \tag{1}$$

From the law of areas, or moment of momentum, we have also

$$xy' - yx' = \sqrt{k^2Ma(\epsilon^2 - 1)},$$

where ϵ is the eccentricity of the hyperbolic orbit. But when very remote from the sun $y' = 0$, $y = D$, $x' = - s$, and therefore

$$D^2s^2 = k^2Ma(\epsilon^2 - 1). \tag{2}$$

Eliminating a between (1) and (2) and then solving for ϵ we have

$$\epsilon = \sqrt{1 + \frac{s^4D^2}{k^4M^2}}, \tag{3}$$

so that the perihelion distance is

$$a(\epsilon - 1) = \sqrt{\frac{k^4M^2}{s^4} + D^2} - \frac{k^2M}{s^2}.$$

If now the radius of the sun is R, then in order that the particle may strike the sun we must have

$$\sqrt{\frac{k^4M^2}{s^4} + D^2} - \frac{k^2M}{s^2} < R,$$

from which it is seen that

$$D^2 < \frac{2k^2MR}{s^2} + R^2. \tag{4}$$

In the case of a body having the mass, speed, and size of the sun the term R^2 is small as compared to $2k^2MR/s^2$, so that if we take a cylinder of radius

$$\rho = \frac{\sqrt{2k^2MR}}{s}, \tag{5}$$

we can say that all of the particles within this cylinder will be swept up by the sun, and the sun's *effective* radius is ρ. For the sun its numerical value is 14,000,000 miles.

Since in one unit of time the sun moves through a distance s, it follows that the rate at which material is swept up is

$$\frac{dM}{dt} = \pi\rho^2 s\delta = \pi\delta\left(\frac{2k^2MR}{s} + R^2 s\right), \tag{6}$$

where δ is the average density of matter in space. This expression has a minimum for $s^2 = 2k^2M/R$, which for our sun gives $s = 383$ miles per second, and for this minimum value $\rho = \sqrt{2}R$. For such speeds as that of the sun the last term of (6) might be neglected in which case we would have the result that the rate at which the mass increases is inversely proportional to the speed.

If we knew the value of δ, which is the amount of matter in a unit volume of space we could compute the rate at which the sun is growing. Without doubt δ varies from one region of space to another. In a recent article in the *Astrophysical Journal*[1] the estimate was made that one particle one one-hundredth of an inch in diameter for every 560 cubic miles of space would account for the blackness of the night sky. Taking this value of the density and the present speed of the sun it would require 1.4×10^{17} years for the mass of the sun to double. If, however, the sun should enter a nebulous region in which the density is as low as 10^{-12} times the density of air at sea level its mass would double in one billion years. A density of this tenuity would be obtained if one cubic foot of air were so expanded as to fill 8 cubic miles. We have of course no means of knowing what the densities of the nebulas are but it is evident that if the sun should penetrate such a region as that of the Orion Nebula its mass would be materially increased in a relatively short time. But independently of such special regions it is clear that not only the sun but all of the stars are gathering in material unto themselves sometimes rapidly but usually slowly.

[1] MacMillan, *Astrophysical Journal*, volume 48, July, 1918.

A closely related subject is the effect of this process upon the planetary system. In order to simplify the discussion let us suppose that the sun and Jupiter in their Keplerian motion pass through a region of space filled with material which they gather in as they journey along, and seek the effect of this growth upon the system. Let ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 be the coördinates of Jupiter and the sun respectively with respect to axes fixed in space; m_1 and m_2 are their masses. The components of momentum at any instant will be $m_1\xi_1'$, $m_1\eta_1'$, $m_1\zeta_1'$; $m_2\xi_2'$, $m_2\eta_2'$, $m_2\zeta_2'$. If either or both of the bodies collide with a stationary mass or particle their masses will be increased and their velocities diminished, but their instantaneous momenta will be unaltered. The differential relations between the changes of velocities and masses are therefore

$$(7)\qquad \begin{aligned} d\xi_1' &= -\frac{\xi_1'}{m_1}dm_1, & d\eta_1' &= -\frac{\eta_1'}{m_1}dm_1, & d\zeta_1' &= -\frac{\zeta_1'}{m_1}dm_1,\\ d\xi_2' &= -\frac{\xi_2'}{m_2}dm_2, & d\eta_2' &= -\frac{\eta_2'}{m_2}dm_2, & d\zeta_2' &= -\frac{\zeta_2'}{m_2}dm_2. \end{aligned}$$

If $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ are the coördinates of the center of mass and M is the sum of the masses, the integrals for the motion of the center of mass are

$$(8)\qquad \begin{aligned} M\bar{\xi}' &= m_1\xi_1' + m_2\xi_2' = \alpha_1 = Ms\cos\lambda_1,\\ M\bar{\eta}' &= m_1\eta_1' + m_2\eta_2' = \alpha_2 = Ms\cos\lambda_2,\\ M\bar{\zeta}' &= m_1\zeta_1' + m_2\zeta_2' = \alpha_3 = Ms\cos\lambda_3, \end{aligned}$$

where s is the speed of the center of gravity and $\lambda_1, \lambda_2, \lambda_3$ are the direction angles of the path of the center of gravity. It is evident that α_1, α_2, and α_3 are unaltered by the collision.

We will take a new set of axes parallel to the fixed axes but with the origin at the sun. Then the coördinates of Jupiter with respect to the sun will be

$$(9)\qquad \begin{aligned} x &= \xi_1 - \xi_2, & y &= \eta_1 - \eta_2, & z &= \zeta_1 - \zeta_2,\\ x' &= \xi_1' - \xi_2', & y' &= \eta_1' - \eta_2', & z' &= \zeta_1' - \zeta_2'. \end{aligned}$$

By differentiation of the second set of (9) we get

$$dx' = d\xi_1' - d\xi_2', \qquad dy' = d\eta_1' - d\eta_2', \qquad dz' = d\zeta_1' - d\zeta_2';$$

and by substitution of values from (7)

$$(10)\qquad \begin{aligned} dx' &= -\left[\frac{\xi_1'}{m_1}dm_1 - \frac{\xi_2'}{m_2}dm_2\right],\\ dy' &= -\left[\frac{\eta_1'}{m_1}dm_1 - \frac{\eta_2'}{m_2}dm_2\right],\\ dz' &= -\left[\frac{\zeta_1'}{m_1}dm_1 - \frac{\zeta_2'}{m_2}dm_2\right]. \end{aligned}$$

After solving (8) and (9) for $\xi_1', \eta_1', \zeta_1'; \xi_2', \eta_2', \zeta_2'$ and substituting in (10) we have

$$
(11) \qquad \begin{aligned}
dx' &= -\frac{\alpha_1}{M}\left(\frac{dm_1}{m_1}-\frac{dm_2}{m_2}\right)-\frac{x'}{M}\left(\frac{m_2}{m_1}dm_1+\frac{m_1}{m_2}dm_2\right),\\
dy' &= -\frac{\alpha_2}{M}\left(\frac{dm_1}{m_1}-\frac{dm_2}{m_2}\right)-\frac{y'}{M}\left(\frac{m_2}{m_1}dm_1+\frac{m_1}{m_2}dm_2\right),\\
dz' &= -\frac{\alpha_3}{M}\left(\frac{dm_1}{m_1}-\frac{dm_2}{m_2}\right)-\frac{z'}{M}\left(\frac{m_2}{m_1}dm_1+\frac{m_1}{m_2}dm_2\right).
\end{aligned}
$$

The change in the relative velocity $v^2 = x'^2 + y'^2 + z'^2$ will be

$$vdv = x'dx' + y'dy' + z'dz',$$

and by means of (11) this expression becomes

$$(12) \quad vdv = -\frac{1}{M}(\alpha_1 x' + \alpha_2 y' + \alpha_3 z')\left(\frac{dm_1}{m_1}-\frac{dm_2}{m_2}\right)-\frac{v^2}{M}\left(\frac{m_2}{m_1}dm_1+\frac{m_1}{m_2}dm_2\right).$$

If μ_1, μ_2, μ_3 are the direction angles of v we have

$$
\begin{aligned}
x' &= v\cos\mu_1, & y' &= v\cos\mu_2, & z' &= v\cos\mu_3,\\
\alpha_1 &= Ms\cos\lambda_1, & \alpha_2 &= Ms\cos\lambda_2, & \alpha_3 &= Ms\cos\lambda_3;
\end{aligned}
$$

and if ω is the angle between the direction of motion of the center of gravity and the direction of the relative motion of Jupiter we have

$$\cos\omega = \cos\lambda_1\cos\mu_1 + \cos\lambda_2\cos\mu_2 + \cos\lambda_3\cos\mu_3$$

and therefore, on removing a factor v, equation (12) becomes

$$(13) \qquad dv = -s\cos\omega\left(\frac{dm_1}{m_1}-\frac{dm_2}{m_2}\right)-\frac{v}{M}\left(\frac{m_2}{m_1}dm_1+\frac{m_1}{m_2}dm_2\right).$$

If the direction of motion of the center of gravity were perpendicular to the plane of relative motion the $\cos\omega$ would always be zero. If it is not perpendicular then $\cos\omega$ varies, as Jupiter completes a circuit about the sun, from a certain positive value to the same negative value; the extreme limits being ± 1 when the direction of motion of the center of gravity lies in the plane of relative motion. If we suppose the added material to be gathered in slowly at all points of the orbit the average value of $\cos\omega$ will be zero, and in the long run this term will be inappreciable. Furthermore it would be exactly zero if the material gathered in by the sun and Jupiter was proportional to their own masses. This term gives the change in relative velocity due to a displacement of the center of gravity. Under the conditions we are discussing its effect is very small and it will be neglected. It is found then that (13) can be written

$$\frac{dv}{v} = \frac{1}{\dfrac{1}{m_1}+\dfrac{1}{m_2}}\, d\left(\frac{1}{m_1}+\frac{1}{m_2}\right),$$

so that if A is the constant of integration we have

$$v = A\left(\frac{1}{m_1} + \frac{1}{m_2}\right).$$

If we suppose that in the accretion of material the ratio of m_1 to m_2 is not altered we will have

$$Mv = B, \tag{14}$$

where B is a constant not altered by the accretion.

The expression for v in the relative motion is

$$v^2 = k^2M\left(\frac{2}{r} - \frac{1}{a}\right).$$

On multiplying through by M^2 we have

$$B^2 = k^2M^3\left(\frac{2}{r} - \frac{1}{a}\right).$$

If a collision occurs r is not altered, nor is B. Hence by di to M and a

$$3M^2\left(\frac{2}{r} - \frac{1}{a}\right)dM + \frac{M^3}{a^2}\,da = 0,$$

which shows how the major axis is affected by the collision. If the process of accretion is a very slow one as we suppose is the case so that all parts of the orbit share alike in the collisions then we can replace $1/r$ by its average value, viz.,[1] $1/a$, after which this differential relation reduces to

$$3\frac{dM}{M} + \frac{da}{a} = 0,$$

from which it follows that

$$M^3a = C, \qquad \text{or} \qquad a = \frac{C}{M^3}, \tag{15}$$

where C is a constant not altered by accretion. The expression for the period $P = 2\pi a^{3/2}/k\sqrt{M}$ becomes

$$P = \frac{2\pi C^{3/2}}{kM^5}. \tag{16}$$

Stated in words these results are that during a process of slow growth in the sun and Jupiter due to stationary material picked up by them in their journey through space, their mean distance apart varies inversely as the cube of the sum of their masses, and their period varies inversely as the fifth power of the sum of their masses.

---

[1] The average value of $\frac{1}{r}$ for a complete revolution is $\frac{1}{P}\int_0^P \frac{dt}{r} = \frac{1}{a}$.

For example, suppose the masses of the sun and Jupiter were each increased to 5 times their present mass, then we would have $P_1 = 3125P_2$. That is the period of Jupiter, 11.86 years, would be reduced to $33\frac{1}{4}$ hours, and its distance would be reduced from 500 millions to 4 millions of miles. It is needless to discuss the fate of the terrestrial planets.

If it is true that the energies of the sun and stars are maintained by the subatomic energies of this added material and if the yielding up of its subatomic energies implies a loss in mass, then it is not necessary that the mass of the sun be increasing even though it is gathering in new material. It may even be declining at the present time. Since Jupiter is not wasting much energy in radiation it would seem as though the mass of Jupiter must certainly be growing. It requires but little imagination to see the possibility of Jupiter eventually rivalling the sun in mass and even brilliancy, and the two together constituting a double star.

To be sure, such a conception involves an enormous range in time, but it will never be possible to understand the astronomical forms which are now presented to our vision in the wide expanses of the heavens until we not only understand the physical processes now at work but also extend their logical consequences to such intervals of time as are necessary to establish a cycle, and undoubtedly such an interval is very great.

CUSPIDAL ROSETTES.[1]

By WILLIAM FRANCIS RIGGE, Creighton University, Omaha, Neb.

The rose, rosette, rosace, Rosenkurve or multifolium, or whatever other name it may have, is a periodic polar curve whose equal sectors may have any angular magnitude. Its general equation, as usually given, $\rho = a + b \sin n\theta$,[2] supposes the tracing point to move with a simple harmonic motion of n cycles along a radial line through the pole, at the same time that it makes one revolution about this pole with uniform angular speed. A simple instance of such a polar curve is the trifolium, Fig. 1 (p. 324), which is drawn by having a tracing pen move with simple harmonic motion of amplitude b in a radial line over a uniformly rotating disk in such a way that the pen just touches its centre without passing beyond, and the disk makes one revolution in three cycles of the pen. The equation is then $\rho = a(1 - \sin 3\theta)$, as is seen by inspection in Fig. 2.

If in the general equation a is greater than b, the pen does not reach the center as in Fig. 3, and if a is less than b, the pen passes beyond it as in Fig. 4 and draws

[1] See "Concerning a new Method of Tracing Cardioids" by William F. Rigge, in the January, 1919, MONTHLY and "On the Construction of Certain Curves Given in Polar Coördinates" by R. E. Moritz, in the May, 1917, MONTHLY.

[2] The two terms may have unlike signs and sin be replaced by cos.

smaller lobes or loops on the other side, which are within or between the larger ones according as n is odd or even, until when $a = 0$, the equation becomes $\rho = b \sin 3\theta$, the loops are equal and we have a trifolium in Fig. 5 somewhat like Fig. 1 but half its size and traced twice by the pen because n is odd, the number of the lobes being doubled when n is even. The rosette is *cuspidal* only when $a = b$ as in Figs. 1 and 2, because only then the pen, after tracing one branch of the curve and coming to a momentary standstill, retreats along another branch, so that both branches have a common rectilinear tangent at the point of rest. This tangent is always between the two branches in the curves treated in this discussion, so that the cusps are of the first species.

All this is well known and has been mentioned merely as an introduction to the subject in hand. As Moritz (l. c.) has very completely treated the case of the pen moving along a *radial* line through the center of the disk, the problem now is to investigate what happens when the pen moves along a *non-radial* line, and when it is set down at any initial phase. One who has tried the experiment will know that he obtained a rosette distorted and somewhat like one of the three mentioned before in Figs. 3, 2 and 4, which we may call rounded or curtate, cuspidal, and looped or prolate. Of these the cuspidal rosettes require certain conditions which it is the specific object of the present article to study.

The Cardioid.—As the cardioid may be defined as a cuspidal rosette with the ratio $n = 1$, the present investigations are based upon the previously cited article by the writer, to which the reader must have recourse for more detailed explanations. The following condensed resumê may however be sufficient.

In Fig. 6, A is the center of the disk which rotates with uniform angular speed in a clockwise direction. B is the point at which the tracing pen is set down in any initial phase of its rectilinear simple harmonic motion which has the same period as the disk, so that, if the disk did not rotate, the pen would move over the line ERG, its distance from R, its middle point, being at any moment the sine of the phase.

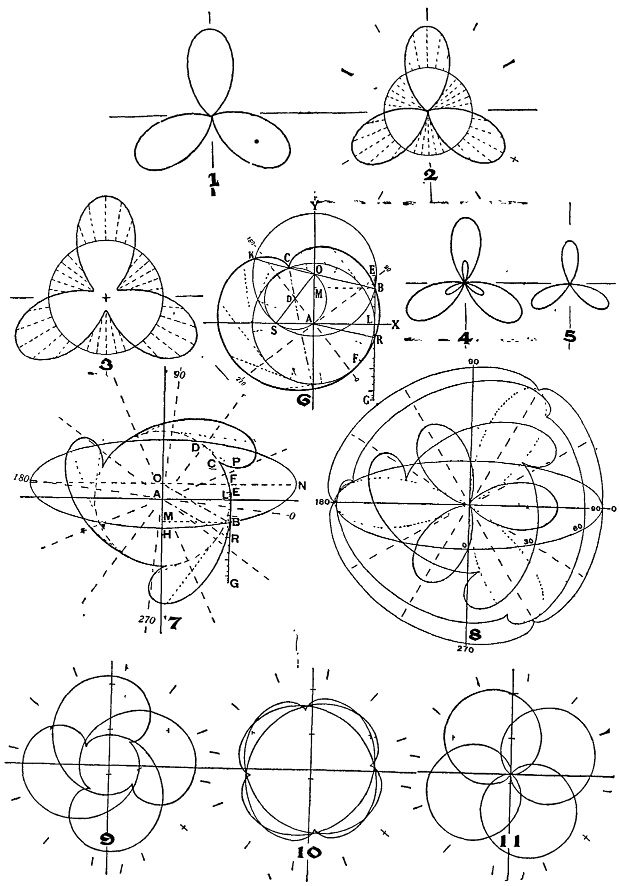
In order to draw a cardioid it is essential that the point B, at which the pen is set down on the rotating disk, should be on the "starting circle" whose radius OB is taken as unity and is equal to the amplitude of the harmonic motion ER or RG, and whose center O is on the Y axis at the distance from the center of the disk equal to $OA = BR = \sin \alpha$ = the sine of the initial phase. The cusp C of the cardioid will then be on the cusp circle, with radius one half of OB, which is internally tangent to the starting circle at S on the axis of X so that the angle $OSA = \alpha$, and which passes through O and A. The selection of the point B on the starting circle fixes the starting angle $\beta = AOB$, which then locates the cusp C on the diameter through B. The distance of the cusp C from A is equal to $AC = AL = MB = \sin \beta$. There is also a second starting point K, diametrically opposite B, so that $\beta' = 180° + \beta$, upon which the pen may be set down at the same initial phase α and trace the same cardioid.

The mechanical method mentioned of tracing the cardioid is what we might call the *tangent* method.[1] A similar method will be seen to apply in a more gen-

[1] Cf. "Concerning a new Method, etc."

eralized form to cuspidal rosettes of all kinds. To facilitate the comparison, the same letters and symbols will be used as much as possible.

Cuspidal Rosettes: *the Ratio* $n = 3$.—We begin with a simple ratio of $n = 3$, or $n = p/q = 3/1$, that is, p or 3 cycles of the rectilinear motion of the pen being equal to q or one of the disk, Fig. 7. It is immaterial whether we suppose both the pen and the disk to move in the manner indicated, or imagine the disk at rest and credit the pen with its motion in the reverse direction. The initial phase α of the rectilinear motion in Fig. 7 has been taken as 20°, so that



$BR = OA = \sin 20°$. We will take $\beta = 30°$. In comparing Figs. 6 and 7 we may note generic similarities and obvious differences. The starting point B and the rectilinear path ERG are the same in both figures, although the scale is not. The starting circle with radius unity in Fig. 6 becomes the large ellipse in Fig. 7, whose center O is as before on the Y axis at the distance $\sin \alpha$ from A. The conjugate axis of the ellipse OH is unity, and its transverse axis $ON = n = 3$, and is parallel to the X axis. AL is now $n \sin \beta = 3 \sin \beta$.

The Starting Point B.—That the point B must be on the ellipse mentioned may be proved by substituting the general ratio n for its value of unity in the cardioid. If in Fig. 6 we draw BO parallel to RA, these lines are equal, and $OA = BR = \sin \alpha$. Drawing BM parallel to AL makes them also equal to one another, and OM equal to LR. As LR is the sine of some phase of the harmonic motion, we may take it as the cosine of the angle ARL or its equal MOB, that is, as $\cos \beta$. Then $MB = AL = \sin \beta$ and $OB = AR = OS =$ unity. Therefore the starting point B is on the circle with radius unity and with its center O the distance $\sin \alpha$ from A.

In the case of a rosette as in Fig. 7, we have MB also equal to AL, $OA = BR = \sin \alpha$ and $LR = OM$. Taking these last equal to $\cos \beta$ as before, we cannot now make AL and MB equal to $\sin \beta$, but must take them equal to $n \sin \beta$, because the variation of the phase of the pen along the line ERG is now n times the angular speed of the disk. Hence the starting point B is on the ellipse whose semi-major axis $ON = n$ and semi-minor axis $OH =$ unity, and center O at distance $\sin \alpha$ from A.

The Locus of the Cusps C.—The rosette in Fig. 7 has three cusps, and in general it is evident that the number of cusps must be equal to p, the number of cycles of the pen, and that their angular intervals must be $360°/p$. For this reason we may confine ourselves to the cusp C that is first drawn after the pen has been set down at B and call it *the* cusp.

To find the phase of the pen when it is at the cusp, we observe that the pen must then be momentarily at rest, so that its rectilinear speed must be equal and opposite to its rotary velocity. As the latter is always at right angles to the radius through A, the harmonic motion can be so only when the pen crosses the X axis at L, so that one value of RL is the sine of the phase of the pen when at the cusp C and the other when it is at F, where the linear and rotary speeds are also equal but in the same direction. The rotary speed of the pen when at C is $ACd\theta = ALd\theta = n \sin \beta d\theta = \sin \beta d\beta$, its rectilinear speed at L is $d(LR)$ so that $LR = \int \sin \beta d\beta$ is equal to $\cos \beta$ or $\sin (90° \pm \beta)$ in absolute value. Of these two, $90° + \beta$ must be the phase for the cusp C, because as the rotary motion carries the pen anticlockwise, the rectilinear motion at L in Fig. 7 must then carry it in the opposite direction, that is, clockwise or downward, so that the phase must be greater than 90°. For the point F the phase must then be $90° - \beta$.

The phase of the harmonic motion at the cusp C being $90° + \beta$, that of the rotary motion must be $(90° + \beta)/n$. The angular position of C on the disk therefore varies directly as β, and as its linear distance from A is $n \sin \beta$, its locus must have an equation like $\rho = n \sin n\theta$, and when $n = 3$ as in Figs. 7 and 8, this is the equifoliated and non-cuspidal trifolium shown in dotted lines in Fig. 8, which is exactly like Fig. 5 but n times as large. The position of the axis of the first lobe is found by making $n \sin \beta$ a maximum, that is, $\beta = 90°$, so that (see Fig. 8 in which four rosettes are given with $\beta = 0°, 30°, 60°, 90°$) the phase of the pen $90° + \beta$ becomes $90° + 90° = 180°$, and the phase of the disk $180°/n$.

This last value directly locates the 0° phase of the disk, which may however be found also from any value of β, since it is equal to $(90° + \beta)/n$ as said before.

The arc AC of the cusp folium is equal to the elliptic arc HB on Fig. 7, and similarly to any corresponding elliptic arc on Fig. 8, that is to say, the distance of the cusp C from the center of the disk A as measured along the arc of the cusp folium is equal to that of the starting point B from the Y axis as measured along the ellipse. In the rectification of a polar curve the length of an arc is

$$s = \int (d\rho^2 + \rho^2 d\theta^2)^{1/2},$$

so that in the cusp folium where $\rho = n \sin n\theta$, we have

$$\begin{aligned} s &= \int (n^2 \cos^2 n\theta \cdot n^2 d\theta^2 + n^2 \sin^2 n\theta \cdot d\theta^2)^{1/2} \\ &= n \int_0^\beta \left(1 - \frac{n^2 - 1}{n^2} \sin^2 n\theta\right)^{1/2} n d\theta, \end{aligned}$$

which is like

$$s = a \int_{\phi_0}^{\phi_1} (1 - e^2 \sin^2 \varphi)^{1/2} d\varphi,$$

the length of an arc of an ellipse with semi-major axis a, eccentricity e, and eccentric angle φ, as given by Byerly in his *Integral Calculus*, page 121, because in our ellipse $a = n$, $b = 1$, $e^2 = (a^2 - b^2)/a^2 = (n^2 - 1)/n^2$. Therefore the cusp folium arc AC equals the elliptic arc HB, and the perimeter of one lobe of the cusp folium is equal in length to the semiperimeter of the ellipse. This expression is true for values of $n > 1$, but when $n < 1$ it takes the form

$$\begin{aligned} s &= \int (1 - (1 - n^2) \cos^2 n\theta)^{1/2} n d\theta \\ &= b \int (1 - e^2 \cos^2 \varphi)^{1/2} d\varphi, \end{aligned}$$

so that as a and b then exchange names and a cosine appears in the place of a sine, the elliptic arc is now reckoned from the end of the major instead of the minor axis, the zero point H however being always on the axis of Y.

The Locus of the Points of Contact F.—As the phase of the pen is $90° + \beta$ at the cusp C and $90° - \beta$ at the point of contact F, all that has been said about the former applies to the latter, provided we reverse the sign of β. The F points are thus seen to lie on the lobes of an equal equifoliated rosette with the equation $\rho = - n \sin n\theta$, these lobes being diametrically opposite to those of the cusp rosette. It has been omitted from Fig. 8.

The Tangent Method of Tracing the Rosette.—The mechanical method of tracing a rosette by having a pen move with a rectilinear simple harmonic motion of n cycles over a uniformly rotating disk, is mathematically a tangent method. If we imagine the two components of the pen's motion, the rectilinear over the

line ERG and the rotary about A, to act successively instead of simultaneously, the pen is first at L in Fig. 7 in its rectilinear motion, in phases $90° + \beta$ and $90° - \beta$, and is then carried to C and F by the rotary motion on a circle with center A and radius $n \sin \beta$. The length of the tangent to this circle is then zero. At any other phase θ the pen in its harmonic motion is, say, at B', which may be anywhere, but which we may place on B in order not to congest the

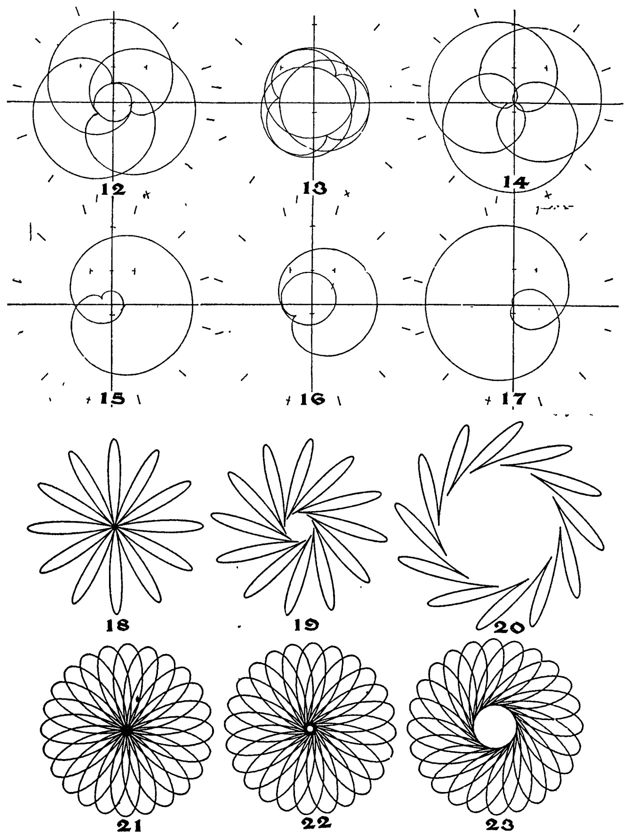


figure and which we accent in order to distinguish the two. While the rotary motion alone may be conceived first to carry the pen to D, the tangential harmonic motion then moves it along the tangent DP which is equal to $LB' = LR - B'R = \cos \beta - \sin \theta$, or rather $\sin \theta - \cos \beta$, because the tangent DP in the case illustrated is really negative.

On Fig. 7 these tangents have been drawn to almost every 30° of the disk. Owing to the small numerical value of LE in the instance presented, the tangents to the *tangent* circle, as we may call it, are positive only between F and C, between phases $90° - \beta$ and $90° + \beta$, which are here 60° and 120°, and negative for all other phases. The points C and F on the curve in phases $90° \pm \beta$ are

thus at minimum, and those in phases 90° and 270° at maximum distances from A. It is obvious, of course, that as the harmonic motion is n times as rapid as the circular, the tangent at any phase angle on the tangent circle runs to a point on the curve in n times that angle, so that 60° on the circle in Fig. 7 is joined to $3 \times 60° = 180°$ on the rosette, 330° on the circle to $3 \times 330° = 2 \times 360° + 270°$ on the curve, and so on.

The Equation of a Cuspidal Rosette.—The tangent DP in Fig. 7 ought to lead us to the equation of the rosette. The radius vector ρ or AP is seen to be such that

$$\overline{AP}^2 = \overline{AD}^2 + \overline{DP}^2$$

or

$$\rho^2 = a^2 n^2 \sin^2 \beta + a^2 (\sin (n\theta + \alpha) - \cos \beta)^2$$

in which a is the amplitude of the simple harmonic motion of the pen ER, which we may take as our unit, n, β, α, have their usual meaning and any assumed values, and θ is the position angle of D from $+X$, or the angle DAL. But it is the angle PAL that we need. Calling this ω and DAP δ, we have $\omega = \theta + \delta$, using the plus sign in general because when DP is negative, as it is in Fig. 7, it will reverse the sign of δ. Now δ is the angle whose tangent is DP/AD. Taking θ as the independent variable, we may find ρ and ω, but it does not seem possible to express the relation between ρ and ω in one equation without the help of θ.

The Three Elements of a Cuspidal Rosette, n, α, β.—There are three elements that determine the shape and position of a cuspidal rosette, not to mention its size which depends upon the amplitude of the harmonic motion that we take as our unit. The first element is $n = p/q$, the ratio of the cycles of the pen p to those of the disk q. While n might be incommensurable, only simple ratios of integral numbers are here considered. The second element is α, the initial phase in its harmonic motion at which the pen is set down on the disk. This may have any value from 0° to 360°. In this paper α is taken as less than 90°, greater values having been treated in the cardioid article. The third element is β, the eccentric angle of the point B on the ellipse at which the pen is set down on the disk.

Variation in the Elements.—The nature of a rosette obviously depends upon $n = p/q$ and β, since p determines the number of lobes and cusps and q the number of its convolutions about the center A, while β, that is, $n \sin \beta$, modifies its shape by determining its distance from the center. There is then nothing left for α to do but to fix the position of the curve. For if the pen is first started in phase 0°, and then in phase α, the advance of the pen on the disk will be α/n, and the whole rosette will be shifted that angular amount forward in a clockwise direction. Hence the disk reading for $+X$ then becomes α/n instead of 0°, as it is in Fig. 8, where $\alpha = 0°$. In Fig. 7 therefore, where $\alpha = 20°$ and $n = 3$, the circle reading for $+X$ is $\alpha/n = 6° 40'$. The four points of the circle, 0°, 90°, 180°, 270° may thus be properly marked and intermediate radii drawn for every 30° at pleasure. Then, as the circle reading for the axis of the first cusp lobe was found to be $180°/n$, its position angle from $+X$ then becomes $(180° - \alpha)/n$.

A variation in α alone shifts the center O of the ellipse along the Y axis to the distance $\sin \alpha$. It has no effect on the nature of the curve, as has been said, so that the rosette in Fig. 7, in which $n = 3$ and $\beta = 30°$ but $\alpha = 20°$, is exactly equal in every respect except as to the axes of X and Y to the second rosette on Fig. 8 in which also $n = 3$ and $\beta = 30°$ but $\alpha = 0°$. Even the position is the same in regard to the circle reading, because this is $(90° + \beta)/n$ for the cusp, independent of α, as we saw before.

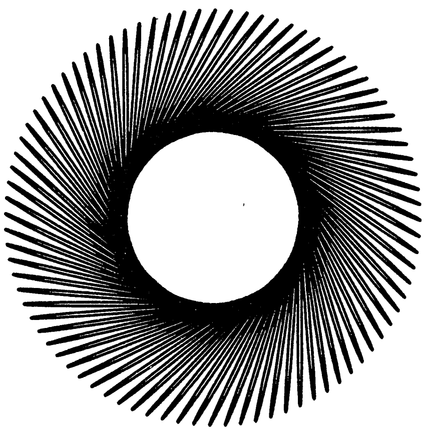
A change of α, as has been said, affects merely the position of the rosette, so that it is drawn sooner or later than it was at the first value of α. While the curve is being drawn there is then no reason why we should not be able to take any instantaneous position of the pen as an initial position. As this initial position must be at the assumed starting point B at the eccentric angle β of the starting ellipse mentioned before whose center is the distance $\sin \alpha$ from A in the direction of B from R (Fig. 7), we may take this assumed point B on the ellipse, move it together with the ellipse along the rosette to a point in another phase, swing the ellipse about this second position of B until its center is at the distance of the sine of this new phase from A on a line parallel to the direction of the harmonic motion of the pen at the moment, and then mark the center on the paper. It we do this for all points of the rosette, we shall find the locus of the ellipse center O to be an equifoliated non-cuspidal rosette exactly like Fig. 5 when $n = 3$, with its lobe axes of unit magnitude lying on those of the contact F folium. The reason is that $AO = \sin \alpha$, and the angular revolution of the ellipse about B must be uniform for equal phase intervals, so that $d\theta$ is constant when $d\alpha$ is, as we always consider it to be. This O-folium may readily be drawn for any value of n, if we place the pen at A when in phase 0°. No illustrations of the O-folia are here given, since they would be for all values of n, as much like the original rosette when $\beta = 0°$ or 180° as Fig. 5 is like Fig. 1, that is to say, the O-folia would be non-cuspidal and of unit magnitude.

A variation in β alone does not affect the nature of the rosette when the old value of β is added to or subtracted from 180° and 360°. For this reason it is most convenient to use values of β less than 90°, and to express greater values in the way indicated. For $180° + \beta$ the new curve is symmetrical to the β rosette with respect to the center of the disk A. For $180° - \beta$ and $360° - \beta$ it is symmetrical to the β curve with regard to the $0° - 180°$ and $90° - 270°$ diameters respectively of the circle reading.

But it is the variations in n that teach us most about rosettes. Accordingly Figs. 9–25 show some typical cases. For Figs. 9–17 the initial phase α has been taken as 52° for the sake of comparison. The starting point B and the center O of the starting ellipse and the extremities of its axes have everywhere been marked. The dozen radiating dashes indicate every 30° of the circle, the 0° being marked by a cross. This last applies also to Fig. 2, in which however $\alpha = 90°$ and $\beta = 0°$. In Figs. 9, 10, 11, $n = 4/3$ and $\beta = 30°, 90°, 180°$, while in Figs. 12, 13, 14, $n = 3/4$ with the same values of β. In Figs. 15, 16, 17, $n = 1/2$ and $\beta = 30°, 60°, 180°$. When the denominator q in n is large, the

number of convolutions of the curve are uninterestingly numerous in proportion.

As α determines only the position of a curve, and this is generally of no consequence, the most convenient value of α to use is 90° when $\sin \alpha$ is a maximum and the pen is at one extremity of its harmonic path. Then when n is large and β small, the starting point is practically on the X axis. Thus Figs. 18, 19, 20 show the ratio $n = 12$ and $\beta = 0°, 3°, 19°$, respectively. In Figs. 21, 22, 23, $n = 24/5$ and $\beta = 0°, 1° 4', 5° 42'$, and in Figs. 24 and 25 $n = 84$, $\beta = 0° 38'$ and $\pm 0° 38'$. This ratio of $n = 84$ is the largest the author's machine is at present able to produce. The machine was described in the *Scientific American Supplement* of February 9 and 16, 1918. Fig. 25 is in a sense merely the double



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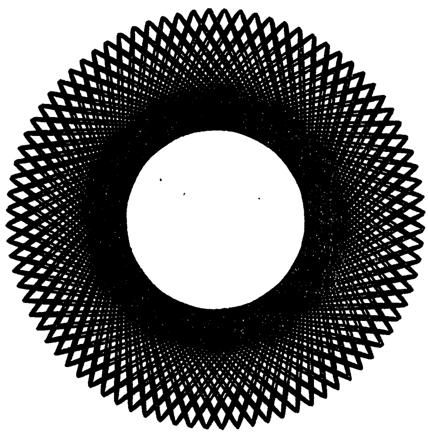

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of Fig. 24. It was drawn by first tracing Fig. 24 with the B point to the right of the center, and then by starting the pen an equal distance to the left. The cuspidal nature of some of the rosettes presented is not conspicuous, especially when β is very small, on account of the closeness and apparent superpositions of the paths of the pen when near the cusp. While other and more beautiful rosettes could have been drawn, had the restriction as to cusps been waived, they would have been foreign to the present study.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

What are the essential ideas which should be retained permanently by a student of the calculus, after the details and the special problems have faded from his mind? That the calculus makes a valuable contribution to the general culture of a student who has no need to make practical use of it, will not be

questioned by any of us who teach it. Too often, however, we fall into a contented routine of meeting our classes with self-satisfied assurance, assigning the regular lists of problems, conscientiously explaining to the students how to surmount their difficulties with Problem 7 or Formula 42, and leaving our teaching dead with lack of emphasis and discrimination. Not seldom we fail to point out the beauties of the forest because we ourselves see only the trees. To bring us back to fundamentals such an article as Professor Rietz's discussion below must be welcomed by every earnest minded teacher of the subject.

The second discussion, by Professor Bennett, points out the disadvantages attendant on each of the usual conventions for attaching an algebraic sign to the distance between two points or between a point and a line, in analytic geometry. He concludes that the plan most commonly found in our text-books (which he calls the transcendental determination) presents more serious difficulties than either of the others. We are glad to endorse his request for "a reply in defense of the accepted methods."

As the last discussion, we present an interesting little study by Mr. B. H. Brown on a subject which has probably not hitherto received so detailed a mathematical treatment. The close agreement between theory and practice in the probabilities of various occurrences connected with "crap-shooting" is surprising. Some of the questions discussed may furnish suitable material for class problems, as for example the verification of the figures in the columns marked "Theoretical" at the beginning of the article. Devotees of the game will scarcely find in the article arguments to support its availability as a rapid road to riches.

I. On the Teaching of the First Course in Calculus.[1]

By H. L. Rietz, University of Iowa.

In his presidential address[2] read before the American Mathematical Society, Osgood expressed the view that "the calculus is the greatest aid we have to the appreciation of physical truth in the broadest sense of the word." This fact should serve to keep before teachers of the first course in calculus the following important question: Can we teach this course so that our students will feel more keenly than at present the use of the calculus in the appreciation of physical truth?

One needs only compare the text-books in calculus at present with those in use thirty years ago to be convinced that much progress has been made along this line. The first course in calculus has been enriched through a broad range of problems from everyday experience and from physical science. Moreover, logical flaws centering around the conceptions of the infinitesimal and differential have been so largely removed that the mental satisfaction derived from a close contact with physical problems has been much increased. This

[1] Read before the Iowa Section of the Mathematical Association of America, April 26, 1919.
[2] *Bulletin of the American Mathematical Society*, Vol. 13, p. 449.

does not mean that time is given in the first course to rigorous demonstrations, but that time is saved through removing mysticism by a clear statement of definitions and assumptions. It goes without saying that further improvement in the problem work of the course will lead towards the attainment of that object of the calculus expressed in the above quotation, and that all teachers of the subject should devote much thought to new problems for the course. In the present paper we shall say very little about the character of problems, as it is our purpose to emphasize mainly another but related phase of the teaching of the calculus.

It has been a matter of interest to me to test the results of my own teaching of the first course in calculus and those of other teachers by asking students some years out of college, say ten years out of college, about the mental picture they retain of the nature and uses of the calculus. Naturally great variations occur in the picture, depending upon the ability and the life work of the man in question. It has been particularly interesting to me to consider on the one hand, the case of the physicist, the chemist, the actuary, and the engineer; and on the other, that of the man whose life work calls mostly for qualitative rather than quantitative thinking. Some of those who are making no professional or occupational use of the methods of the calculus seem to retain only a general and rather vague notion of certain formulas and transformations applied to the finding of areas and volumes. Indeed, with respect to most of the men who have little quantitative thinking to do the formulas are forgotten. But it has been a source of great satisfaction to find that many of these men long out of college have a picture of the method of the calculus that shows something of how it has entered into their thinking. They still picture the differential calculus as a method which gives us a precise conception of the rate of change of functions and quantities in the phenomenal world. Some have told me that in the integral calculus, they learned to add an infinitely large number of infinitely small quantities. One of my students who took calculus several years ago and who has to make no direct use of the calculus told me recently he could never forget this as fundamental in the calculus and that he appreciates to some extent the importance of this problem because most of the things about us seem to be made up of an infinitely large number of infinitely small parts. Although this conception of adding an infinitely large number of infinitely small parts may savor of an archaic view of the infinitesimal and involve a certain form of mysticism, still I would far rather find that the student has retained such a picture of the calculus than a picture of forgotten formulas.

For our consideration as teachers of the first course in calculus let me submit a brief description of what seems to me should be the minimum of detail in the mental picture retained by the average student ten or more years after studying a first course in the calculus. It seems to me the picture should include at least the following:

(1) That the determination of the rate of change of any quantity is a problem of the differential calculus and that this problem is of almost universal interest in relation to quantitative phenomena.

(2) That by the methods of the differential calculus approximate rates of change of quantities are replaced by exact formulas.

(3) That any limited physical quantity can be represented, with an approximation as close as you please, as a sum of simple and similar component elements, and that by the methods of the integral calculus such approximate sums are replaced by exact formulas.

(4) That physical laws are very generally formulated as differential equations.

The man who has formed a picture, with these notions as the main lines, to aid in the appreciation of the physical world may well be said to have formed "the calculus habit" in his thinking. It is our problem to teach the first course in calculus so as to establish this calculus habit of thought in our students. The important question in this connection is: By what means can we more successfully accomplish this end in our teaching? We would no doubt agree that each of the four points named above should be illustrated many times by concrete problems representing situations in everyday life and in the phenomenal world. We are no doubt doing much at the present time in our teaching of the first course in calculus that tends to carry out the purpose which I have in mind. Thus, we are all accustomed to impressing upon our students the necessity of looking upon a rate of change with respect to time as the limiting value of an average rate of change. We thus replace the approximate rate by an exact conception. We are also no doubt carrying our problems on areas, volumes, masses, moments of inertia, energy, and work back to the central point of looking upon the quantities as limits of the sum of simple and similar component elements. And yet, with all the beautiful and interesting problems that are given and will be given in the first course in the calculus, some of us have a strong feeling that the appreciation of the subject could be increased if more emphasis were placed on the fact that the solution of these problems may well be regarded as interesting details of a picture whose main lines are in a few fundamentals such as the four points mentioned above. The habits of the average student and perhaps even those of the exceptional student are formed largely through repetition. If we would have our students form "the calculus habit" of thinking or would make the calculus such an aid for them as it should be in their appreciation of physical truth, we must recall the main features of our picture whenever we fit into it the various applications to problems of physics, chemistry, geometry, engineering, and everyday life. Otherwise the calculus habit in thinking is not likely to be formed.

In thus contemplating the picture which we would have our students form of the calculus as a fundamental aid in the appreciation of physical truth, it is not to be inferred that the technique or formal side of the calculus may be neglected as unessential. Knowledge of the elements of the calculus is acquired largely through formal work. It is only after attaining considerable proficiency on the algebraic side of the calculus that one is ready to form the picture that should be retained. In this connection it is sometimes maintained that a knowledge of the derivatives and integrals of a few simple functions is adequate prepara-

tion in the calculus for an engineer. This position is not without a certain element of truth. It is possible to state the integrals necessary to follow an elementary treatment of physics and mechanics in brief space, but a well-grounded knowledge of the subject that will abide with the student is not to be obtained by confining the algebraic work within very narrow limits. It seems to me that with respect to technique neither the extreme view of algebraic formalism that was in vogue twenty-five years ago in the first course in calculus based on certain English textbooks very generally in use in this country nor the practical lack of formalism advocated by those who would popularize the calculus, makes a satisfactory contribution to our picture of the first course. Surely without approximately the amount of technique given in our recent textbooks we should give totally inadequate preparation for the serious study of mechanics, electricity, thermodynamics, rigid dynamics, hydrodynamics, the theory of elasticity, or for the reading of much of the recent literature of physics and chemistry. Indeed, a second course in calculus has been so strongly urged by certain prominent electrical and mechanical engineers that a course has been introduced into the junior year in certain schools to meet the demand. It goes without saying that technique is indispensable to the student who is to become a mathematician and it is clear that we should take due account of this class of men in looking to the future of education along mathematical lines. It seems to me that some of our recent textbooks in the elements of calculus exhibit rare judgment on the part of the authors in their position with respect to the amount of formal work to be done in the first course in calculus.

In conclusion, let me submit that we can make the first course in calculus a more important aid in the appreciation of physical truth by aiming more directly at establishing in the minds of our students what I have tried to indicate as the calculus habit of thinking about phenomena and that this result is most likely to be obtained by creating for the student a mental picture with only a few main lines.

II. The Sign of the Distance in Analytical Geometry.

By Albert A. Bennett, University of Texas.

The pupil in a course in analytical geometry comes very early across two formulas concerning distances. These yield the following schematic equations:

$$[\text{distance from } (x', y') \text{ to } (x, y)]^2 = (x' - x)^2 + (y' - y)^2,$$

$$[\text{distance from } (x', y') \text{ to } ax + by + c = 0]^2 = \frac{(ax' + by' + c)^2}{a^2 + b^2}, \qquad (a^2 + b^2 \neq 0).$$

No discussion of the derivation of these formulas need be made here. The question of present interest is what is to be done in determining the algebraic sign of the distance itself, that is the square root of each of the above expressions. Now the pupil has become accustomed in dealing with square roots to employ conventions of three types with respect to the determination of signs:

1. Numerical determination, where the positive sign only is used.
2. Transcendental determination, where both positive and negative signs occur, but the choice in any given instance is determinate.
3. Algebraic determination, where the sign is indeterminate, and both possibilities as to sign are retained throughout.

The numerical determination is that regularly occurring in elementary arithmetical and geometrical problems, and in many algebraic examples. The transcendental determination is used in the basic conventions of analytical geometry and trigonometry, and the algebraic convention is the rule in algebra except in certain numerical problems. The existence of these distinct conventions is only vaguely present to the pupil's mind but is a frequent source of misconception and error, and merely adds to the student's conviction of the arbitrary and blind character of mathematical notations. But the different points of view are not simultaneously consistent and each has its place, a fact which should at least be appreciated by the teacher.

Each of these three possibilities as applying to the sign of the distance will be treated in turn.

1. **Numerical Determination.** In elementary arithmetic, subtraction is taught, but negative numbers are not usually admitted. A quadratic equation as such does not arise, but the notion of square root is frequently in evidence. One thinks in every case of but a single square root, which is positive. In geometry a construction is given for the mean proportional, and usually mention is not made of two mean proportionals of opposite sign but of one only. Problems in mensuration such as: define the altitude of an equilateral triangle with an integral side, the radius of a circle of given area, etc., lead to square roots, but always to positive quantities. And so the fact that lengths and distances in everyday life are never negative should not surprise the mathematician. The distance from New York to Chicago is equal to the distance from Chicago to New York, the shortest distance from Denver to the Canadian border is equal to the shortest distance from the Canadian border to Denver, etc.

Thus when the student is given an elementary locus problem it might appear natural to regard distances as inherently positive. If a straight line be drawn in a plane, where no reference is made to coördinates, and it be desired to find the locus of points one foot from the line, it seems obvious that the locus will be two lines, not one only. If two points be given, the length of the segment joining them should be the distance between the two points, and the length is inherently positive as usually treated. The locus of points in a plane equally distant from two given intersecting straight lines, each line being viewed as extending indefinitely in both directions, is of course the pair of bisectors of the angles, a pair of perpendicular straight lines. It seems hardly necessary to reinvestigate the question as to whether the center of the inscribed circle of the triangle is equally distant from the sides, and the centers of the escribed circles equidistant from one side and the extensions of the other two sides, or whether the center of the circumscribed circle is equidistant from the vertices. To check up such facts may

serve as good pedagogical material but that a convention should retain these relations as still true can occasion no comment.

The numerical determination is, however, subject to the following serious drawback in the elementary treatment of analytical geometry: It requires every algebraic solution to be supplemented by a more or less tedious examination of extraneous solutions for which no adequate preparation is usually made. A few examples will be cited which by their simplicity serve to illustrate sharply the awkwardness inherent in this convention.

1. Find the locus of a point such that the sum of its distances from the rectangular axes is equal to unity.

The locus is a square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. Here the algebraic work leads to four lines of which only the segments forming the sides of the square satisfy the conditions. The extensions of the sides are extraneous.

2. Find the locus of a point such that the sum of its distances to the two rectangular axes is equal to the distance to the line whose equation is $4x + 3y - 5 = 0$.

The locus is an irregular quadrilateral with vertices on the axes. Only the points of the segments between adjacent vertices satisfy the problem.

3. Find the locus of a point such that the sum of its distances to the two axes and its distance to the line $4x + 3y - 5 = 0$, is unity.

The locus consists of a single point, the origin; all other parts of eight lines are extraneous.

4. Find the locus of a point such that the sum of its distances to the two axes and its distance to the line $4x + 3y - 5 = 0$, is two.

The locus consists of an irregular convex hexagon, whose opposite sides are not equal in length but are parallel.

5. Find the locus of a point such that its distance from $(0, 1)$ exceeds by unity its distance from the X-axis.

The locus consists of one entire parabola tangent to the X-axis at the origin together with the negative half of the Y-axis.

6. Find the locus of a point such that its distance from $(0, 1)$ exceeds by two its distance from the X-axis.

The locus consists of parts only of two parabolas, both having $(0, 1)$ as focus, but with different latera recta, the curves having a common chord on the X-axis and being joined there, but not crossing each other, the portions nearer the focus being extraneous.

7. Find the locus of a point such that the sum of its distances from $(1, 0)$ and $(-1, 0)$ is two.

The locus consists of the segment of the X-axis joining these two points. For the remainder of the X-axis the condition is that the difference of the distances be two, and if the subtrahend and minuend be specified, that one of the portions which is intended is given.

8. Find the locus of a point such that the sum of its distances from the three sides of an equilateral triangle is a given constant.

When the given constant is less than the altitude of the triangle, the locus contains no points.

When the constant is equal to the altitude of the triangle, the locus contains every point within and on, but no point exterior to, the triangle. The locus is in this case not a curve but a portion of the plane.

When the given constant is greater than the altitude of the triangle, the locus consists of a hexagon surrounding the triangle.

In conclusion, the numerical determination while entirely natural and self-consistent, appears to introduce unnecessary complications and non-algebraic investigations not particularly adapted to an elementary course in analytical geometry.

2. **Transcendental Determination.** Analytical geometry became possible by a conscious selection of special conventions facilitating the treatment of negative quantities. Having selected an axis for X's, no algebraic study could determine which direction would be the positive and which the negative. By a non-algebraic and essentially transcendental step one side is adopted as the positive half and the other as the negative. Even when the X-axis is agreed upon, the convention for Y's remains arbitrary. The question of sense is not settled by symmetry. The choice adopted makes the equations for rotation of axes appear in the unsymmetrical form

$$x' = \quad x \cos \theta + y \sin \theta,$$

$$y' = - x \sin \theta + y \cos \theta,$$

rather than in the other possible but equally unsymmetrical form. Seeing what advantages are secured by this arbitrary choice, it is but natural that similar advantages be expected from an arbitrary convention with respect to distances throughout. The entire theory of a coördinate system is to regard given distances as associated with opposite signs when measured in opposite directions, and more than this, to select one direction in each case as positive. Now a line separates the rest of the plane into two parts. If a point be on one side, it cannot move continuously to the other side without passing across the line. What is more obvious than to assign to the distance an algebraic sign, such that one side is reserved for positive distances and the other for negative, these being separated by the line itself whose points are at zero distance? This convention is made in nearly all, if not all, standard American textbooks in analytical geometry. The choice used in the various particular cases is easily suggested by the fact that the origin is always present as a datum point for any line not containing it, and for lines through the origin, a convention consistent with that for the axes is desirable. The convention as to distance between points is usually the numerical determination already mentioned, since otherwise the points of a circle might not all be equally distant from the center.

The transcendental conventions are open to very serious objections, however. For example, the distance of a point from a straight line even if it be selected for conve ence as an axis of reference is not independent of the rotation of axes,

despite the explicit statements in most of the books concerning the invariance of distance, since the sign changes in certain cases. But consider some selected examples:

1. Find the locus of a point whose distance from the origin is equal to its distance from the Y-axis.

The locus consists of the origin and the positive half only of the axis of X's. Points on the negative half of the X-axis have the opposite sign.

2. Find the locus of a point the sum of whose distances from the two axes is equal to its distance from the origin.

The locus consists of the two positive half axes with the origin.

3. Find the locus of a point at a given constant distance from a straight line.

The result of this problem is somewhat involved. If the equation of the line as well as the sign of the distance be known, there is but one answer: a single line parallel to the given line and on the determinate side of it. If the equation of the line is not given, one of two possible lines is clearly the answer, but which one? The transcendental theory prohibits two answers, and requires that one be selected in a prescribed manner, but does not furnish the data. If a student does as he is told and arbitrarily selects the coördinate system in some convenient way, he will obtain a unique and supposedly correct answer. But two different students may each assert that his own solution is unique and correct, while these may correspond to opposite determinations of sign.

4. Find the locus of a point such that the distance between the point and a given line is a given constant.

This is merely a catch question and has no answer, because the distance is either from the line to the point or from the point to the line and these are distinct. There is no single distance, unless it be zero, between the point and the line.

5. Find the locus of points equidistant from two given intersecting lines.

This problem is impossible unless the position of the origin be known, and should the origin be on one of the lines, the position of the axes must also be given. These data known, the answer is a single line.

6. Find the center of the inscribed circle of a given triangle.

This is but occasionally a point equidistant from the three sides. This being a point determined by the figure and not by the coördinate system, all special transcendental conventions are useless. The point equidistant from the three sides by the transcendental convention, may be tested algebraically or graphically and has roughly only one chance out of four of being the center of the inscribed circle.

7. Find the locus of points equidistant from two given parallel lines.

The locus consists of no points, unless the origin should fall within the two lines, or in certain cases upon one of the lines.

8. Given AB perpendicular to BC, find the distance from A to BC.

This problem is impossible unless the position of the origin be known. The distance may or may not be equal to the positive length AB.

9. Calling the locus of points one unit distant from a given line, the line one

unit distant from the given line, find the envelope of the lines one unit distant from the lines through the origin.

The envelope is not a circle, but half a circle.

10. Given a fixed point, (x_0, y_0), other than the origin. Any line through (x_0, y_0) will have an equation of the form, $(x - x_0) \cos \theta + (y - y_0) \sin \theta = 0$. Determine the character of the distance from any point, (x', y'), to the line, $(x - x_0) \cos \theta + (y - y_0) \sin \theta = 0$, as θ varies.

The distance is a periodic function of θ. If (x', y') is not collinear with the origin and (x_0, y_0), the distance has a finite discontinuity as $\tan \theta$ passes through the value $- x_0/y_0$. When (x', y') is on the line joining (x_0, y_0) and the origin, the distance is constantly positive or constantly negative, depending upon the order of the points $(0, 0)$, (x', y'), (x_0, y_0), on this common line.

The transcendental definition has the great advantage of emphasizing the fact that the sign of the distance from a line may be made of use in determining whether or not two points given by their coördinates are on the same or opposite sides of a given line, but in so far as it specifically determines a positive or negative sign, there seems to be no advantage derived. One might protest in behalf of the fundamental conventions of analytical geometry with regard to cöordinates. A reply is, however, obvious. The coördinates may be held to involve more than distances, they involve directions and this distinction is regularly observed at the start. The simpler geometric notions of distance are independent of the position of coördinate axes, and only suffer by special conventions attempting to relate them.

3. **Algebraic Determination.** As is usual in algebra one can regard distances as inherently either plus or minus as far as algebraic distinctions are concerned. It is possible to explain and emphasize the fact that in any particular example a fixed sign may be given to the expression, which sign is selected being clearly immaterial, and so the fact as to whether two points are on the same or opposite sides is readily determined. Mention may be made with regard to the convenient character of the origin as a reference point when it happens to be given. At the same time any complete answer to an exercise must take into consideration the possibility that the student has a free choice in sign and the solutions will be as numerous as the possible combinations of signs. This is just what is done throughout in algebra. For example, if $x^2 - 3x + 2 = 0$, be an equation proposed and it is desired to find the solution x intended by the proposer, the problem is not possible algebraically. One can only say that x must have been either 1 or 2, and more cannot be determined except possibly by other methods. This does not mean that the proposer had necessarily these two solutions in mind. He may have intended one or the other or both, but which possibility cannot be found by algebra alone if at all. Thus the locus of points at one unit's distance from a given line, may be viewed as a single line on one side, or one on the other side or as both lines. The only legitimate answer appears to be a pair of lines, with the understanding that either one alone might be the answer intended by special conventions. The algebraical determination has all the completeness of

the numerical determination, and all the flexibility of the transcendental. It has also its own objections, however. The algebraical determination always gives complete algebraic curves instead of portions as sometimes given by each of the other conventions. Its disadvantage is that plus a distance and minus a distance are not distinguished. Let us examine some instances.

1. Find the locus of a point such that the sum of its distances from two given points is a constant.

The solution is either an ellipse, an hyperbola, or a straight line, according to the relation of the constant to the distance between the two given points. By the numerical definition, the ellipse and part of the line correspond to sums, and the hyperbola and the rest of the line to differences, the two branches of the hyperbola, and the two rays of the line corresponding to the two choices of subtrahend and minuend.

2. Find the locus of a point such that the sum of its distances from the rectangular axes is equal to unity.

The locus is four complete straight lines, viz.,

$$x + y = 1, \qquad x - y = 1, \qquad -x + y = 1, \qquad -x - y = 1,$$

parts of the lines corresponding in the numerical definition to distances.

3. Find the center of the inscribed circle of a given triangle.

The three lines of the triangle have each two sides which are half planes. The two sides may be at once distinguished algebraically, if the opposite vertex be in each case compared with any point under consideration. Thus of the four points obtained as equidistant from the lines, that one which is simultaneously on the same side of each line as the opposite vertex is the center of the inscribed circle. If an origin be given not on any of the lines, or any other point be convenient, this point might be used as reference for all three lines if preferred.

Conclusion. No teacher or good student is badly misled by the transcendental conventions as to distances in the analytical geometry textbooks. He uses the convention when convenient and rejects it otherwise. The definitions there given are, however, baffling to the poor student, and either examples must be skilfully selected in the textbook or by the teacher to avoid trouble, or else the conventions which have been so carefully drilled into the memory of the unreasoning must be denied at just those moments when the pupil feels most helpless. It seems time to reject the transcendental concepts which are neither simple nor tenable, in behalf of either the obvious but difficult numerical determination or the simple but perhaps over-inclusive algebraic determination. The author realizes, however, that this cannot be a new question and it is one which must have often been examined by teacher and by textbook writer. A reply in defense of the accepted methods might prove of general interest.

April, 1919.

III. Probabilities in the Game of "Shooting Craps."

By Bancroft H. Brown, Harvard University.

Webster's *International Dictionary* gives the following explanation: "The caster throws or 'shoots' the dice, and wins if the throw is 7 or 11 (called a *nick* or *natural*), but loses if it is 2, 3, or 12 (called a *crap*). If 4, 5, 6, 8, 9, or 10 is thrown it becomes the caster's 'point,' and the caster continues to throw until he wins, by again throwing his point, or loses, by throwing 7. The odds are 251 to 244 against the caster."

This probability and the incidence of various points, craps, and naturals have been tested in a series of 9,900 games. Theoretical and actual incidences follow:

		Theoretical.		Actual.	
		Won.	Lost.	Won.	Lost.
Naturals	7	1,650		1,635	
	11	550		553	
Craps	2		275		259
	3		550		508
	12		275		293
Points	4	275	550	267	565
	5	440	660	481	662
	6	625	750	567	787
	8	625	750	620	738
	9	440	660	451	685
	10	275	550	297	532
Totals		4,880	5,020	4,871	5,029

In the further discussion of this game, use is made of the following theorem, which is believed to be new:

Theorem: *In any series of games where the probability of winning is constantly p, the average number of games won, up to and including the first lost game, is the reciprocal of the probability of losing.*

For the average number, A, is equal to the limit of the sum of the various probabilities that the first $(n-1)$ games are won and the nth lost multiplied by n, where n increases without limit, that is:

$$A = 1\cdot q + 2qp + 3qp^2 + \cdots + nqp^{n-1} + \cdots \text{ (where } q \text{ is the probability of losing)}$$

$$= q\cdot \sum_0^\infty (n+1)p^n, \qquad 0 < p < 1,$$

$$= q\cdot \left[\sum_0^\infty p^n\right]^2 = q\cdot\left[\frac{1}{1-p}\right]^2 = \frac{1}{q}$$

We employ this theorem to determine the average number of rolls needed to decide a game. For example, if "6" is a point, the point is won or lost as "6" or "7" is thrown, and since of the 36 ways in which 2 dice may fall, 5 ways give "6," and 6 give "7," the probability of deciding the point on the next cast is 11/36. Hence 36/11 throws, on the average, are required to decide the point "6." Exactly the same reasoning holds if "8" is the point. For "5" or "9,"

A is 18/5; for "4" or "10," A is 4. The average of these, allowing for their relative incidence is $3\frac{31}{55}$; and the average number of rolls per game (including those games won or lost by naturals or craps) can now be shown to be $3\frac{62}{165}$ In general, we may say that in 557 rolls of the dice: 55 are naturals or craps, 110 make points, and 392 complete these points.

This result may be shown independently of the theorem quoted. Let x equal the average number of rolls which are needed to complete a point *inclusive* of the first roll. Then in N games, where N is very large:

$$\frac{N}{3}+\frac{2Nx}{3} = N', \qquad \text{where } N' \text{ is the total number of rolls,}$$

$$= 6\cdot s, \qquad \text{where } s \text{ is the total number of "7's" rolled,}$$

$$= 6\cdot\left(\frac{N}{6}+\frac{N}{9}+\frac{2N}{15}+\frac{5N}{33}\right)$$

whence x equals $4\frac{31}{55}$, and the conclusion follows as above.

According to best Army traditions, the caster retains the dice, and has the privilege of naming the size of the next bet as long as he wins, and also if he throws a crap; giving the dice and the privilege to his opponent only if he loses a point. The probability that he holds the dice is obviously 244/495 augmented by the probability of throwing a crap (4 chances in 36), that is, 299/495. The average number of games which he will roll is then 495/196 or 2.5255, of which he will win 1.2449 games, and lose 1.2806 games.

De Morgan[1] indicated the incidence of unusual runs of luck in several series of 2048 trials where the probability p was 1/2. In this question of holding the dice we have a probability of 299/495, practically 3/5. The first 2,471 games in the test mentioned above comprise exactly 1,000 turns, and an integral approximation of the theoretical together with the actual incidence is tabulated below:

	Theoretical.	Actual.
Total number of games	2,526	2,471
Total number of turns	1,000	1,000
Turns ending with 1 game	396	423
" " " 2 games	239	210
" " " 3 "	145	142
" " " 4 "	87	93
" " " 5 "	53	70
" " " 6 "	32	23
" " " 7 "	19	14
" " " 8 "	12	9
" " " 9 "	7	3
" " " 10 "	4	6
" " " 11 "	2	1
" " " 12 "	2	4
" " " 13 "	1	2
" " " 14 " (or more)	1	0
	1,000	1,000

CAMP DEVENS, MASS., May 5, 1919.

[1] Augustus de Morgan, *A Budget of Paradoxes*, 2d ed., 1915, Vol. 1, pp. 281–3.

RECENT PUBLICATIONS.

REVIEWS.

Elementary Mathematics for Field Artillery.

Elementary Mathematics for Field Artillery. By Lester R. Ford. (Prepared and published by direction of the Chief of Field Artillery.) Field Artillery Central Officers' Training School, Camp Zachary Taylor, Kentucky (Instruction Memo. No. 20.). Louisville, Ky., Courier Journal Press, 1919. 8vo. 72 pp.

Educators have recently become critical. Before deciding how a subject ought to be taught, it is now customary to question whether it ought to be taught at all, and to admit that the aim to be achieved may have an influence upon the methods to be employed. Elementary mathematics has been under fire in many quarters, and while the word "mathematics" has long ceased to be used as a plural noun efforts are but now being exerted in numerous directions to impress upon the pupil the unity of the science, particularly in coördinated freshman courses.

The war presented an exceptional although fleeting opportunity to emphasize the fundamental necessity of mathematics in the technical equipment of the individual. This was most conspicuous perhaps in the case of artillery and naval units. The student officer was required to learn and use many mathematical ideas that he had never acquired, or had frequently long forgotten. The result at least in many of the training camps was rather pitiable. The instructors themselves usually knew the formulas, but not the reasoning, and blind confusion was the rule rather than the exception among the students. In the S. A. T. C. program, the situation was nearly as bad. Even had classes not been interrupted, the students would never have acquired many of the essentials, for two reasons: much of the time was spent with unnecessary details, and again the instructors were not prepared in many cases to give what was needed. They did not know what concepts were employed, what problems were to be faced, and frequently what were the mathematical facts of the subjects even were these named. Such terms as mil, grad, deflection, azimuth, dispersion diagram, probable error, center of impact, nomogram, or alignment chart, meant little or nothing to many college instructors in mathematics, while most college graduates could not approximate a square root by the algorithm, determine visibility on a map with the contour lines given, interpolate with second differences, and the like. But all of these things had to be taught, not to a few only but to many. Not having been given even in a college course, these topics had to be learned from men of action rather than teachers, from soldiers who had never been concerned with mathematical theory.

It was in the face of an emergency of this sort that the present pamphlet

was prepared and issued, and although the military urgency has passed there is much suggestiveness in this little text-book, for the teacher of collegiate mathematics. The reviewer has learned that "about 15,000 students took the course in the three months prior to the signing of the armistice. The staff of instructors, recruited chiefly from the candidates who had mathematical training, was ordinarily well over a hundred, and at one time numbered as many as one hundred sixty-nine."

This pamphlet comprises sections entitled "arithmetic," "algebra," "geometry," "trigonometry," "approximate methods," "coördinates," "aids to calculation," and "probability," and has also an appendix of tables.

In the section on arithmetic the most interesting features emphasized are the conversion of metric into English units and *vice versa*, various units of angular measure, mil, grad, degree, the process of extracting the square root, and interpolation. The algebra and geometry sections cover customary topics; the geometry section is particularly rich in theorems, but many of the proofs are omitted. Under trigonometry, seven pages altogether, the sine, cosine, tangent, and cotangent are mentioned, but not the secant or cosecant. The law of sines and the law of cosines are derived and applied, but triangles with three sides given are not discussed and the law of tangents is not mentioned. Under "aids to calculation," logarithms, the slide rule, and a nomographic logarithmic table are explained. Under the topic of probability, the probability curve and the dispersion diagram are illustrated and applied. The tables are to four decimal places, and comprise (I) square roots of numbers, (II) logarithms of numbers, (III) natural trigonometric tables—mils, (IV) logarithmic trigonometric tables—mils, (V) natural and logarithmic tables—degrees.

This memorandum as an official publication contains of course neither preface nor introduction, but was obviously written with a single purpose in view—to serve as a text in teaching the field artillery officer the mathematics he must know. No topic that is necessary is omitted on account of its difficulty, nor is any topic of clearly minor importance for this object introduced because of its symmetry or historical interest. Nearly every problem mentioned in the exercises bears on its face the answer to the common query, "But why must I study this?" The work is written for adults, and is so condensed as to serve well for a reference book and source of problems but scarcely as a guide for self instruction of the mentally deficient. From a logical point of view, the book is surprisingly satisfactory. The definitions are clear and cover the terms used, the arrangement is progressive, where proofs are omitted there is no attempt to conceal the fact, processes are always explained in general terms, as well as applied to numerical examples, emphasis is placed on the operations used rather than on formulas—in short the book is written by a mathematician. In his experience in teaching from American text books designed for standard first year college courses, the reviewer has seldom found a treatment with so few logically objectionable elements; of the numerous mathematical memoranda he has seen, used in the army training schools, this is the first which shows ease and clarity of treatment.

A few of the exercises may be cited:

1. Near the muzzle of the three-inch gun the rifling makes one turn per 25 calibers. If the projectile leaves the muzzle with a velocity of 1,700 feet per second, how many rotations about its axis does it make per second? Ans. 272.

2. The total length of the French 155 mm. cannon is 2.332 m. Express this in calibers. Ans. 15.05.

3. In rising above the earth's surface the temperature falls about 6° C. for each kilometer of altitude up to eleven kilometers and thereafter remains practically stationary. What increase in elevation (to the nearest ten feet) will lower the temperature 1° F.? Ans. 300 ft.

4. For a range of 3,000 yards the 3-inch gun is elevated at an angle of 5° 5.3′ above the line joining gun and target. When fired the projectile actually leaves the gun at an angle of 92.3 mils about this line. How much is the former angle increased by the "jump" of the gun as it is fired? Ans. 1.8 mils.

5. A scout measures the angle between a line to an object C and the straight road along which he is passing. He proceeds along the road until, 1,300 yards farther, a line to C makes with the road an angle twice as great. How far is he then from C? Ans. 1,300 yds.

Note. This is the method used by sailors in "doubling the angle on the bow."

6. A reconnaissance officer has an accurate map of the surrounding region but does not know his exact position on the map. He sees two objects A and B, which are shown on the map. He measures the angle between the two points and draws lines through A' and B', the map positions of the two points, to meet at the angle found at some point C' on the map. Show that the true position is on the circle drawn through A', B' and C'. If he recognizes a third point whose map position is known he can get a second circle on which his position lies. His true position is at the intersection of these circles.

Note. This is the method of "Italian Resection" used to locate a position with a plane table.

7. In maps based on English units 1 in. to the mile, 3 in. to the mile, 6 in. to the mile, and 12 in. to the mile, are commonly used scales. Find the representative fractions of these maps? Ans. (*a*) 1/63360, (*b*) 1/21120, (*c*) 1/10560, (*d*) 1/5280.

8. Find the area of a shrapnel pattern 100 yds. long and 35 yds. wide. Ans. 2,749 sq. yds.

9. Find the danger space for a horse 15 hands high when the trajectory makes an angle of 252 mils with the ground. Ans. 20 ft.

10. An observer 2,000 yds. from the battery and 3,540 yards from the target finds the angle between the battery and the target to be 114 degrees. Find the range from the battery to the target. Ans. 4,721 yds.

11. Find the range and deflection components of a wind of ten miles per hour blowing from the southwest if the target is due east of the battery.
Ans. range component = deflection component = 7.1 mi. per hr.

12. How far is the B. C. station from the right gun if the wheel of the gun carriage, which is 56 in. in diameter, subtends an angle of 4 mils? Ans. 390 yds.

13. Of ten shots fired at a range of 2,500 yds. eight are to the right of the target and two to the left of it. What change should be made in the deflection? Ans. Increase 7 mils.

Only five or six misprints were noted.

ALBERT A. BENNETT.

UNIVERSITY OF TEXAS,
May, 1919.

A First Course in Calculus. By W. P. MILNE and G. J. B. WESTCOTT. Part 1: Powers of x. London, G. Bell & Sons Ltd., 1918. 12mo. 20 + 196 pp. Price 3s. 6d.

This little book (namely part 1) duodecimo, and only about half an inch thick, covers roughly a half of what is usually found in American texts on the calculus. The treatment is very well graded, the important ideas are approached gradually and developed at length. Long or troublesome proofs are avoided throughout.

This is made possible by deferring or omitting such topics as the differentiation and integration of transcendental analytic functions, the general evaluation of indeterminate forms, the evolute and involute, general criteria for maxima and minima, mean value theorems, partial differentiation, envelopes, series, singular points and asymptotes, hard definite multiple integrals, and the like; secondly, by explicitly assuming formulas without proof in certain cases, as in physical formulas, and geometric formulas, for example that for radius of curvature, etc.; thirdly, by outlining rather than completing the method of proof in a few instances in which the details of rigor would not appeal to the pupil. The authors state in the preface "there is no attempt whatever to discuss the philosophy of 'limits' and such like" and "It is hoped, therefore, that the treatment will be found suitable for boys and girls in the upper classes of the Secondary Schools, for the pupils in Army Classes, and for students in the Technical schools and colleges."

The book is exceedingly rich in interesting and suggestive exercises, a feature characteristic of most British texts. These exercises are perhaps even too numerous to be useful. They show a strong inclination to be "practical" as is evidenced by the fact that coefficients are seldom selected as integers, and results do not "come out exact" but are usually desired correct to two decimal figures, and that instead of remaining within the subject of algebra, they roam over all branches of elementary physical science. A student completing the course will have not only a facility in handling elementary derivatives, definite integrals and a few types of differential equations, but should have a fair grasp of the significance of many of the laws of mechanics. The print and the make-up of the book are attractive and misprints appear to be rare. The figures are poorly drawn; the perspective ellipses, pages 134, 135, are obviously drawn as circular lunes with sharp corners. The parabola (Fig. 68) has a point of inflection.

A complete set of answers and a usable index are found in the back of the book, and an historical introduction of an entertaining nature opens the text. The wholesale acceptance by the authors of J. M. Child's view that Barrow was the discoverer of the calculus is interesting, but the authors do not pretend to attempt to lend scholarly weight to this view, nor to enter into polemics. They accept what has appeared in print without being sufficiently interested to give the exact reference for their quotation. The book is one of a series of which the best known to American readers is undoubtedly Sommerville's *Elements of Non-Euclidean Geometry.*

It is unfortunate that in their quest for methodological excellence, the authors have paid so little regard to questions of exactness in phraseology. One can hardly read a half dozen consecutive pages without noting some instance of looseness in the language or notation employed. This lack of precision while always irritating is in most of the individual instances trifling, and merits mention only on account of its frequency. Unlike the situation familiar to us in numerous elementary American text-books, the current ambiguity is usually avoided in the more important definitions and explanations. But one or two instances need be cited of features almost characteristic of the book. After giving a satis-

factory definition of maximum, and after having previously treated irregular physical curves, the statement is made (page 30), "then at points on the graph at which either y is a maximum or a minimum, the tangent is parallel to the axis of x." The possibility of a maximum being either a corner point or an end value, while illustrated in figures used in other connections (*e. g.*, Fig. 27, Fig. 46 and Fig. 57) is not even acknowledged throughout the whole book. Again, the sixth chapter bears the title "Differential Equations and Indefinite Integrals," but except for their place in the title the terms "integral," and "indefinite integral" do not appear in the text of the chapter, although assumed as familiar, page 68, example 9.

The term "derivative" appears nowhere in the book, the phrase "differential coefficient" being defined instead, although this in turn is casually referred to as "the differential" in example 40, page 35. The notion of a differential as distinguished from a derivative, according to the usage familiar to American readers, is never suggested.

It has been a tradition that even an approach to the meaning of a differential equation is in the nature of things inaccessible to a student who has not been drilled in so elementary a subject as trigonometry. This tradition is completely ignored in the present little treatise. Here a student with but the rudiments of algebra may easily acquire a grasp of what is meant by a differential equation, a definite integral and other fundamental terms, although applying these only to the "powers of x." It is true that the free use of technical nomenclature in some of the exercises might more than appal the American college freshman whose mathematical foundations have carefully shielded him from the harsh realities of "applications." Just how successfully from the pedagogical viewpoint the subject of the calculus may be split in a not unfamiliar fashion according to the functions used rather than the fundamental concepts employed, can only be examined in the light of the whole subject, and so far as this work is concerned judgment must be withheld since part two is not yet published. The efficiency of the part here at hand however raises squarely the issue as to whether the claims of the calculus as a freshman subject can be ignored even in institutions whose students are generally poorly prepared.

To the attention of all college teachers interested in the closer relationship between elementary mathematical technique and physical applications, as taught with simplicity and directness, this little book is recommended.

ALBERT A. BENNETT.

Des phénomènes gyroscopiques et de leurs principales applications à la navigation. By A. LUCAS. Paris, Challamel, 1918. Royal 8vo. 110 pp. Price in boards 7.80 francs.

Contents—Chapter I, Préliminaires, 1–20: Projections de la vitesse et de l' accélération d'un mobile sur trois axes rectangulaires; Equations du mouvement d'un point matériel et d'un corps solide libre; Mouvement d'un corps solide par rapport à des axes animés d'un mouvement de translation avec lesquels il est entraîné; Rotations; Composition des rotations; Problème sur le mouvement d'un corps ayant un point fixe; Moments d'inertie; Mouvement d'un corps

solide autour d'un axe fixe. Chapter II, Corps de révolution en rotation autour de son axe, 21–38: Problème sur les projections de l'accélération d'un point matériel; Effets produits par les accélérations; Etude de l'effet combiné de deux rotations; Généralisation pour un corps libre de révolution; Effets des forces qui agissent sur un corps en rotation; Etude analytique de l'effet du couple P; Cas ou le corps de révolution a un point de son axe fixe. Chapter III, Applications, 39–107; Gyroscope de Foucault; Gyroscope-marin de E. Dubois; Boussole gyroscopique de Foucault; Compas gyroscopique; Perturbations produites à bord sur l'appareil; Influence de la variation de vitesse du tore; Horizontalité de la rose; Amortissement; Compas gyroscopique Sperry; Emploi du gyroscope sur les torpilles; Projectile oblong; Toupie ordinaire; Balance gyroscopique; Toupie gyroscopique; Gyroscope-collimateur de l'amiral Fleuriais; Effet des frottements; Effet des mouvements du navire; Effet de la rotation de la terre; Utilisation des principes précédents pour déterminer à bord avec un sextant la hauteur d'un astre; Erreur d'horizontalité ou collimation du repère; Autre procédé d'observation; Mouvement de la terre autour de son centre de gravité; Effets gyroscopiques sur les turbines; Gyroscope stabilisateur.

Rehabilitation Monographs. Joint Series No. 26. Unit Course.—Mathematics I. Use of the Slide Rule. [By W. E. BRECKENRIDGE.] Issued by the Federal Board for Vocational Education in coöperation with the Surgeon General's Office and the Bureau of War Risk Insurance. February, 1919. (Trial Edition.) Washington, Government Printing Bureau. 27 pp.

Extract from the preface: "For the purpose of insuring a continuous program of education for wounded and sick soldiers during the time they are in the general hospitals and after their discharge the Surgeon General's Office and the Federal Board for Vocational Education coöperated in the preparation of a series of courses of study. These courses are tentative and suggestive. They may, however, be accepted as models so far as their form and principles which govern their organization are concerned, for all courses whether academic or vocational, where the adjustment of materials is intended to be made to suit the individual capacities and attainments of students."

NOTES.

Il Bollettino di Matematica, 1917–18 (anno 15), contains a sketch of Giuseppe Veronese by P. Gazzaniga (pages 53–65), and an Italian translation with elaborate notes by M. Domenico, of the first of J. W. Young's lectures on *Fundamental Concepts of Algebra and Geometry* (pages 161–183). The complete translation of this work into Italian was referred to in our issue for June.

Vuibert, of Paris, has announced that publication of the following mathematical periodicals is to be resumed this month: *L'Education Mathématique* (which last appeared in July, 1914) and *Revue de Mathématiques Spéciales* (last published in September, 1914).

In our issue for April we had occasion to refer to the publication of the first five parts of *Materialien für eine wissenschaftliche Biographie von Gauss*, edited by Klein, Brendel and Schlesinger. In the latter part of 1918 a sixth part, of 46 pages, by P. Maennchen, was published. It was entitled: *Die Wechselwirkung zwischen Zahlenrechnen und Zahlentheorie bei C. F. Gauss.*

To take the place of *Minerva*, long sent out from Germany, Gauthier-Villars advertises the first volume of an annual, published under the direction of the mathematician Dr. R. de MONTESSUS DE BALLORE, with the following title: *Universitatum et Eminentium Scholarum, Index Generalis. Annuaire Général des Universités. The Yearbook of the Universities.* (Paper, 18 francs; bound, 21 francs.)

The second edition, revised and enlarged, of F. CAJORI's *History of Mathematics* was published in August (Macmillan, price \$4.00)—Teacher's Manual, *First Course in Algebra* by W. B. FORD and C. AMMERMAN appeared in July (Macmillan, price \$2.00)—Recently announced thoroughly revised editions: *College Algebra* by H. L. RIETZ and A. R. CRATHORNE (Holt, price \$1.60) and *Complete School Algebra* by H. E. HAWKES, W. A. LUBY and F. C. TOUTON (Ginn, price \$1.40)—In July the University of Chicago Press published E. R. Breslich's *Correlated Mathematics for Junior Colleges* (price \$1.25), "comprising college algebra, analytic geometry and some differential calculus." This work is the fourth volume in Breslich's course of mathematics for high schools—In April Heath brought out *New High School Arithmetic: Academic, Industrial, Commercial* by W. WELLS and W. HART (price \$1.20).

The Education of Henry Adams. An autobiography, was privately printed, to the number of one hundred copies, in 1907. In September 1918 it was published for the general reader by the Massachusetts Historical Society and became one of the most notable books of the year. The following quotations are made;

"In any and all its forms, the boy detested school, and the prejudice became deeper with years. He always reckoned his school days, from ten to sixteen years old, as time thrown away. Perhaps his needs turned out to be exceptional, but his existence was exceptional. Between 1850 and 1900 nearly everyone's existence was exceptional. For success in the life imposed on him he needed, as afterwards appeared, the facile use of only four tools: Mathematics, French, German, and Spanish. With these he could master in very short time any special branch of inquiry, and feel at home in any society" (page 38).

"The four years passed at college were, for his purposes, wasted. Harvard College was a good school, but at bottom what the boy disliked most was any school at all. He did not want to be one in a hundred—one per cent. of an education. He regarded himself as the only person for whom his education had value, and he wanted the whole of it. He got barely half of an average. In the one branch he most needed—mathematics—barring the few first scholars, failure was so nearly universal that no attempt at grading could have had value, and whether he stood fortieth or ninetieth must have been an accident or the personal favor of the professor. Here his education lamentably failed. At best he could never have been a mathematician; at worst he would never have cared to be one; but he needed to read mathematics, like any other universal language, and he never reached the alphabet" (pages 59–60).

"Socially or intellectually, the college was for him negative and in some ways mischievous. . . . The habit of looking at life as a social relation—an affair of society—did no good. It cultivated a weakness which needed no cultivation. If it had helped to make men of the world or give the manners and instincts of any profession—such as temper, patience, courtesy, or a faculty of profiting by the social defects of opponents—it would have been an education better worth having than mathematics or languages" (page 65).

"He supposed that, except musicians, everyone thought Beethoven a bore, as every one except mathematicians thought mathematics a bore" (page 80).

"Few men in Washington cared to overstep the school conventions, and the most distinguished of them, Simon Newcomb, was too sound a mathematician to treat such a scheme seriously. The greatest of Americans, judged by his rank in science, Willard Gibbs, never came to Washington, and Adams never enjoyed a chance to meet him. . . . By chance it happened that Raphael Pumpelly . . . fell to talking with Adams about these matters [race and sex], and said that Willard Gibbs thought he got most help from a book called the *Grammar of Science* by Karl Pearson. To Adams's vision, Willard Gibbs stood on the same plane with the three or four greatest minds of his century, and the idea that a man so incomparably superior should find help anywhere filled him with wonder. He sent for the volume and read it. . . . Here came in, more than ever, the fatal handicap of ignorance in mathematics. Not so much the actual was needed, as the right to judge the product of the tool" (pages 377, 449).

satsen" by V. Lenander, 68–71; "Relativitetsprinciperna" by H. Fixén, 82–100; Mathematical questions and solutions, 118–128—No. 3, March, 1919: "Om seriesummeringar" by A. Meyer, 129–158; "Tillämpad matematik i skolans astronomikurs" by C. Lönnquist, 159–173; Mathematical questions and solutions, 181–192.

Transactions of the American Mathematical Society, volume 20, no. 2, April, 1919: "Memoir on the general theory of surfaces and rectilinear congruences" by G. M. Green, 79–153; "Modular concomitant scales, with a fundamental system of formal covariants, modulo 3, of the binary quadratic" by O. E. Glenn, 154–168; "Concerning a set of postulates for plane analysis situs" by R. L. Moore, 169–178; "On the limit functions of sequences of continuous functions converging relativity uniformly" by E. W. Chittenden, 179–184.

AMERICAN DOCTORAL DISSERTATIONS.

E. D. Grant, 1873– , *Motion of a flexible cable in a vertical plane.* Lancaster New Era, 1918. 4to. 28 pp. (Chicago, 1916).

J. E. McAtee, —— - 1918, *Modular invariants of a quadratic form for a prime power modulus.* [Reprinted from *American Journal of Mathematics*, volume 41, 1919]. Pp. 225–242. (Chicago, 1917).

UNDERGRADUATE MATHEMATICS CLUBS.

Edited by U. G. Mitchell, University of Kansas, Lawrence.

CLUB ACTIVITIES.

Denison Mathematics Club, Denison University, Granville, Ohio.
[*1918*, 403-04.]

The Denison Mathematics Club had a somewhat unique scheme for their first annual banquet which was held on May 20, 1919. The Club walked in a group about a mile to a farmhouse where the meal was served. The twenty-five who attended were seated at two long tables. In the center of each table was a vase holding a large bouquet of various mathematical symbols, cut from white paper, each having a stem of slender wire. Cardioids cut from green paper and mounted in the same way supplied the leaves. The candle shades were encircled with simple equations cut in the cardboard of the inner shade which formed the support for the outer tissue shade, so that, when the candles were lighted, these equations stood out distinctly from their darker surroundings. Other symbols were suspended by means of fine thread from the lower edges of the shades.

Below is given a cut of the menu and program pages.

Since the reader can not have the opportunity of checking his interpretation of the menu by the food served the following key is given:

Course 1.

n_1: Escalloped potatoes.
n_2: Sliced veal loaf.
n_3: Rolls with butter.
n_4: Olives.
n_5: Coffee with cream and sugar.
(The two things selected in advance were the water and the the coffee berry.)

Course 2.

n_1: Fruit salad with dressing poured over the top and the whole "over" a lettuce leaf.

n_2: Wafers.

Course 3.

n_1: Ice cream.

n_2: Cakes as parallelopipeds.

Course 4.

Mints.

Curriculum.

Course 1.

n_1: ∥ sections of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

n_2: Proper Fractions

n_3 Roulettes with applications for Slide (rule) Purposes.

n_4 Rounded Numbers

n_5 ${}_nC_r$, where $n \cdot 4$, $r = 2, 3, 4$ (two things selected in advance)

Course 2.

n_1: $\left\{\sum_{i=1}^{n} \Delta x_i \, \Delta y_i \, \Delta z_i \, \Delta u_i \Delta v_i + \int_{cap} dv\right\}$ $\times$ [letuce leaf]$^{-1}$

n_2 Parallelepiped volumes of dimensions: $a \times b \times \Delta z$

Course 3

n_1: H.C.F. at $+32°$ F

n_2: $[abc]$

Course 4

$x^2+y^2=r^2 + \rho = r.$

Class Hour

Time .... Moment of Inertia

Instructor .... R. H. Howe '20.

Reports due as follows

R_1 $I=f(t)$; $t=t_0$ ...... A Shumaker, '22

R_2 $I=f(t)$; $t=t_l$ ...... Ruth Phillips. '19.

R_3 Groups principally those defined by S. at. C." ...... Prof. K. D Swartzel, O.S.U.

R_4 "The First Derivative"—Prof. C. L. Arnold O.S.U.

R_5 The Unknown Quantity—Prof. F. B. Wiley.

We $\doteq$ Limit as $t \doteq 7^{\underline{30}}$;

then

translate back to origin.

The guests of honor were Professors Karl D. Swartzel and Charles L. Arnold of Ohio State University. Professor Swartzel was a district examiner for the S. A. T. C. which fact suggested the topic for his toast. Professor Arnold is the main sponsor of the mathematics club at the Ohio State University.

The copy of the program given above was lettered by Mr. Richard H. Howe '20, president of the Club, who acted as toastmaster at the banquet.

THE MATHEMATICS CLUB OF NORTHWESTERN UNIVERSITY, Evanston, Ill. [*1918*, 132–134, 409.]

The officers of the Club for the year 1918–19 were: President, Franklin Mohr '18; vice-president, Frank Danielson '18; secretary, Helen Kelly '19; treasurer, Margaret Walker '19; faculty adviser, Mrs. Mayme Irvin Logsdon, instructor.

The following programs were given during the year.

September: "Some mathematics of artillery fire" by Theodore Doll Gr.; Informal talk on radio telegraphy, by Franklin Mohr '18.

October: "The game of nim" by Mrs. Mayme Irvin Logsdon, Instructor.

November: "Mercator's projection" by William H. Burger, Professor of Civil Engineering.
December: "Applications of algebra to arithmetic" by Margaret Walker '19.
January, 1919: "Finger-counting among the ancients" by Christine McMartin '19; "A short-hand system for advanced mathematical analysis" by Gladys Williams '19.
February: "Rotating fluid bodies" by Professor Elton J. Moulton.
March: "Integral right triangles" by Helen Kelly '19; "Fermat's last theorem" by Professor David R. Curtiss.
April: "Some indefinite locations at sea" by Leonard Janes '17.
May: "Communication with other planets by means of mathematics" by Eleanor Olmstead '19; "Problem of two bodies under the action of gravity" by William Janes '19.

Near the close of the year, a very pleasant social gathering was held at the home of Professor Curtiss. The reading of humorous sketches and participation in guessing contests furnished a good part of the entertainment.

THE PENTAGRAM, University of Texas, Austin, Texas. [*1918*, 273–274.]

The *University of Texas Bulletin* for May 10, 1919, gives the following resume of the Pentagram's activities during the year 1918–19:

"The Pentagram, the mathematics club at the University of Texas, is in its third year. It has proved itself to be profitable and stimulating to all its members. The influenza epidemic caused the suspension of the activities of the club the greater part of the first half of the present academic year. The work was resumed, however, about the middle of the winter term and the usual interest is being taken by both students and faculty. Under the leadership of Professor Albert A. Bennett, who organized the club in 1916, and who has returned from government service, the Pentagram looks forward to even greater success than it has had thus far.

"The officers of the Pentagram for the year are: President, Bert McDonald '19; vice-president, Georgie Savage '20; secretary-treasurer, Goldie P. Horton, instructor in pure mathematics; student member of the executive council, Warren Simonds '20; faculty member of the executive council, Professor Albert A. Bennett."

The meetings of the year were as follows:

November 15, 1918: "Derivativeless continuous functions" by Professor M. B. Porter.
January 9, 1919: "Homogeneous coördinates" by Professor J. W. Calhoun.
February 18: Social meeting.
March 5: "The graph of $y = f(x)$ for complex values" by Alice Ballard '21.
March 21: "Continued fractions" by Ernest Normand '18.
April 4: "A comparison of formulas used in computing parabolic arches" by Warren Simonds '20.

THE MATHEMATICS CLUB OF GOUCHER COLLEGE, Baltimore, Maryland. [1918, 357–358.]

During the year 1918–19, Professor Florence P. Lewis, who has heretofore had charge of the Mathematics Club at Goucher, was Exchange Professor at Wellesley College and Misses Margaret Amig '19 and Marguerite Lehr '19 acted as program committee. Below is given a complete list of the programs given during the year.

December 16, 1918: "Report of the summer meetings of the Mathematical Association of America and the American Mathematical Society" by Professor Clara E. Smith, Exchange Professor from Wellesley College.

January 6, 1919: "Report of a meeting of the New England Association of Teachers of Mathematics" by Professor Florence P. Lewis.

January 20: "Map-making" by Marie Whelan '18.

February 3: "Empirical equations" by Professor Clara L. Bacon.

February 19: "Paper folding" by Margaret Amig '19.

March 5: "Persons and anecdotes in mathematics" by Ethel Carrol '20.

March 17: "History of the squaring of the circle" by Loretta Whelan '20; "Simple Hints on plotting graphs" by Varina Davis '20; "Magic squares" by Deldee Groff '20.

April 14: "Russian peasant method of counting" by Eleanor Norris '20; "The planimeter" by Louise Ellery '20 and Mary Brower '20.

May 5: "The golden mean" by Jean Burke '20; "Zeno's paradoxes" by Ethel Fox '19.

May 13: Picnic.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems to B. F. FINKEL, Springfield, Mo.

NOTE ON THE NUMBER OF SOLUTIONS OF LINEAR INDETERMINATE EQUATIONS.

BY D. N. LEHMER, University of California.

The solution of problem 2700 on page 215 of the May issue of the MONTHLY suggests the question as to whether a process may be found for obtaining, at least approximately, the number of solutions of an indeterminate equation in several variables. In the *Proceedings of the National Academy of Sciences*, April, 1919, I have given a method for getting, by a simple and direct process, the general form of the solution of any linear indeterminate equation in any number of variables. Such equations have been solved by Jacobi (*Werke*, Vol. 6, p. 355) but his process does not lead to the general form directly.

One finds for the equation $6w + 4x + y = 875$, which is the one treated in the Artemas Martin problem, the following general solution:

$$x = X - Y,$$
$$y = 875 - 4X - 2Y,$$
$$w = Y;$$

which give, since $w + x + y + z = 250$,

$$z = 3X + 2Y - 625.$$

All values of x, y, z, w, may be obtained from these by assigning integral values to X and Y. In order that x, y, z, w, shall be positive certain restrictions must be put on the values of X and Y. These restrictions are most easily understood when the pairs (X, Y) are taken as the coördinates of a point in the plane. It thus appears that the points (X, Y) available for our problem must lie in or on the boundaries of a quadrilateral enclosed by the four lines: $Y = 0$, $X - Y = 0$, $875 - 4X - 2Y = 0$, $3X + 2Y - 625 = 0$. The number of points with integer coördinates in this region will be approximately equal to the area of the quadrilateral which is found to be 2930. If Mr. Uhler's enumeration is correct this result is in error by 36. The actual number might be obtained by counting the number of lattice points in this quadrilateral.

The method applies as easily to problems involving a greater number of variables. Instead of areas one will have to consider volumes in space of three or more dimensions.

PROBLEMS FOR SOLUTION.

2784. Proposed by T. H. GRONWALL, New York City.

Show that all solutions in integers of $y^2 = 1 + x + x^2 + x^3 + x^4$ are given by

$$\begin{array}{llll} x = & -1, & 0, & 3; \\ y = & \pm 1, & \pm 1, & \pm 11. \end{array}$$

2785. Proposed by W. H. ECHOLS, University of Virginia.

If on the sides, as bases, of any closed plane polygon, there be constructed similar triangles similarly placed, all outward or all inward, then the centroid of the vertices of these triangles coincides with the centroid of the corners of the polygon.

2786. Proposed by R. A. JOHNSON, Hamline University.

At what angle should the face of a golf club (driver) be laid back, in order to secure the maximum distance of flight? Show that, under certain assumptions, the angle should be 15°.

2787. Proposed by WARREN WEAVER, University of Wisconsin.

In the gambling game known as "craps," two dice are thrown. The one throwing the dice makes a bet which is covered by an equal amount by one or more players opposing him. If he throws on the first throw a sum of seven or eleven he wins at once; if he throws a sum of two, three, or twelve he loses at once; if he throws any other sum he continues throwing until he either duplicates his original sum, and wins, or throws a sum of seven and loses. Show that the probability that the one who is to throw the dice will win is 0.49847, so that the game would be very nearly fair if one person were to throw the dice continuously instead of changing about, as is actually done.

Note.—Compare the article on "Probabilities in the game of 'Shooting Craps'" published elsewhere in this issue of the MONTHLY.—Editor.

SOLUTIONS OF PROBLEMS.

245 (Number Theory) [May, 1916; May, 1917]. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that $x^2 + y^2 = (a_1a_2 \cdots a_m)^n$ has $4(n+1)^m$ solutions in integers, in 2^{m+2} of which x and y are relatively prime, the a's being primes of the form $4k+1$ and n a positive integer.

I. Solution by Frank Irwin, University of California.

In Dirichlet's *Zahlentheorie* (4th edition, page 164) it is shown that the equation $x^2 + y^2 = M$ has 2^{m+2} solutions in integers x, y relatively prime, where M has m different prime factors (all of them being of the form $4k+1$). It may be expressly noted that the solutions, if any, in which x or y are zero are not counted in the number 2^{m+2}.

The solutions of our given equation in which x, y are not relatively prime may evidently be obtained by taking relatively prime solutions of equations of the form $x'^2 + y'^2 = N$, where N is the quotient of the right side of our equation by any square integer d^2; then $x = dx'$, $y = dy'$. The number of these solutions may be found as follows:

(i) n odd. N may contain a_i 1, 3, $\cdots$ or n times, $i = 1, 2, \cdots m$, and for each such choice of N we have 2^{m+2} solutions; altogether, then, $[(n+1)/2]^m 2^{m+2} = 4(n+1)^m$ solutions.

(ii) n even. Here N need not contain all the m a's as factors, and we must consider separately the cases thus arising. If N has m different prime factors, we get $(n/2)^m 2^{m+2}$ solutions; if N has $m-1$ different prime factors, we get $m(n/2)^{m-1}2^{m+1}$ solutions; if N has $m-2$ different prime factors, we get $\binom{m}{2}(n/2)^{m-2}2^m$ solutions; etc., etc.; that is, in all

$$4\left[n^m + mn^{m-1} + \binom{m}{2}n^{m-2} + \cdots + 1\right] = 4(n+1)^m$$

solutions.

II. Solution by C. F. Gummer, Queen's University.

It is well known (see, for instance, Barlow's *Theory of Numbers*, Chapter IX) that for every prime a_i of the form $4k+1$ positive integers x_i, y_i, relatively prime, may be found such that $x_i^2 + y_i^2 = a_i$. Let the even one be x_i, and let us write $x_i = \sqrt{a_i}\cos\theta_i$, $y_i = \sqrt{a_i}\sin\theta_i$. Then we can prove that a solution in integers of

$$x^2 + y^2 = a_1^{n_1}a_2^{n_2}\cdots a_m^{n_m} = A, \tag{1}$$

where the a's are primes of the form $4k+1$, is given by

$$x = \sqrt{A}\cos\Theta, \qquad y = \sqrt{A}\sin\Theta, \qquad \Theta = k\pi/2 + \sum_i r_i\theta_i, \tag{2}$$

where r_i is any one of the integers n_i, $n_i - 2$, $n_i - 4$, $\cdots$, $-n_i$, and k is an integer.

There are no solutions of (1) other than those included in (2). For let $x^2+y^2=x'^2+y'^2=B$, where B is odd, x, y, x', y' positive or zero, x and x' even, and $x > x'$. Since

$$\tfrac{1}{2}(x+x')\cdot\tfrac{1}{2}(x-x') = \tfrac{1}{2}(y'+y)\cdot\tfrac{1}{2}(y'-y),$$

and neither member is zero, we infer that every divisor of $\frac{1}{2}(x+x')$ is a divisor of either $\frac{1}{2}(y'+y)$ or $\frac{1}{2}(y'-y)$, and so for $\frac{1}{2}(x-x')$. Hence, we may write

$$\tfrac{1}{2}(x+x') = pr, \qquad \tfrac{1}{2}(x-x') = qs, \qquad \tfrac{1}{2}(y'+y) = ps, \qquad \tfrac{1}{2}(y'-y) = qr,$$

and consequently

$$x = pr + qs, \qquad y = ps - qr, \qquad x' = pr - qs, \qquad y' = ps + qr,$$

and

$$B = x^2 + y^2 = (p^2+q^2)(r^2+s^2).$$

Since no one of p, q, r, s can be zero, it follows that $x^2 + y^2 = B$ can admit two essentially distinct solutions only if B is composite, and therefore that the θ_i are unique. Also, on writing

$$p = \sqrt{C}\cos\theta, \qquad q = \sqrt{C}\sin\theta, \qquad r = \sqrt{D}\cos\phi, \qquad s = \sqrt{D}\sin\phi,$$

we see that if $x^2 + y^2 = B$ has two such solutions they are deducible from integral solutions of

$x^2 + y^2 = C$ and $x^2 + y^2 = D$ $(CD = B)$ by the rule $x = \sqrt{B}\cos(\theta \mp \phi)$, $y = \sqrt{B}\sin(\theta \mp \phi)$. If we allow for the interchange of x and y and for negative values, the rule becomes

$$x = \sqrt{B}\cos(l\pi/2 \pm \theta \pm \phi), \qquad y = \sqrt{B}\sin(l\pi/2 \pm \theta \pm \phi);$$

and on applying this method successively to A and its divisors we get the formula (2).

We wish to know how many distinct solutions are given by (2). Two solutions given by

$$\Theta' = k'\pi/2 + \sum_i r_i'\theta_i \qquad \text{and} \qquad \Theta'' = k''\pi/2 + \sum_i r_i''\theta_i$$

are equal only if

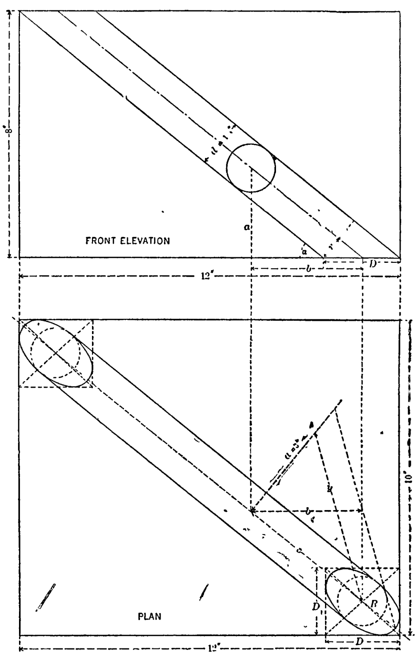
$$(k' - k'')\pi/2 + \sum_i (r_i' - r_i'')\theta_i$$

is a multiple of 2π. If this is so and $r_1' - r_1'' \neq 0$, it follows that

$$\cos(r_1' - r_1'')\theta_1 = \cos\left\{K\pi/2 + \sum_{i=2}^{m}(r_i' - r_i'')\theta_i\right\}.$$

But the left member, being a polynomial of degree $|r_1' - r_1''|$ in $\cos\theta$ with integral coefficients of which the highest is a power of 2, is a fraction with denominator $a_i^{|r_1'-r_1''|}$, while the right member cannot be of this form. Hence $r_1' = r_1''$. Two solutions are then identical only if $r_i' = r_i''$ $(i = 1, \cdots, m)$ and $k' - k''$ is a multiple of 4. The number of effectively distinct combinations of values of $k, r_1, \cdots, r_n$ is therefore $4(n_1 + 1)(n_2 + 1)\cdots(n_m + 1)$.

In some of the solutions given in (2), x and y will admit the common divisor $E = a_1^{s_1}a_2^{s_2}\cdots a_m^{s_m}$. These are found by multiplying by E the roots of the equation $x^2 + y^2 = a_1^{n_1-2s_1}a_2^{n_2-2s_2}\cdots$, and therefore occur in the cases where $|r_i| \leqq n_i - 2s_i$ $(i = 1, 2, \cdots m)$. The solutions have no common divisor when there is no number of the form E other than unity for which this condition holds, that is when $r_i = \pm n_i$ $(i = 1, \cdots m)$. The number of relatively prime solutions is therefore $4.2^m = 2^{m+2}$.



2696 [April, 1918]. **Proposed by L. E. LUNN, Heron Lake, Minnesota.**

An air pipe 18 inches in diameter passes diagonally through a room from one lower corner to the opposite upper corner leaving through elliptical openings in the floor and ceiling, so that the ellipses are tangent to two boundaries of the floor and to the two opposite boundaries of the ceiling. If the room is $10 \times 12 \times 8$ feet, find the remaining cubic capacity of the room.

Solution by A. R. Nauer, St. Louis, Missouri.

Make D the apparent width of the floor or ceiling contact as seen at the front elevation, and D_1 the same for the side elevation. Then

$$D = d/\sin\alpha = -\frac{27}{61.75} + \frac{\sqrt{468 \times 61.75 + 27^2}}{61.75} = 2.35025+$$

and

$$D_1 = -\frac{22.5}{61.75} + \frac{\sqrt{369 \times 61.75 + 22.5^2}}{61.75} = 2.10715+$$

a is the length of a perpendicular to the floor from an arbitrary point A on the center line of the pipe; a is here taken 3 feet.

r is the radius and d the diameter of the pipe.

R is the half major axis of the bounding ellipses, and

b is the distance, in front elevation from the foot of perpendicular a, to center of D, which is also the center of contact ellipses;

$$b = \frac{a}{8}(12 - D) = \tfrac{3}{8}(9.64975) = 3.61865, \qquad b^2 = 13.095 +$$

c is the distance from the foot of perpendicular a to center of ellipse.
y is the distance from A to center of ellipse.

$$c^2 = b^2 \frac{(12 - D)^2 + (10 - D_1)^2}{(12 - D)^2} = 21.8552 +$$

$$y = \sqrt{a^2 + C^2} = \sqrt{30.8552} = 5.5547 +$$

$$R = \frac{r}{a} y = \tfrac{1}{4} y = 1.3887 +$$

Volume of pipe is $8\pi Rr = 2.51328 \times 1.3887 \times 0.75 = 26.17644 +$ cubic feet.
Hence remaining capacity of room is 960 cu. ft. $-$ 26.17644 cu. ft. equal to 933.82356 cubic feet.

2722 [September, 1918]. Proposed by FRANK IRWIN, University of California.

The number of terms in the general polynomial of the nth degree in m variables and in that of the mth degree in n variables is the same. It would be interesting to devise schemes which, without assuming this result, should exhibit the terms of these polynomials in one-to-one correspondence with each other.

Solution by C. F. Gummer, Queen's University.

Consider first the polynomial $P_n(x_1, x_2, \cdots, x_m)$ of degree n, with coefficients all equal to 1. The general term is

$$x_1^{p_1} x_2^{p_2} \cdots x_m^{p_m} x_{m+1}^{p_{m+1}}, \qquad (\Sigma p = n),$$

where $x_{m+1} = 1$. Let the term be written at length, and a y $(= 1)$ inserted after each group of like x's except after the one for x_{m+1}, the y appearing even when the corresponding p is zero. The term is completely defined by the positions of the y's, so that the subscripts may be dropped, and the term written $xx \cdots yxx \cdots yxx \cdots$. Thus, in a polynomial of degree 4 in 5 variables, $x_1^2x_2$ will be denoted by $xxyxyyyyx$, the last x representing $x_6 = 1$. The various terms of P_n then correspond to the permutations of n x's and m y's. In the same way the terms of $P_m(x_1, x_2, \cdots x_n)$ of degree m may be made to correspond to the permutations of m x's and n y's. We may now put into one-to-one correspondence the terms of P_n and P_m which differ by interchange of the letters x and y.

Since the choice of a term in P_n corresponds to the choice of positions for the m y's, this method furnishes a direct explanation of the fact that the number of combinations of $m + 1$ kinds of thing taking n things at a time and allowing repetition is $\binom{m+n}{n}$.

2729 [November, 1918]. Proposed by N. P. PANDYA, Sojitra, India.

Solve in integers $x^3 + 3y^4 = z^2$.

Solution by S. A. Corey, Des Moines, Iowa.

Having obtained by any means one solution x, y, z, it is easily seen that a^4x, a^3y, a^6z is a solution, where a may be any integer. Since 1, 2, 7 and 1, 1, 2 are solutions, a^4, $2a^3$, $7a^6$ and a^4, a^3, $2a^6$ are solutions, whatever the value of a.

2731 [November, 1918]. Proposed by J. K. WHITTEMORE, Yale University.

A bowl is in the form of a paraboloid of revolution. If for a given volume the surface is a minimum, prove that the ratio of the diameter of the top to the depth is approximately 1.86.

Solution by Elijah Swift, University of Vermont.

Let the parabola have the equation, $y = kx^2$. Call the depth of the bowl, l. The required ratio is

$$\frac{2\sqrt{\frac{l}{k}}}{l} = \frac{2}{\sqrt{kl}}; \text{ the volume, } V, = \frac{\pi}{2k}l^2; \text{ the surface, } S, = \frac{\pi}{6k^2}\{(1 + 4kl)^{3/2} - 1\}.$$

The necessary condition for a minimum may be found by equating to zero the partial derivatives of $S + \lambda V$ (λ a parameter) with respect to k and l. We thus obtain, after simplification, $\sqrt{1 + 4kl}(-2 - 2kl) + 2 - 3kl^2\lambda = 0$, and $\sqrt{1 + 4kl} + l\lambda = 0$.

Eliminating λ, there results $\sqrt{1 + 4kl}(-2 + kl) = -2$.

Rationalizing and reducing, we obtain a cubic equation with the roots $kl = 0$, $1.1569 +$. $2.5931 +$. Of these, the last is not a root of the last equation given above, but is extraneous, The first corresponds geometrically to an infinite ratio, *i. e.*, a "flat bowl," or a maximum. If there be a minimum, as seems evident from geometric considerations, it must correspond to the remaining root, $kl = 1.1569 +$. The ratio corresponding is $\frac{2}{\sqrt{kl}} = 1.8595$ or $1.86 -$.

Also solved by A. M. HARDING, GERTRUDE I. MCCAIN, and the Proposer.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Dr. W. D. MACMILLAN has been promoted from assistant professor to associate professor of astronomy at the University of Chicago.

Mr. PAUL HEMKE, of Northwestern University, has accepted a position as instructor in mathematics at the U. S. Naval Academy at Annapolis.

Dr. OTTO DUNKEL has been promoted from assistant professor to associate professor of mathematics at Washington University, St. Louis.

Dr. L. C. KARPINSKI has been promoted from associate professor to professor of mathematics at the University of Michigan.

Recent appointments as instructors in mathematics at Yale in the Sheffield Scientific School for next year are: Dr. T. M. STETSON, Mr. EUGENE TAYLOR, Mr. HERMAN BETZ.

Captain J. V. MCKELVEY, of Cornell University, has been appointed assistant professor of mathematics at Iowa State College, Ames, Iowa.

Mr. F. L. KERR, of Northwestern University, has resigned his position as instructor in mathematics to become registrar of the university.

At Cornell University, Mr. H. S. VANDIVER and Dr. G. M. ROBISON have been appointed instructors in mathematics.

Owing to an unexpected increase in the number of students, the regular staff of the department of mathematics in the summer session of Cornell University was augmented by the appointment of Professor M. G. GABA of the University of Nebraska, Professor F. W. BEAL of the University of Tennessee, and Mr. ARTHUR HARMAN.

D. H. R. HASSÉ, late fellow of St. John's College, Cambridge, and senior lecturer in mathematics at the University of Manchester, has been appointed professor of mathematics at the University of Bristol.

J. HAAG, who received his doctorate from the University of Paris in 1910, has been appointed professor of differential and integral calculus at the University of Hancy.

It is planned to organize at the University of Strassburg conditions which will be attractive for more advanced American graduate students of mathematics. Except at the University of Paris the department of mathematics will have the largest number of instructors of any university in France. For the coming year these are: PÈRÈS, professor of general mathematics; VALIRON, professor of differential and integral calculus; VILLAT, professor of mechanics; FRÉCHET, professor of higher analysis; ESCLARGON, professor of astronomy; and three maîtres de conferences, ANTOINE, DARNOIS, and VÈROMET.

Monsieur CHAZY has been appointed professor of general mathematics at the University of Lille, in place of Professor JEAN CLAIRIN, killed in battle in 1914.

CLAUDE GUICHARD, professor of general mathematics at the University of Paris, has been appointed to the chair of higher geometry there, made vacant by Darboux's death. Professor E. P. J. VESSIOT, of the University of Lyons, has been appointed to succeed Professor Guichard as professor of general mathematics.

Monsieur ROY has been appointed professor of rational mechanics at the University of Toulouse, to replace the late Professor LATTÈS.

Professor UGO AMALDI, of the University of Modena, has been appointed professor of descriptive geometry at the University of Padua.

Professor TULLIO LEVI-CIVITA, of the University of Padua, has been appointed professor of higher analysis at the University of Rome.

Professor CARLO SEVERINI, of the University of Catania, has been appointed professor of infinitesimal analysis at the University of Genoa.

A. MOHRMANN has been appointed ordinary professor of mathematics at the University of Basel.

RODOLPHE GUIMARÃES, the Portuguese engineer, died in 1918 aged 52 years. His 160-page work on *Les Mathématiques en Portugal au XIX^e Siècle*, prepared for the Paris Exposition in 1900, was enlarged by about 500 pages in the second edition (Coïmbre, 1909) entitled *Les Mathématiques en Portugal.*

MATTEO BOTTASSO, chargé du cours of rational mechanics at the University of Messina, died at Turin, October 3, 1918, aged 40 years. He was the author of *Analyse vectorielle générale,* volume 4: *Astatique* (Pavia, 1915), in the work of Burali-Forti and Marcolongo.

Dr. C. BRANDENBERGER, professor of mathematics in the canton school, Zürich, and professor of mathematical didactics and methodology in the normal section of the Polytechnikum, died January 2, 1919, in the forty-sixth year of his age. He contributed, to the International Commission on the Teaching of Mathematics, the long report (1912) on mathematical instruction in Swiss gymnasia and "real schools."

Professor MAXIME BÔCHER's mathematical library has been purchased for the library at the Proving Ground, Aberdeen, Md.

The state legislature of California has established a Southern California branch of the University of California in connection with the Los Angeles State Normal School. Dr. E. C. MOORE, who is known to our readers as a philosopher who has taken active part in recent discussions on mathematics, will be the director of this branch in addition to his duties as president of the Normal School. Miss MYRTIE COLLIER who was head of the department of mathematics in the Normal School has been made head of that department in the new school. Mr. G. E. F. SHERWOOD, who was formerly associate professor of mathematics at the Colorado College of Mines, and who has been a graduate student at the University of Chicago during the past year, will take charge of the collegiate work in mathematics.

In the MONTHLY for September, 1916, reference was made to the last will and testament of Professor MITTAG-LEFFLER and his wife, with its generous provisions for the founding of a Mathematical Institute. It was originally planned that the Institute should not be put into operation until after Professor Mittag-Leffler's death. But Mr. Jourdain announces in the last issue of *Science Progress* that Professor Mittag-Leffler "has handed over a capital sum to the Academy of Sciences so that the activity of the Institute can begin at the present time on a modest scale. 'I already have,' writes Prof. Mittag-Leffler, 'two scholars endowed with travelling fellowships, and will send them to England as soon as circumstances permit.'"

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

FOURTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The Association held its fourth summer meeting by invitation of the University of Michigan at Ann Arbor, Michigan, on Thursday, Friday and Saturday, September 4–6, 1919, in conjunction with and following the meetings of the American Mathematical Society and the American Astronomical Society. One hundred ninety persons were in attendance, including official representatives of three institutions and the following 91 members of the Association:

R. C. ARCHIBALD, Brown University.
G. N. ARMSTRONG, Ohio Wesleyan University.
NORMAN ANNING, University of Maine.
L. A. BAUER, Carnegie Institution.
MRS. ETHELWYNN R. BECKWITH, College for Women, Western Reserve University.
W. S. BECKWITH, Ohio Northern University.
W. W. BEMAN, University of Michigan.
SUZAN R. BENEDICT, Smith College.
G. A. BLISS, University of Chicago.
HENRY BLUMBERG, University of Illinois.
J. D. BOND, A. and M. College of Texas.
R. L. BORGER, Ohio University.
J. W. BRADSHAW, University of Michigan.
W. C. BRENKE, University of Nebraska.
E. W. BROWN, Yale University.
W. D. CAIRNS, Oberlin College.
A. L. CANDY, University of Nebraska.
E. H. CLARKE, Hiram College.
MYRTIE COLLIER, Southern Branch, University of California.
G. H. CRESSE, U. S. Naval Academy.
D. R. CURTISS, Northwestern University.
MARIAN E. DANIELLS, Iowa State University.
S. C. DAVISSON, Indiana University.
F. F. DECKER, Syracuse University.
L. W. DOWLING, University of Wisconsin.
JOHN EIESLAND, West Virginia University.
L. P. EISENHART, Princeton University.
L. C. EMMONS, Michigan Agricultural College.
FAY FARNUM, Iowa State College.
PETER FIELD, University of Michigan.
B. F. FINKEL, Drury College.
J. A. FOBERG, Crane Junior College.
W. B. FORD, University of Michigan.
C. F. GUMMER, Queen's University.
W. A. HAMILTON, Beloit College.
E. R. HEDRICK, University of Missouri.
G. W. HESS, Bethany College.
T. H. HILDEBRANDT, University of Michigan.
H. A. HOWE, University of Denver.
E. V. HUNTINGTON, Harvard University.
W. J. HUSSEY, University of Michigan.
H. R. KINGSTON, University of Manitoba.
A. E. LAMPEN, Hope College.
FLORENCE P. LEWIS, Goucher College.
G. H. LING, University of Saskatchewan.
A. C. LUNN, University of Chicago.
E. B. LYTLE, University of Illinois.
W. D. MACMILLAN, University of Chicago.
J. L. MARKLEY, University of Michigan.
JOHN MATHESON, Queen's University.
J. V. MCKELVEY, Iowa State College.
G. A. MILLER, University of Illinois.
E. J. MOULTON, Northwestern University.
F. R. MOULTON, University of Chicago.
A. L. NELSON, University of Michigan.
C. A. NELSON, University of Kansas.
H. L. OLSON, University of Wisconsin.
F. W. OWENS, Cornell University.
C. I. PALMER, Armour Institute.
A. D. PITCHER, Adelbert College.
L. C. PLANT, Michigan Agricultural College.
R. G. D. RICHARDSON, Brown University.
H. L. RIETZ, University of Iowa.
MARIA M. ROBERTS, Iowa State College.
E. D. ROE, JR., Syracuse University.
W. H. ROEVER, Washington University.
J. R. SAGE, JR., Iowa State College.
G. T. SELLEW, Knox College.
J. B. SHAW, University of Illinois.
H. E. SLAUGHT, University of Chicago.
E. R. SLEIGHT, Albion College.
G. W. SMITH, University of Kentucky.
P. F. SMITH, Yale University.
G. G. SPEEKER, Michigan Agricultural College.
W. M. STEIRNAGLE, Jonesboro, Ark.
R. P. STEPHENS, University of Georgia.
E. B. STOUFFER, University of Kansas.
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MARY L. WELTON, Ann Arbor High School.

L. C. KARPINSKI, University of Michigan.
A. J. KEMPNER, University of Illinois.
A. M. KENYON, Purdue University.
E. C. KIEFER, State College of Iowa.
K. P. WILLIAMS, Indiana University.
W. H. WILSON, University of Iowa.
B. F. YANNEY, College of Wooster.
J. W. YOUNG, Dartmouth College.
ALEXANDER ZIWET, University of Michigan.

Institutional representatives:

CLIFF GUILD, Illinois Wesleyan University.
W. F. RIGGE, Creighton University.
TWO SISTERS OF ST. FRANCIS, College of St. Teresa.

The attendance was very gratifying, though smaller than the offer of reduced railroad rates might have induced had not the minimum number for the securing of such rates been placed so high by the railroad administration. It would seem most appropriate if this minimum were greatly reduced in the case of purely scientific organizations such as were meeting on this occasion. Nevertheless, a goodly number of members made long journeys to Ann Arbor and all who came declared their great satisfaction with the benefits derived from their attendance.

Members and their friends were comfortably housed either in the Newberry Residence or in the Michigan Union. The latter provided accommodations for all meals and the club room in Memorial Hall afforded a convenient and commodious gathering place for social purposes and committee meetings. The representatives of the departments of mathematics and astronomy had, under the chairmanship of Professor Beman and the secretaryship of Professor L. A. Hopkins, made elaborate preparations for entertainment and for the efficient conduct of all matters connected with the various programs. A booklet gave in compact form the combined programs of all three organizations, together with a description of buildings of special interest and a map of Ann Arbor. The reception for all members and friends Tuesday evening at the Observatory, the reception for the ladies at Professor Markley's and the smoker for men at the club on Wednesday evening, the complimentary luncheon Thursday noon, the observation parties conducted about the campus, the automobile drive Thursday evening, and the numerous groups privately entertained in the homes of the faculty, all attested the fine social spirit that prevailed throughout. By a rising vote of the 170 persons at the joint dinner thanks for all these courtesies were embodied in a resolution to be presented to the university authorities.

At the joint dinner, President Slaught acting as toastmaster called upon Regent Beal and Professor Beman who gave an official welcome to the members and friends. Response was made by President Schlesinger for the Astronomical Society, by Professor Eisenhart for the Mathematical Society and by Professor Rietz for the Mathematical Association. Other speakers were Dr. Klotz of the Dominion Observatory of Ottawa, Canada, who pleaded for more cultural studies along with the sciences; Professor Joel Stebbins who gave a lively account of the experiences of the American delegates to the Brussels meeting; and Professor Hedrick who gave a vivid account of the work of the American University in France for the soldiers. This gathering was pronounced on all sides as one of the most successful occasions of its kind and many wishes were expressed for like opportunities in the future.

The discussion of the proposed Mathematical Dictionary Friday morning was in no sense in the nature of a report of the committee which was appointed some three years ago and which has made on other occasions preliminary reports. The object of this discussion was avowedly to secure a consensus of opinion among a large number of mathematicians and to gather data both pro and con, for the guidance of the committee in its future deliberations. To this end numerous individuals, including some members of the committee, presented tentative proposals concerning different phases of the project, all of which were solely on the responsibility of the individuals concerned. All of these suggestions will be of the utmost usefulness to the committee when it comes to the formulation of definite plans for carrying forward the project, such as the final determination of the scope and content of the dictionary and the distribution of the editorial responsibility. Abstracts of the papers and discussions are given below. The most important phase at this time was the determination of the desirability and feasibility of the project. On this point a vote was taken in the form of a resolution which was carried with but one dissenting voice. The chief questions of doubt raised in the discussion were (1) whether such work would interfere seriously with the output of research; (2) whether some larger project should not be undertaken, such, for example, as the founding of a journal for abstracts in the English language to fill the place formerly occupied by the *Fortschritte*. These objections were met by the various speakers as shown in the abstracts of their remarks given below. A significant fact was that in the case of several who expressed a doubt they hastened to say that they should of course want a copy of the dictionary for their own use; in the last analysis the fact that both students and experts want the dictionary is a controlling reason why it should be prepared.

The program accompanied by numbered abstracts is grouped in four parts. Professor W. B. Ford was chairman of the program committee.

Joint Session of the Association with the American Mathematical Society and the American Astronomical Society.

(1) "Mathematics and Statistics"—Retiring address of the President of the Association, Professor E. V. Huntington, Harvard University.

(2) The Work of the National Research Council with reference to Mathematics and Astronomy.—Professor E. W. Brown, Yale University.

(3) Report on the Meeting of the International Astronomical Union at Brussels.—Director Frank Schlesinger, Allegheny Observatory, University of Pittsburgh.

(4) Report on the Meeting of the International Union of Geodesy and Geophysics at Brussels.—Director L. A. Bauer, Department of Terrestrial Magnetism, Washington, D. C.

(1) The address is to be printed in the December issue of the Monthly.

(2) Professor Brown stated that when the modern scientific societies were formed, they were devoted less to forwarding the work of the various sciences

than to providing that those interested might meet one another and confer about their results. The National Research Council is to a considerable extent the result of the great war. Just as the analogous organization in Great Britain endeavored to bring the resources of English science into effective relations with the government, so in the United States certain leading scientists foresaw that it might become needful for the country to provide for certain contingencies and these men planned to provide an organization which would in case of need provide the means of marshalling the forces of science for the service of the government.

Very early the government acquired from the allies a great body of useful information on ordnance, signalling, development of aërial navigation, etc., and within but a few months after our declaration of war the work of the Research Council came into evidence. The organization is still in its original form but is now devoted to purposes of peace. One of its important activities may be characterized roughly as the encouragement of research; this may be accomplished by assistance rendered to individuals engaged in research, the removal of hindrances to such research, the provision of adequate subsidies for the apparatus of research, greater coöperation between groups of workers and between those working in different but related lines, the preparation of a national census of research, etc.

A second phase of the work of the Council is that of coöperation in activities which demand international efforts. This has been done previously in astronomy more fully than in any other subject. But there is need everywhere of a great advance. For instance, the questions of units and of notation are of great interest to astronomers and indeed to all scientists; one may cite further exchanges of scientific publications, information service, exchange of professors and of research students, collaboration on international catalogs and periodicals. Practically every one of the old organizations is dead; the International Research Council affords the agency for bringing about anew meetings of scientific men and the reading of scientific papers.

The National Research Council proposes in no sense to control research or to force coöperation as taking the place of any desired individual activity; it seeks rather to encourage research, to obtain increased assistance when required and to make a survey of the larger possibilities of research. The selection of members of the Council and subdivisions will be made mainly by the national societies of America. The subdivisions of the Council will ultimately be very numerous and will be adequate to handling these problems through sections of physicists, mathematicians, astronomers, etc.

(3) Dr. Schlesinger referred to two meetings held in the fall of 1918 at Paris and London; those in attendance were the foreign secretaries or other representatives of the national academies of the principal allied nations. Under the authority of these meetings there were to be formed international unions, as well as national sections of such unions in the individual countries. Lest such a section should assume the character of a national society in a country where

the latter is already organized the members of the American delegation to the International Astronomical Union have recommended the only permanent feature of the section shall consist of an executive committee of seven members whose chief function would be to prepare for the triennial meeting of the International Union. This is to be done by making a survey of the state of the science through the appointment of small temporary committees in specific branches of the science.

With regard to the Brussels meetings in July of this year, the American delegates were prepared to discuss technical and scientific plans, but the European delegations had considered only matters of organization. The general plan of the latter was to subdivide astronomy into a number of divisions and to set up for each of these a central bureau, each of which was to have a president, a secretary and an executive committee. The American delegation was of opinion that the Union should be made up of committees which were to deal with specific problems that require international coöperation. This latter plan was finally adopted. Thus the Carte du Ciel will henceforth be directed through a committee of the International Union, and similarly the determination of Latitude Variations and many analogous projects.

It was decided to admit neutral nations to the Union on the same basis as the allied countries; for this reason the definite organization of the Union will not begin until January 1, 1920; those of the allied and neutral nations who have signified their adherence to the Union at that date will be considered as charter members of the Union.

The number of votes for each country and the financial contributions from each were apportioned as follows:

	Votes.	Units for Financial Contributions.
Under 5 millions population	1	1
Between 5 and 10 millions	2	2
" 10 " 15 "	3	3
" 15 " 20 "	4	5
Over 20 "	5	8

The unit for financial contribution was recommended as $400 for the present. The next meeting of the Union will be held at Rome in the spring of 1920, and the next after that probably at Cambridge, England, in 1922.

(4) As finally organized, this Union, Doctor Bauer reported, consists of the following six sections: Geodesy, seismology, meteorology, terrestrial magnetism and electricity, physical oceanography, and vulcanology. The opinion was expressed generally at that meeting that in the organization of work for the various sections the endeavor should be to distribute the work among the various committees rather than centralize the investigational work at the central bureau.

In addition to the two unions described in these two addresses, there were established under the auspices of the International Research Council international unions of mathematics, of physics, of chemistry, and of scientific radio-telegraphy.

TOPICS ON THE SEPARATE ASSOCIATION PROGRAM.

(5) "Continuity in Synthetic Geometry"—PROFESSOR JOHN MATHESON, Queen's University, Kingston, Canada.

(6) "Some Aspects of Mathematics in Biology"—DR. R. B. ROBBINS, University of Michigan, by invitation.

(7) "Mathematical and Astronomical Rarities in the Library of the University of Michigan"—PROFESSOR L. C. KARPINSKI, University of Michigan.

(5) The geometry of Euclid had no generalizations but was characterized by the logical sequence of its propositions. A certain amount of generalization was attained through the so-called principle of continuity as introduced by Kepler and more fully developed by Poncelet. But a higher degree of generalization may be reached through the application of the notion of continuity as applied to ordinary mathematical functions. For this purpose definitions may be given of the limit of a point which traces out a curve and of the continuous motion of the point, of the continuous motion of a line, and of the continuous variation of quantities such as length, angle and area. Based on such definitions, there is the fundamental principle that if a constant relation holds for a number of continuous variables for all sets of simultaneous values approaching a limit, the relation holds true also at the limit. On this principle we find all through geometry groups of theorems which are forms of a common principle. Such generalizations add much both to the interest in the subject on the part of college students.

(6) The Mendelian theory of inheritance is a clear cut hypothesis of the same nature as the statement of an urn problem. It opens up for those interested in the theory of probability a variety of types of problems with the added attraction that they are suggested by the needs of another science. The purpose of this paper is to show by means of a few examples the kinds of mathematical tools needed in these problems. The problems so far considered have involved the solution of systems of recurrence relations, subject to initial conditions determined by the nature of the original matings.

Most of the mathematical work so far done along this line has been on what we call one factor problems. The two factor problems are more complicated to about the same extent that geometry of two dimensions is more complicated than that of one dimension. The problems are further complicated when we consider linked hereditary traits. The present author's contribution to the literature of this subject consists of articles in Volumes 2, 3 of *Genetics* (Princeton University Press) under the heading "Applications of mathematics to breeding problems" and a short article in *Journal of Genetics,* Vol. VII, entitled "Partial self-fertilization contrasted with brother and sister mating."

(7) The mathematical and astronomical collections of the Library of the University of Michigan compare most favorably with any collections in America to be found outside of New York City. In the Exhibit, which was arranged for the Association meeting and which was visited by many during the sessions and on Friday afternoon when Professor Karpinski personally explained the collection, were placed all the works on the history of mathematics and astronomy, a

fairly complete collection, the bibliographies, the dictionaries of mathematics and astronomy, the mathematical tables, and all mathematical and astronomical works in the library which were published before 1800.

Of particular interest is the collection of photographs (rotographs) of manuscripts from European libraries dating from the twelfth to the sixteenth centuries. These photographs include a large number of arithmetics or algorisms, as they are called after Al-Khowarizmi, in which the Hindu art of reckoning was first taught to Europeans; early Latin algebras are included, notably two Latin versions of Al-Khowarizmi's algebras and one of Abu Kamil. Recent additions to this collection include the collected works of Richard Wallingford, Abbot of St. Albans, one of the most famous English mathematicians and astronomers of the fourteenth century. Wallingford's *Quadripartitum de sinibus et chordis* has been transcribed by Professor J. D. Bond of Texas A. and M. College, a fellow in mathematics at the University of Michigan. Professor Bond made the interesting discovery that Wallingford proposed a new radius of 150, to be used in the computation of sines, instead of the radius of 60 used by Ptolemy in his table of chords. The tables exhibited include several editions of Ptolemy's tables in his *Almagest;* of particular interest is the complete five volume official edition of the Alfonsine tables, made under the direction of King Alfonse the Wise of Spain, about 1275. A number of various works of Peurbach are included, notably his *De sinibus et chordis* with tables of sines by Regiomontanus (1436–1476) to the radius 6,000,000 and 10,000,000. After the introduction of decimal fractions the tables to the radius 10,000,000 adapted themselves to the unit radius.

Of the seven incunabula exhibited three are not found elsewhere in America. These are the following:

1490, Regiomontanus, *Tabule directionum profectionum famosissimi viri Magistri Ioannis Germani de Regiomonte.* Augsburg, Ratdolt, 2. Jan, 1490; written 1467.

1495, Bradwardine, *Geometria Thome brauardini cum tractatulo de quadratura circuli bene reuisa a Petro sanchez ciruelo:* operaque Guidonis mercatoris diligentissime impressa Parisiis in campo gaillardi Anno domini 1495, die 20 maij.

1499, Sacrobosco, *Opusculum Ioannis de sacro busto spericum cum notabili commento. atque figuris textum declarantibus vtilissimis.* Leipzig, Wolfgang Stöckel, 1499.

By the courtesy of the University Library pamphlets with a brief description by Professor Karpinski of the University collections in mathematical lines were available for the visitors.

Discussion of the Proposed Mathematical Dictionary.

(8) Its General Desirability and Feasibility. Professor E. R. Hedrick, University of Missouri.

(9) Its Possible Scope and Content. Professor G. A. Miller, University of Illinois.

(10) Possible Distribution of Editorial Responsibility. PROFESSOR R. C. ARCHIBALD, Brown University.

(11) General discussion, in person or by letter, by PROFESSORS D. E. SMITH, Columbia University, L. E. DICKSON, University of Chicago, L. P. EISENHART, Princeton University, and E. W. BROWN, Yale University, and others.

(8) Professor Hedrick rehearsed briefly the organization, previous reports, and present organization of the Committee on a Mathematical Dictionary. The general need for such a work and the activity of the Committee both in outlining the work and in seeking adequate financial support were mentioned. Recently, through the activity of Professor D. E. Smith, definite proposals looking toward financial aid have been made to organizations disposing of funds.

The relation of this work to the larger field of activity of the Committee on the Apparatus of Research of the American Association of University Professors was mentioned. This other committee has already initiated important work for which financial support has been secured; and it stands ready to aid the dictionary project directly, if funds can be secured for such general purposes. The committee is also interested in such works of reference as those formerly published in Germany in which abstracts of articles appeared. It is understood that this phase, so far as mathematics is concerned, is already under consideration by the American Mathematical Society.

The view was expressed that all such work, throughout the field called the Apparatus of Research, will devolve more and more upon America. This dictionary project, on account of its relative simplicity and on account of its natural terminability, offers a safe means of initiating such work in this country, to train men toward such work and to determine its feasibility here.

Another sort of relation concerns possible interference with scientific research on the part of men engaged in it. To avoid this, a very detailed plan, involving the extensive use of clerks to do much clerical work, was proposed.. While this plan cannot be repeated in detail here, it may be said that its purpose would be to relieve research men entirely of needless clerical work.

The conclusion drawn was that the entire project is desirable and entirely feasible, and that without undue interference with research activities, provided always that funds can be secured. Finally, to uphold the hands of those engaged in the attempts to secure such funds, it was urged that unequivocal resolutions be passed in support, not only of this dictionary project, but also, in a much wider field, of all those projects under consideration by the Committee on the Apparatus of Research.

(9) In Professor Miller's judgment all terms which are usually sufficiently defined in the ordinary dictionaries and encyclopedias should be excluded from this dictionary. Personal names should not be included among the terms to be explained in view of the fact that the number of these names is so large; this can be more adequately provided in a separate work, which indeed this Association might well look forward to undertaking at some future time, especially so far as it relates to American mathematicians.

If, to be concrete, we speak of group theory, it would be unwise to start the student with the original memoirs of Cauchy; these are for the most part difficult to follow and have been put into better form in modern textbooks on group theory. The large mathematical encyclopedias are invaluable to those who aim to use original sources to the best advantage, and the speaker believes that a mathematical dictionary, properly constructed, will be of equally great value to those who wish to make the most effective use of original material.

All technical mathematical terms found in the reputable literature should be included, the varied needs of the first year graduate student being especially considered. This would probably make a work of 1,500,000 words in three or four volumes comprising the definition of, say, ten thousand terms.

Professor Miller gave a somewhat detailed plan whereby 100 to 200 terms might be treated in expository articles by experts in the several fields, and the definitions of the remaining terms could be prepared first by assistants and later by associate editors in such a manner as to relieve research men of the bulk of the demands which might otherwise make serious inroads upon their research.

(10) Professor Archibald stated that of several important undertakings that demand intense coöperative effort on the part of mathematicians it is the dictionary project alone which we, as a nation, are equipped to carry through unaided. The great need of a new mathematical dictionary was expressed twenty-five years ago, especially by the French,[1] existing dictionaries being more than fifty years old and Müller's French-German and German-French mathematical vocabulary not performing adequately the function of a dictionary. In comment upon Professor G. A. Miller's article published in this MONTHLY[2] to illustrate the nature of a possible general article for the proposed dictionary, Professor Archibald called attention to a criticism of this article by P. E. B. Jourdain[3] and expressed his judgment that the order and extent of inclusion of foreign languages should be French, German, Italian, Swedish, Dutch, and possibly Russian or Spanish. He would greatly increase the value of this collection of terms by arranging those in each language alphabetically with the English equivalents opposite; this could be done for 12,000 terms in each of five foreign languages in an extra volume of 500 pages.

He then presents a plan whereby not two editors, as already proposed, but four editors should devote to the work all their time for two years, these four to select a consulting board of ten whose chief duty would be to read, and to suggest possible revisions of, all articles written. Professor Archibald thus indicates his conviction that the labor of drawing up first drafts of thousands of articles is sure to devolve upon the editors; herein his plan differs materially from some of the others. A suggestion is made of the possibility of requesting some scholars from the British Empire to share in the undertaking, although the bulk of preparation would still fall to Americans. Such a plan as here set forth, it is estimated, would require a budget of sixty-three thousand dollars.

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[1] AMER. MATH. MO., Vol. I, p. 368.
Paul Tannery, *Bulletin des sciences mathématiques*, 1899, vol. 22, pp. 165–167; see also vol. 24, pp. 25–27.

[2] November, 1918, pp. 383–387.

[3] *Science Progress*, April, 1919.

(11) Various estimates of the financial needs in the preparation of the Dictionary were proposed, ranging from $30,000 by Professor Smith to $63,000 by Professor Archibald, depending on the number of paid experts who might be employed on salaries and on the length of time that might be involved. But it was the consensus of opinion that it would surely prove to be an expensive undertaking and that its high character should in no way be compromised by stinting the expenditure of funds in its preparation. Moreover it was pointed out that a project whose far reaching importance demands large financial backing will be much more likely than one of smaller nature to appeal favorably to those who are in a position to provide such funds. To quote Professor Smith, "It is evident that a dictionary could be prepared for less money. If it cost half as much, I should think it would be worth about one tenth as much. If we go into the matter, I feel that we should produce a work that will be a standard in Europe as well as America for many years to come,—one which will be an honor to American scholarship and to American book manufacturing."

Professor Dickson in a written communication said: "As to the proposed Mathematical Dictionary, I have been convinced by reflection and conversations with others that it would prove to be a valuable aid to mathematics, partly to experts when reading outside their specialties, but mainly to amateurs, to students, and to persons early in their mathematical careers. I believe many overlook the great extent of technical mathematical language in the literature and the fact that difficulties of students are very largely due to their failure to understand this technical language. Here is where the main service of the dictionary would come in. The fact that our technical words are in English causes us to focus attention on them less than would be the case if they were in Latin as in botany and medicine, with the result that we do not really learn thoroughly most of the technical shades of meaning.

"A mathematical dictionary which would collect by aid of experts all the *technical* uses of words and phrases in mathematics would be of great value to research men and of still greater value to all students of mathematics.

"No project is ever endorsed unanimously. The fact that the Dictionary is so generally approved seems to me a proof that the idea is sound. Nor do I fear any dissipation of mathematical energy in the preparation of the manuscript. There are enough well equipped men not putting their time mainly on research to do the work without lessening the research output. As there is no other big mathematical project afoot, it seems to me that there is nothing to lose in this Dictionary project, while I am one of those who strongly believe that there is considerable serious value to gain in general and that the pedagogical value is undisputed."

Professor Eisenhart suggested that some valuable aid in the compilation of definitions might be secured from experts in dictionary making. It is important that Americans do much more than merely prepare a limited number of papers each year; judging by these only a small number of our mathematicians are doing real research work. Mathematicians should be so organized that they

might be stimulated to do further work and to acquaint themselves with particular fields of study; for example, in the making of this dictionary many such might write up the definitions, the experts in various fields criticizing and selecting these.

Professor Shaw expressed his belief that we have a large body of men capable of doing this work. His belief is based on an experience with a successful plan carried out at the University of Illinois whereby the staff coöperate in keeping the run of mathematical journals, each one reporting on the matter found in some particular journal. Incidental to such reading as this, one often needs a mathematical dictionary when reading in a field to one side of his own special field.

Professor Ford said that the dictionary makers must exercise care and see to it that the definitions are of the sort that the users need,—simple, pithy definitions with clear examples. He showed several dictionaries, among these the American Medical Dictionary which in four or five lines for each word gives its derivation, and an English-German-French Technological Dictionary.

Professor Huntington instanced the attitude which the non-mathematician often holds regarding mathematics, quoting a dismayed reviewer of a book on statistics, who said that the technical terms in whose scientific exactness mathematicians pride themselves are forbidding and impossible. A mathematical dictionary would help such outsiders.

Professor Richardson raised doubt whether the dictionary project is the task for American mathematicians for the immediate future. He stated that we should attempt something at once to replace the *Fortschritte,* and that this would not be possible if we were to attempt the dictionary; a translation of Weber-Wellstein or of the new edition of Pascal's *Repertorium* would supply the need for the longer article which is the more important aspect of the project.

Professor Brown stated that he would undoubtedly order three copies of the dictionary for himself and the libraries, but he finds (1) a scientific difficulty in defining many words which have both a mathematical and a popular meaning, e.g., addition, and (2) business difficulties, particularly as to the relative importance of this project and the plan to institute a proposed journal to replace the *Fortschritte* and an Anglo-American or an English encyclopedia.

In answer to the last two speeches, Professor Lunn told of the scheme for abstracts now being used by the *Physical Review,* whereby the author makes his own abstract and this is revised by the editor, such a plan serving to save effort in making reviews of the type of those in the *Fortschritte* and to avoid the foregoing criticism of the loss to research; and Professor Hedrick stated his sympathy with the *Fortschritte* plan but desired to see the dictionary scheme completed which now seems immediately feasible.

Mr. Webb expressed his hope for an early publication of a smaller edition of the proposed dictionary.

At the completion of the discussion, as already stated, the following resolution was adopted:

Be it resolved by the Mathematical Association of America that the prepara-

tion and publication of a dictionary of mathematical terms is not only most desirable, but also entirely feasible, provided that financial aid for editorial guidance and clerical assistance can be secured; and

Be it resolved, that the Association views with great interest and hope the efforts of a national committee on the Apparatus of Scholarship, for the promotion of publication of works of a reference nature, including works for abstracting and reporting progress in the various sciences, for which there is and promises to be the most pressing need in all scientific, scholarly, and educational work.

Symposium upon the Present Day Relations and Tendencies between the High Schools and the Colleges as regards Mathematics.

(12) From the standpoint of the High School as a fitting school for College. Professor H. E. Webb, Central High School, Newark, N. J.

(13) From the standpoint of the College as a training school for High School teachers. Dr. E. B. Lytle, University of Illinois.

(14) From the standpoint of coördinated institutions in a general system of education. Professor B. F. Yanney, College of Wooster.

(15) The Work of the National Committee on Mathematical Requirements. Professor J. W. Young, Dartmouth College.

(12) Ten years ago much ado was made of the inutility of the traditional canon of secondary mathematics. It was fondly hoped that problems of real import to the external world would bring to mathematics an added interest. The undesirable results are too well known to bear recounting. Yet these were not altogether disastrous. We are, for example, in a fair way to eradicate the endless chain of competition between the examiner who invents new and unusual algebraic forms for solution and the tireless private tutor who carefully codifies all those forms for future recognition by his pupils. The speaker believes that one of the present tendencies in secondary teaching with a view to college preparation is toward a greater emphasis upon clarity and accuracy in the formulation of fundamental principles of algebra and geometry as opposed to a constantly advancing standard of intricacy in illustration.

Again the insistence upon real problems has convinced those students who have a native interest in abstract mathematics that this science is primarily intended for use rather than for amusement.

A consideration of the curriculum of the secondary schools for the past generation leads the speaker to the conclusion that one and only one foreign language for entrance credit shall be sufficient save for the possible exception of one or two years of Latin preceding a romance language. This would allow time for a thorough mathematical preparation and for adequate drill in English. A detailed plan for four years of mathematics on this basis was then given with due attention to continuity in these courses.

On the other side of the question the requirements for college entrance should include a certain degree of maturity of mind, and possibly of body, and such specific demands in particular subjects as are prerequisite for further study.

Thus the responsibility for secondary training is placed exactly where it belongs, —upon the authorities in the secondary field. The earnest desire was expressed that this Association might put itself upon record as against the utter looseness of election of secondary studies. Finally, in view of the extension of high school instruction into the domain of college work under competent instruction, Professor Webb pled for greater liberality in admission to advanced standing and for a standardization of freshman and sophomore mathematics.

(13) Besides calling attention to the importance of high-school teacher training, and giving some facts and data concerning the present status of teacher training in our colleges and universities, Dr. Lytle developed for discussion suggestions that American standards of secondary mathematics teacher training may be raised (1) by emphasizing the desirability of the greatest possible scholarship in mathematics, (2) by developing stronger courses in theory of equations, advanced geometry, fundamental concepts and history, (3) by emphasizing in all our courses perspectives and large ideas, (4) by developing the use of libraries, knowledge of the best literature and ability to prepare and forcefully present papers on mathematical topics, (5) by making one member of the staff responsible for investigation in the field of teacher training, (6) by encouraging professional training specially in adolescent psychology, (7) by taking constructive and not destructive attitudes toward scholarly investigation in education, (8) by demanding that supervisors of mathematics teacher training work have the degree of Ph.D. or the equivalent in mathematics as well as special training in education, (9) by studying the Wisconsin "directed teaching" plan with the purpose of modifying it to apply to college classes, and (10) by writing and translating books or articles which appeal to the present interests and attainments of high-school teachers.

(14) Professor Yanney first gave a historic review, beginning with the Colonial period and ending with the world war. Starting with no mathematics on the boundary line between the two units of our educational system, the varying changes were noted, culminating with one unit of algebra and one unit of plane geometry as fairly well established on the high school side of this boundary line.

Then followed a reference to organizations, institutions and individuals interested in various problems of relations between the two educational units, together with a description of the activities now engaging the attention of these organizations and institutions. These include both entrance problems and problems connected with the preparation of high school teachers in the colleges.

The present relations were characterized as cordial, coöperative, and constructive, with a number of details yet to be worked out before standard practices now well started become generally prevalent. There are standard colleges now articulating their collegiate courses in mathematics with preparatory courses comprising two, or three, or four units of mathematics. Two distinct current tendencies were mentioned. (1) One was the breaking up into smaller units, along the border line, of secondary school work and college work,—on the one side the junior and senior high schools, and on the other the junior college.

This was regarded as favorable to a still better articulation of mathematics between high school and college. (2) Another tendency referred to was that of the breaking up of the traditional compartments in mathematics, a movement in evidence on both sides of the boundary line already referred to. A third strong tendency, affecting both high school and college mathematics, was pointed out to be that of reappraising educational values as a result of experiences in the recent upheaval, educational aspects having become more or less international. This would eventuate, it was predicted, in increased interest in the study of mathematics in both the high school and the college.

(15) Professor Young reported that it was early found by the National Committee on Mathematical Requirements that it would be impossible to get their plans under way without larger funds and that finally, last spring, the General Education Board interested itself in this work. In the endeavor to reorganize and to strengthen the secondary school curriculum, the committee now purposes to obtain the names and addresses of teachers of high schools and junior high schools, to interest them in the activities of the committee, to make a study of the educational systems in the various states, to furnish information and speakers for the various organizations interested. It is expected that contact will be made with perhaps fifty or sixty such organizations through a representative officer of each.

A bulletin is projected which shall serve as a means of intercommunication between the committee and the great body of teachers. The committee looks to the members of the Association and others over the country for all possible assistance and suggestions as to the furtherance of its purposes. A fuller report in these pages may be expected at an early date.

Meeting of the Council of the Association.

Ten members of the Council attended the two sessions held on Thursday and Friday.

The following twenty-two persons and one institution, on applications duly certified, were elected to membership:

To individual membership.

C. R. Adams, A.B. (Brown). Instr., Brown University, Providence, R. I.

Maelynette Aldrich, Ph.D. (Yale). Prof., Martha Washington College, Abingdon, Va.

Stuart Ballantine. Radio Research Engineer, U. S. Navy Dept., Philadelphia, Pa.

Sarah Beall. Computer, Coast and Geodetic Survey, Washington, D. C.

Florence L. Black, A.B. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans.

T. M. Blakslee, Ph.D. (Yale). Retired, Des Moines College, now of Ames, Ia.

J. G. Coffin, Ph.D. (Clark). Asso. prof., Coll. of City of New York, on leave of absence; Asst. dir. of research, Curtiss Engg. Corp., Garden City, L. I.

J. M. Foster, A.M. (Rochester). Supt. of Schools, Corning, N. Y.

W. W. HART, A.M. (Wisconsin). Asst. prof., Univ. of Wisconsin, Madison, Wis.

C. A. ISAACS, A.M. (Columbia). Head of dept. of math., State Coll. of Washington, Pullman, Wash.

M. F. JOHNSON, C.E. (Mich. Agric. Coll.), M.S. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich.

E. C. KIEFER, M.S. (Michigan). Asst. prof., Iowa State Coll., Ames, Ia.

G. R. LIVINGSTON, B.S. (California). Head of dept. of math., Junior Coll., Santa Barbara, Calif.

ANNA MARM, A.M. (Kansas). Instr., Univ. of Kansas, Lawrence, Kans.

E. W. MARTIN, S.B. (Chicago). Prof., Wilmington Coll., Wilmington, Ohio.

ANNIE McK. PEGRAM, A.M. (Trinity Coll., N.C.; Columbia). Prof., Coll. for Women, Greensboro, N. C.

J. R. SAGE, M.S. (Rose Polytechnic). Asst. prof., Iowa State Coll., Ames, Ia.

Mrs. CATHERINE SELVES, B.S. (Kirksville State Normal). Instr., La Grange Coll., La Grange, Mo.

H. B. SMITH, A.B. (Kenyon). Lt., U. S. Coast Artillery, Fort Caswell, N. C.

C. N. STOKES, A.M. (Illinois). Head of dept. of math., McKendree Coll., Lebanon, Ill.

R. R. TILESTON, A.M. (Dartmouth). Prof. of physics, Colorado Coll., Colorado Springs, Colo.

ARTHUR WALTER, A.M. (Stanford). Asst. and graduate student, Stanford Univ., Salinas, Calif.

To institutional membership.

WEST CHINA UNION UNIVERSITY, Chengtu Sze Chwan, W. China.

It was voted to be the sense of the Council that the annual meeting of the Association be held at New York City provided that like action is taken by the Society.

It was voted that a committee, with Professor Eisenhart as chairman, be appointed by the president to investigate the question of junior branches or of associate membership in the Association.

The Council considered informally the financial statement of the treasurer and the question of the extension of the arrangement between the Association and the *Annals of Mathematics* beyond the original three-year term.

It was voted to increase to four dollars the subscription price of the MONTHLY to non-members, beginning with January, 1920, in view of the notification from the printers of an increased charge for printing and in view of the reasonableness of distinguishing between the general subscription price and a reduced price to members of the Association.

It was voted that the Council request the president and the chairman of the Committee on Dictionary to present the matter of the dictionary project to the Council of the Society and to request their endorsement of the project.

W. D. CAIRNS, *Secretary-Treasurer.*

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

In the first discussion this month Professor Moritz shows a method of deriving the area bounded by a parabola and certain straight lines by principles of elementary character such as are used in obtaining the area of a circle in plane geometry. The method applies with equal ease to the ordinary quadratic parabola and to parabolas of higher order. It is not likely that this proof will be entirely intelligible to all the students of an ordinary high-school class; but the same statement holds in connection with the area of the circle. It is worthy of note that the validity of Professor Moritz's proof depends on the possibility of saying that the quantity which he calls F^2 remains finite for all choices of r and n; this is easy to see on writing out the value of F^2.

Mr. Haldeman shows how to solve a cubic equation graphically by means of ruler, compass, and an appropriately chosen equilateral hyperbola, and applies the result to the graphical construction of the side of a regular heptagon inscriptible in a given circle. Since equilateral hyperbolas differ only by translations, rotations, and similarity transformations, it seems possible that the construction could be performed by the use of ruler, compass, and a given fixed equilateral hyperbola.

Professor Schmiedel indicates a method for obtaining the sum of a definite number of terms of certain types of series, and gives some interesting interpretations of the results. All the series considered are of frequent occurrence in connection with Fourier series and other allied developments.

The fourth discussion is a short note by Mr. M. W. Jacobs on the reason for the occasional success of a false rule for finding the hypotenuse of a right triangle in terms of the two sides. The case actually considered, even in the extended form of the corollary, is so simple as to be almost obvious. If we ask when it is possible for the hypotenuse to be represented linearly with rational coefficients (neither being required to be unity) in terms of the perpendicular sides, we have a slightly more complicated case, which leads to a familiar type of Diophantine equation. May we have the discussion of this case also?

I. ON THE QUADRATURE OF THE PARABOLA.

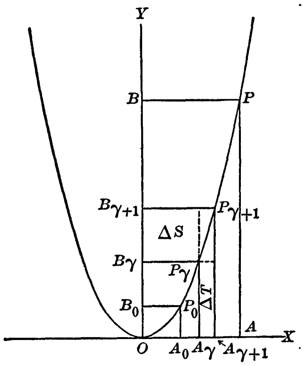
By R. E. MORITZ, University of Washington.

While most books on analytical geometry derive the formula for the area of an ellipse from its relation to the area of the circumscribed circle, the attempt is but seldom made to derive the formula for the area of a segment of the parabola.

The reason for this must be sought in the fact that the classic proofs of the formula in question require principles which are foreign to the methods of analytical geometry. Thus the famous proof by Archimedes involves, besides ten or eleven preliminary propositions, the sum of the infinite series $\Sigma(1/n^2)$. On the other hand, the method of infinitesimals (indivisibles) first applied to the quadrature of the parabola by Cavalieri and later extended by Wallis to the quadrature of the curve $y = x^p$, involves the limit of the sum $(1^p + 2^p + 3^p + \cdots + n^p)/(n^{p+1})$ as n approaches infinity.

The following proof is probably not new though the writer is not aware that it is recorded in any of the sources available to him. It is certainly far more simple and direct than any of the classic proofs. In fact it involves no principles except such as are familiar to every student of elementary algebra and geometry. Even when extended to the quadrature of the general parabola, $y = px^m$, it comes well within the range of the average high-school student.

Quadrature of the Parabola $y = px^m$. Let $P_0(x_0, y_0)$, $P(x, y)$, represent any two points on the parabola which lie on the same side of the vertex O, and let it be required to determine the area included between the curve, either axis, and the perpendiculars from these points on the y-axis.



From P_0 and P draw the perpendiculars P_0A_0 and PA to the x-axis. Divide A_0A into n equal parts and call the length of each part h. At the points of division, A_r, erect perpendiculars and through the points P_r, in which these perpendiculars meet the curve, draw lines parallel to the x-axis intersecting the y-axis in the points B_r. Denote the area B_0P_0PB by S, the area APP_0A_0 by T; the area of the rectangle P_rB_{r+1} by ΔS, and the area of the rectangle P_rA_{r+1} by ΔT. Then

$$\Delta S = x_r(y_{r+1} - y_r) = px_r(x_{r+1}{}^m - x_r{}^m),$$

$$\Delta T = y_r(x_{r+1} - x_r) = px_r{}^m(x_{r+1} - x_r),$$

$$\frac{\Delta S}{\Delta T} = \frac{x_{r+1}{}^m - x_r{}^m}{x_r{}^{m-1}(x_{r+1} - x_r)} = \frac{x_r{}^{m-1} + x_r{}^{m-2}x_{r+1} + x_r{}^{m-3}x_{r+1}{}^2 + \cdots + x_{r+1}{}^{m-1}}{x_r{}^{m-1}}$$

$$= 1 + \left[1 + \frac{x_{r+1} - x_r}{x_r}\right] + \left[1 + \frac{x_{r+1}{}^2 - x_r{}^2}{x_r{}^2}\right] + \cdots + \left[1 + \frac{x_{r+1}{}^{m-1} - x_r{}^{m-1}}{x_r{}^{m-1}}\right]$$

$= m + hF^2$, since each of the binomials is positive and divisible by $x_{r+1} - x_r$ which is equal to h.

Now as n is increased indefinitely, h approaches 0, hence $\Delta S/\Delta T$ approaches m, the sum of all the ΔS's approaches S, the sum of all the ΔT's approaches T, hence in the limit

$$S = m \cdot T.$$

Adding first mS and then T to both sides of the equation we find

$$(1 + m)S = m(S + T), \qquad S + T = (1 + m)T,$$

hence

$$S = \frac{m}{1 + m}(S + T), \qquad T = \frac{1}{1 + m}(S + T).$$

Now obviously $S + T = xy - x_0y_0 = p(x^{m+1} - x_0^{m+1})$, so that we may write

$$S = \frac{mp}{1 + m}(x^{m+1} - x_0^{m+1}), \qquad T = \frac{p}{1 + m}(x^{m+1} - x_0^{m+1}).$$

If the foregoing proof seems to lack rigor, the objection may be easily removed as follows. We have shown that $\Delta S/\Delta T = m + hF^2$ and $\Delta T = phx_r^m$, hence

$$\Delta S = m \cdot \Delta T + hF^2 \cdot \Delta T = m \cdot \Delta T + ph^2F^2x_r^m,$$

$$\Sigma \Delta S = m \cdot \Sigma \Delta T + ph^2\Sigma(F^2x_r^m).$$

In the limit $\Sigma\Delta S = S$, $\Sigma\Delta T = T$, hence in the limit

$$S = mT + \lim\,[ph^2\Sigma(F^2x_r^m)],$$

and it remains to show that the second term on the right is 0.

Let G^2 represent the greatest of the n values of $F^2x_r^m$, then $\Sigma(F^2x_r^m) < n \cdot G^2$, and $ph^2\Sigma(F^2x_r^m) < ph^2nG^2 = ph(x - x_0)G^2$, since $hn = x - x_0$, and therefore

$$\lim\,[ph^2\Sigma(F^2x_r^m)] = \lim\,[ph(x - x_0)G^2] = p(x - x_0)G^2 \cdot \lim h = 0.$$

In the case of the common parabola, $m = 2$, $F^2 = 1/x_r$, and $G^2 = x$. In that case, too, the foregoing proof holds for any segment of the parabola. For if the line through the middle point of the chord and parallel to the axis of the parabola be chosen for the y-axis, and the tangent parallel to the chord for the x-axis, the equation of the parabola remains unchanged, and areas of the parallelograms corresponding to ΔS and ΔT are equal to $\Delta S \cdot \sin\theta$ and $\Delta T \cdot \sin\theta$ respectively, θ being the angle between the coördinate axes. The ratio of these areas remains therefore unchanged and the conclusion regarding the ratio of S to T remains valid.

II. Geometrical Construction of the Roots of a Cubic, and Inscription of a Regular Heptagon in a Circle.

By C. B. Haldeman, Ross, Butler County, Ohio.

1. The equation

$$q(x^2 - y^2) + 2x\sqrt{-(r^2 + q^3)} + 2ry = 0 \tag{1}$$

represents a real equilateral hyperbola when q^3 is negative and numerically greater than r^2, and

$$x^2 + y^2 = -4q \tag{2}$$

is an equation of a circle, which will be real when q is negative. Eliminating x from these equations, we have

$$(y^3 + 3qy + 2r)(qy - 2r) = 0.$$

The first of these factors is a form to which the general cubic equation may be reduced. From this it appears that the three real roots of a cubic equation, reduced to the above form, may be represented by the ordinates of the intersections of an equilateral hyperbola and a circle. The three intersections, whose ordinates are the three real roots of this equation, are the vertices of an equilateral triangle; because the roots are

$$y = -2\sqrt{-q}\sin\tfrac{1}{3}\sin^{-1}\frac{r}{q\sqrt{-q}},$$

$$y = 2\sqrt{-q}\sin\tfrac{1}{3}\left(\pi + \sin^{-1}\frac{r}{q\sqrt{-q}}\right),$$

$$y = -2\sqrt{-q}\sin\tfrac{1}{3}\left(\pi - \sin^{-1}\frac{r}{q\sqrt{-q}}\right);$$

since $y = -2S\sqrt{-q}$ substituted in $y^3 + 3qy + 2r = 0$ gives $3S - 4S^3 = \dfrac{r}{q\sqrt{-q}}$; and comparing this with

$$3\sin A - 4\sin^3 A = \sin 3A,$$

which is well known, we obtain the values of y.

2. The equation $z^3 + Rz^2\sqrt{7} - R^3\sqrt{7} = 0$ gives the value of z which represents the side of a regular heptagon inscribed in a circle whose radius is R. The transformation

$$z = y - \frac{R\sqrt{7}}{3}$$

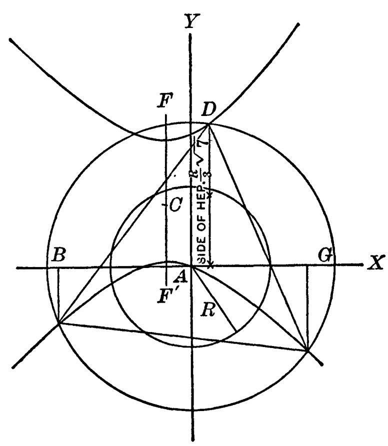
will reduce this equation to

$$y^3 - \frac{7R^2y}{3} - \frac{13R^3\sqrt{7}}{27} = 0.$$

Comparing this with the equation

$$y^3 + 3qy + 2r = 0,$$

we have $q = -7R^2/9$, $r = -13R^3\sqrt{7}/54$; also we have the radius of (2) equal to $2R\sqrt{7}/3$.



With A as a center and $2R\sqrt{7}/3$ as a radius describe a circle BGD and with A as the origin of coördinates, after substituting the values of q and r just found in (1), draw the equilateral hyperbola

$$x^2 - y^2 + \frac{R\sqrt{21}}{7}x + \frac{13R\sqrt{7}}{21}y = 0$$

with the diameter BG prolonged as the axis of X and AY as the axis of Y. Then will the ordinate of intersection at D minus $R\sqrt{7}/3$ be the side of a regular heptagon inscribed in a circle whose radius is R; that is,

$$z = \frac{R\sqrt{7}}{3}[2\sin\tfrac{1}{3}(\pi + \sin^{-1}\tfrac{13}{14}) - 1]$$

is the side of the regular heptagon required.

III. On the Summation of Certain Series.

By O. Schmiedel, Parsons College.

The summation of a finite number of terms of series such as

$$1 + 3x + 6x^2 + 10x^3 + \cdots,$$

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cdots,$$

$$1 + x\cos\theta + x^2\cos 2\theta + x^3\cos 3\theta + \cdots,$$

$$1 + 3x\cos\theta + 6x^2\cos 2\theta + 10x^3\cos 3\theta + \cdots,$$

and the evaluation of the complete series is a problem of frequent occurrence, and is easily solved by the use of De Moivre's Theorem;[1] but greater insight to the student into their nature and mutual relation is had by direct application of the more common processes of analysis.

It is proposed to produce a formula sufficiently comprehensive to embrace all series of the types suggested, including as particular cases the binomial formula for negative integral exponents and the theorem that the difference of the same powers is divisible by the difference of the first.

Let the second of the series above be deduced first, for illustration.

Assuming the identity

$$2\sin\theta/2\cos t\theta = \sin(t + 1/2)\theta - \sin(t - 1/2)\theta,$$

assigning to t values from 0 to n, and adding the results, gives directly the expression for the sum of the series of cosines, namely:

$$\sum_{t=0}^{t=n}\cos t\theta = 1/2 + 1/2\frac{\sin(n + 1/2)\theta}{\sin\theta/2} = \frac{1}{\sin\theta/2}\sin\frac{n+1}{2}\theta\cos\frac{n}{2}\theta. \tag{1}$$

In like manner is found the sum of sines:

$$\sum_{t=0}^{t=n}\sin t\theta = \frac{1}{\sin\theta/2}\sin\frac{n+1}{2}\theta\sin\frac{n}{2}\theta; \tag{2}$$

[1] Chrystal, *Algebra*, Vol. II, 2 ed., p. 273.

and the quotient of (2) by (1) gives the interesting relation

$$(3)\qquad \frac{\sum_{t=0}^{t=n} \sin t\theta}{\sum_{t=0}^{t=n} \cos t\theta} = \tan \frac{n}{2}\theta,[1]$$

an extension of the familiar formula $\tan \theta/2 = \sin \theta/(1 + \cos \theta)$.

Formula (1) may also be written thus:

$$(4)\qquad \sum_{t=0}^{t=n} \cos t\theta = 1/2 + \frac{\cos n\theta - \cos(n+1)\theta}{2(1-\cos\theta)},$$

$$(5)\qquad -\sum_{t=-n+1}^{t=-1} \cos t\theta = 1/2 + \frac{\cos n\theta - \cos(n-1)\theta}{2(1-\cos\theta)}.$$

Analogously the summation of the other series is effected, as:

$$(6)\qquad \sum_{t=0}^{t=n} x^t \cos t\theta = \frac{1 - x\cos\theta - x^{n+1}\cos(n+1)\theta + x^{n+2}\cos n\theta}{1 - 2x\cos\theta + x^2},$$

$$(7)\qquad -\sum_{t=-n+1}^{t=-1} x^t \cos t\theta = \frac{1 - x\cos\theta - x^{-n+1}\cos(n-1)\theta + x^{-n+2}\cos n\theta}{1 - 2x\cos\theta + x^2},$$

one or the other of which is convergent for a given $x \neq \pm 1$, as n becomes infinite. Using the following definition and identities:

$$c_{a,\,b} = \frac{a(a-1)\cdots(a+1-b)}{1.\ 2.\ \cdots b};$$

$$c_{a,\,b} + c_{a,\,b+1} = c_{a+1,\,b+1}, \qquad c_{a,\,b} = \sum_{t=0}^{t=n} (-1)^t c_{n,\,t} c_{a+n-t,\,b-t};$$

$$2\cos u \cos v = \cos(u+v) + \cos(u-v),$$

$$\cos r\theta(1 - 2x\cos\theta + x^2)^m = \sum_{t_1=0}^{t_1=2m} \sum_{t=0}^{t=t_1} c_{m,\,t_1-t} c_{m,\,t} (-x)^{t_1} \cos(r + 2t - t_1)\theta;$$

the sums to n terms are found successively by the same process for all the series under this head, as:

$$(8)\qquad \begin{aligned}(1 - 2x\cos\theta + x^2)^2 \sum_{t=0}^{t=n} c_{1+t,}\, x^t \cos t\theta &= \sum_{t=0}^{t=2} (-1)^t c_{2,\,t} x^t \cos t\theta \\ &+ \sum_{t_2=0}^{t_2=3} \sum_{t_1=0}^{t_1=1} \sum_{t=0}^{t=1} (-1)^{t_2+t} c_{2,\,t_2-t_1} c_{2,\,2+t_1-t} c_{n+t,\,1} x^{n+1+t_2} \cos(n + 1 + 2t_1 - t_2)\theta,\end{aligned}$$

$$(9)\qquad \begin{aligned}(1 - 2x\cos\theta + x^2)^3 \sum_{t=0}^{t=n} c_{2+t,\,2} x^t \cos t\theta &= \sum_{t=0}^{t=3} (-1)^t c_{3,\,t} x^t \cos t\theta \\ &- \sum_{t_2=0}^{t_2=5} \sum_{t_1=0}^{t_1=2} \sum_{t=0}^{t=2} (-1)^{t_2+t} c_{3,\,t_2-t_1} c_{3,\,3+t_1-t} c_{n+t,\,2} x^{n+1+t_2} \cos(n + 1 + 2t_1 - t_2)\theta,\end{aligned}$$

[1] *Ibid.*, p. 327 ff.

etc., and finally the general formula sought:

$$(10)\quad \begin{aligned}(1 - 2x\cos\theta + x^2)^m \sum_{t=0}^{t=n} c_{m-1+t,\, m-1}x^t \cos t\theta &= \sum_{t=0}^{t=m} (-1)^t c_{m,\,t} x^t \cos t\theta \\ &+ (-1)^m \sum_{t_2=0}^{t_2=2m-1} \sum_{t_1=0}^{t_1=m-1} \sum_{t=0}^{t=m-1} (-1)^{t_2-t} c_{m,\, t_2-t_1} c_{m,\, m+t_1-t} c_{n+t,\, m-1} \\ &\times x^{n+1+t_2} \cos(n + 1 + 2t_1 - t_2)\theta,\end{aligned}$$

$$(11)\quad \begin{aligned}-(1 - 2x\cos\theta + x^2)^m \sum_{t=-n+1}^{t=-1} c_{m-1+t,\, m-1}x^t \cos t\theta &= \sum_{t=0}^{t=m} (-1)^t c_{m,\,t} x^t \cos t\theta \\ &+ (-1)^m \sum_{t_2=0}^{t_2=2m-1} \sum_{t_1=0}^{t_1=m-1} \sum_{t=0}^{t=m-1} (-1)^{t_2-t} c_{m,\, t_2-t_1} c_{m,\, m+t_1-t} c_{-n+t,\, m-1} \\ &\times x^{-n+1+t_2} \cos(-n + 1 + 2t_1 - t_2)\theta,\end{aligned}$$

where n and m are positive.

Of the two expressions in the second member of (10), which constitute the sum of the series in the first, one is free from n, hence constant as to n; the other contains n, and hence changes with it. If for increasing n the second becomes less and less, with zero as limit, the series is convergent.

A rigorous proof of (10) by the $(n + 1)$-rule, though somewhat lengthy, is not difficult to give.

With θ made zero, (10) properly reduced, assumes the special form:

$$(12)\qquad (1 - x)^m \sum_{t=0}^{t=n} c_{m-1+t,\, m-1}x^t = 1 - \sum_{t=0}^{t=m-1} c_{n+t,\, t}(1 - x)^t x^{n+1},$$

which can be viewed in three different aspects.

Written in the form:

$$\sum_{t=0}^{t=n} c_{m-1+t,\, m-1}x^t = \frac{1}{(1 - x)^m} - \frac{1}{(1 - x)^m} \sum_{t=0}^{t=m-1} c_{n+t,\, t}(1 - x)^t x^{n+1},$$

it represents in its second member, in a definite number of terms, the sum of the series in the first member, consisting of any number of terms. The first expression of the second member being invariant as to n, becomes the sum of the series when the second, dependent upon n, vanishes by special values of n or x. Thus, for $m = 2$, this second expression becomes zero when $x = (n + 2)/(n + 1)$; and this, substituted in (12), gives the identity:

$$(13)\qquad 1 + 2\left(\frac{n+1}{n}\right) + 3\left(\frac{n+1}{n}\right)^2 + \cdots + n\left(\frac{n+1}{n}\right)^{n-1} = n^2.$$

If the second expression in the previous formula becomes indefinitely small as n indefinitely increases, we have the summation of the infinite convergent series.

Written in the form:

$$\frac{1}{(1 - x)^m} = \sum_{t=0}^{t=n} c_{m-1+t,\, m-1}x^t + \frac{1}{(1 - x)^m} \sum_{t=0}^{t=m-1} c_{n+t,\, t}(1 - x)^t x^{n+1},$$

it represents the development of the binomial with negative exponents, *i.e.*, $(1 - x)^{-m}$, into a finite series with a remainder term. The development is therefore good whether the series is finite, convergent or divergent.

If considered in the order:

$$\frac{1 - \sum_{t=0}^{t=m-1} c_{n+t,\,t}(1 - x)^t x^{n+1}}{(1 - x)^m} = \sum_{t=0}^{t=n} c_{m-1+t,\,m-1} x^t,$$

it presents an extension of the theorem, that the difference of the same powers is divisible by the difference of the first powers.

Let, for illustration, $n = 3$, $m = 1, 2, 3, \cdots$, and we shall have successively:

$$\frac{1 - x^4}{1 - x} = 1 + x + x^2 + x^3,$$

$$\frac{1 - x^4 - 4(1 - x)x^4}{(1 - x)^2} = 1 + 2x + 3x^2 + 4x^3,$$

$$\frac{1 - x^4 - 4(1 - x)x^4 - 10(1 - x)^2x^4}{(1 - x)^3} = 1 + 3x + 6x^2 + 10x^3,$$

etc.; where the ubiquitous binomial coefficients are recognized in their usual rôle.

IV. An Erroneous Rule for Finding a Hypotenuse, with a Corollary.

By M. W. Jacobs, Jr., Harrisburg, Pa.

To find the hypotenuse of a right-angled triangle, take half the length of the longer leg and add the length of the shorter leg. This was the rule nearly all the boys of an arithmetic class were found to be using. Much to the instructor's surprise, the correct answer was thus obtained to almost all of the examples. An inspection of the problems correctly solved in this manner showed that the ratio of the longer to the shorter leg was 4 : 3.

That this is the only ratio for which the boys' rule holds true may be demonstrated as follows: Let a = longer leg; b = shorter leg; h = hypotenuse. By the boys' rule

$$h = \frac{1}{2}a + b = \frac{a + 2b}{2}.$$

Then

$$h^2 = \frac{a^2 + 4ab + 4b^2}{4}.$$

But $h^2 = a^2 + b^2$. Hence

$$a^2 + b^2 = \frac{a^2 + 4ab + 4b^2}{4}.$$

Then $4a^2 + 4b^2 = a^2 + 4ab + 4b^2$; whence $3a^2 = 4ab$; therefore $a/b = 4/3$. The

ratio a/b has the *single value* 4/3 (when in lowest terms); the rule holds for this but for no other value.

Corollary: Generalizing by substituting the rational fraction t/u in place of 1/2 in the boys' rule, we get:

$$h = \frac{t}{u}a + b = \frac{ta + ub}{u};$$

then

$$h^2 = a^2 + b^2 = \frac{t^2a^2 + 2tuab + u^2b^2}{u^2}.$$

Whence $u^2a^2 - t^2a^2 = 2tuab$, and $a(u^2 - t^2) = 2tub$.

Consequently

$$\frac{b}{a} = \frac{u^2 - t^2}{2tu}.$$

The two legs are therefore in the proportion: $(u^2 - t^2) : 2tu$. Let us take exactly the values $u^2 - t^2$ and $2tu$ for the sides.

Hence the hypotenuse $= \sqrt{(u^2 - t^2)^2 + (2tu)^2} = u^2 + t^2$. Therefore

$$a = u^2 - t^2; \qquad b = 2tu; \qquad h = u^2 + t^2, \tag{1}$$

where t and u are integers, to be chosen at pleasure, (except that $u > t$, since a is to be positive). More generally we may write

$$a = c(u^2 + t^2); \qquad b = c(2tu); \qquad h = c(u^2 + t^2), \tag{2}$$

which is the well-known formula for Pythagorean triangles in its most general form. The particular case, $c = 1$, gives the primitive types.

RECENT PUBLICATIONS.

REVIEWS.

History of the Theory of Numbers.

History of the Theory of Numbers. Volume I: *Divisibility and Primality*. By LEONARD EUGENE DICKSON. Carnegie Institution of Washington, 1919, publication number 256, Vol. I. 14 + 486 pages. Price, in paper, \$7.50; in cloth, \$8.00.

There are at least three great ends to serve by the history of science: to enrich the general culture and intellectual life of cultivated people; to enable a scientific worker quickly to orient himself in a chapter of science so as to proceed most readily to a detailed mastery of its literature; to enable a scientific worker to ascertain with completeness what has already been attained in a given subject. To serve any one of these three purposes requires a very different

treatment from that needed in the case of the others. It is probably impossible to contribute satisfactorily at the same time and by the same work to any two of them. The present literature of each class is altogether inadequate in the case of science in general and in each particular discipline.

It is hard to say which of these three aspects of the need of scientific history presses hardest upon our attention. Each one would probably decide in favor of that which most vitally affects his own particular interest. Probably the last of the three purposes named is realized in the fewest instances. This work of Dickson, of which the first volume is now before us, embodies an attempt to serve this purpose adequately so far as the theory of numbers is concerned. The work is arranged so as to be suitable not only to the needs of the researcher but also to the interests of the amateur who studies number theory in the spirit of play and with restful delight.

From the preface we quote: "The theory of numbers is especially entitled to a separate history on account of the great interest which has been taken in it continuously through the centuries from the time of Pythagoras, an interest shared on the one extreme by nearly every noted mathematician and on the other extreme by numerous amateurs attracted by no other part of mathematics. This history aims to give an adequate account of the entire literature of the theory of numbers. The first volume presents in twenty chapters the material relating to divisibility and primality. The concepts, results, and authors cited are so numerous that it seems appropriate to present here an introduction which gives for certain chapters an account in untechnical language of the main results in their historical setting, and for the remaining chapters the few remarks sufficient to clearly characterize the nature of their contents."

To give an adequate account of the entire literature of so vast a subject and one of such long history as the theory of numbers is an undertaking of enormous magnitude; and it is carried through in this work with a marvelous success in the presence of which one must pause in admiration. Henceforth this history will be indispensable to all investigators in the theory of numbers. No one will be justified in publishing his researches on this subject till he has compared them with the relevant chapters of this work. Frequent duplications, all too common in the past development of the subject, will have far less excuse for their continuance. A great deal of scientific energy will be saved through this storehouse of exact information about the development of the theory of numbers and the actual present state of knowledge.

As indicated already, the subject is well suited to the interests of amateurs in mathematics. The reviewer's private correspondence shows that not a few such people, interested in number theory, are to be found throughout our country (and presumably in other countries as well). Many of these are far separated from adequate scientific libraries. To such this book will now be indispensable. With it in hand, even though they are separated from library sources, they will be able to get a fair estimate of the novelty of their work. The book will certainly add greatly to the delight and the value of the work of such people. Since

the theory of numbers is precisely that part of mathematics in which the amateur has the greatest opportunity of making a contribution of interest it may be hoped that one of the pleasing fruits of publishing such a work as the present will be the inspiration afforded to amateurs to pursue their work and to make known their discoveries in this field of fascinating interest which has inspired the enthusiasm of people of every century for some millenniums.

Where the detailed results are so numerous and are distributed widely through so varied a literature it is impossible that the first adequate research into the matter should uncover all the information and that no errors should be made in the presentation of what was found. Realizing this, the author has put in his preface the statement: "Readers are requested to . . . increase the usefulness of this work by sending corrections, notices of omissions, and abstracts of papers marked not available for report, for insertion in the concluding volume." It is to be hoped that this request may have wide notice and that every addition or correction found to be needed will be sent to the author at the University of Chicago. Besides several minor corrections which the reviewer is sending to the author in this way mention should now be made of the following changes needful to avoid confusion or annoyance on the reader's part: p. 131, l. 4, change "simple" to "prime power cyclic"; p. 381, l. 11, change k to m in two places; p. 403, l. 22, change $U_{2\ +1}$ to U_{2n+1}; p. 413, l. 13, read J. Perott[8] noted that, if $p_1, \cdots, p_n$ are all the primes and their product $\leqq N \cdots$.

Probably no one sufficiently interested in the theory of numbers to read this review will be satisfied without examining the volume itself. He will naturally begin with the ten-page preface or prefatory introduction, where we have a rapid review of some of the main features of the development of the theory covered by the present volume. Not a few people will wish that this had been more extended, especially since the immense amount of detail necessary in the body of the book makes it hard sometimes to select the salient matters. Necessarily the classification is not into rigidly separated portions so that matters more or less closely related will sometimes be found in different chapters. The subject index will be found to bring these together in a helpful way.

The book opens with a forty-eight page history of perfect, multiply perfect, and amicable numbers, a group of closely related topics which have engaged the attention of arithmeticians of every century of the Christian era. If the sum $\sigma(n)$ of the divisors of n is a multiple mn of n then n is said to be a multiply perfect number of multiplicity m. If $m = 2$, so that n is the sum of its aliquot divisors, we say that n is perfect. Twelve perfect numbers are known as such. The list of multiply perfect numbers is much larger. In comparing the two new multiply perfect numbers due to Cunningham and given on page 37 of Dickson's *History* with the table of Carmichael and Mason I discovered in the latter the error of taking 137561 [= 151.911] to be a prime. This requires the exclusion from their table of the 19 numbers which have the factor 19^4 (but not 19^5). But the two numbers of Cunningham show that if one of the numbers $C \cdot 19^2 \cdot 127$ and $C \cdot 19^4 \cdot 151 \cdot 911$ is multiply perfect and C is prime to $19 \cdot 127 \cdot 151 \cdot 911$ then the

other of these numbers is also multiply perfect of the same multiplicity. From this principle and the table of Carmichael and Mason it is easy to find (in a manner now obvious) 47 additional multiply perfect numbers. Moreover I have found also another number of multiplicity 7:

$$2^{32}\cdot 3^{11}\cdot 5^{4}\cdot 7^{8}\cdot 11^{2}\cdot 13^{2}\cdot 17^{2}\cdot 19^{3}\cdot 23\cdot 31\cdot 37^{2}\cdot 61\cdot 67\cdot 71\cdot 73\cdot 89\cdot 181\cdot 307\cdot 1063\cdot 2141$$
$$\cdot 599479.$$

In Carmichael and Mason's list the first number containing the factor 2^{46} should have the factor 67 inserted into it. With these additions and corrections there are now 282 known perfect and multiply perfect numbers.

Two numbers m and n are said to be amicable if their sum is equal to the sum of the divisors of each. Sixty-eight such numbers are known of which fifty-nine are due to Euler, three having been discovered before his time and six since. We propose to extend the notion of amicable numbers (in a way different from those indicated on page 50 of Dickson's *History*) and to say that two numbers m and n are multiply amicable of multiplicity k if k times their sum is equal to the sum of the divisors of each, so that we have $\sigma(m) = \sigma(n) = k(m + n)$. The usual amicable numbers are those for which $k = 1$. That multiply amicable numbers exist is shown by the following examples for $k = 2$:

$$2^5\cdot 3^2\cdot 7\cdot 13\cdot\begin{cases}5\cdot 11\\ 71\end{cases}\qquad 2^2\cdot 3^2\cdot 5\cdot 7\cdot 13\cdot\begin{cases}11\cdot 19\\ 239\end{cases}.$$

The last is one of eleven pairs of multiplicity 2 which may be obtained from this one by replacing the factor $2^2\cdot 7$ by any one of the eleven known perfect numbers other than 6. Other similar sets may also be made up by multiplying certain amicable odd numbers of Euler's list by perfect numbers. It would be rather interesting to have a systematic determination of multiply amicable numbers after the method employed by Euler for $k = 1$ and so well described by Dickson on pages 42–46. [A similar generalization of Dickson's amicable triples (p. 50) would also be of some interest.]

After a short chapter on the number and sum of divisors and certain related problems of Fermat and Wallis, we have the extensive third chapter (pp. 59–103) on Fermat's and Wilson's theorems, their generalizations and converses, and symmetric functions modulo p of the positive integers less than p and prime to p; and this is followed by a closely related short chapter on the residue of $(u^{p-1} - 1)/p$ modulo p. In the preface it is pointed out that it was while investigating perfect numbers that Fermat was led to the theorem which bears his name and which forms the basis of a large part of the theory of numbers. Numerous properties and generalizations of Euler's φ-function are given in the long chapter V (pp. 113–158). Owing to the frequent occurrence of the φ-function in many parts of number theory the reader should employ the index to find accounts of its use in other ways than in those mentioned in chapter V. In this connection special attention should be given to the chapters on Lucas's functions and on

recurring sequences of integers. The great amount of important detail included in such major chapters as III and V is such as to render an adequate summary impossible.

After an elementary chapter dealing with periodic fractions and factors of $10^n \pm 1$ we have in VII (pp. 181–222) and VIII (pp. 223–262) two other of the more fundamental chapters. The first of these deals with primitive roots and binomial congruences; the second with higher congruences. Together they form one of the most valuable sections of the book. Following them we have a chapter of sixteen pages on divisibility of factorials and multinomial coefficients. Then comes chapter X (pp. 279–325) on the sum and number of divisors of an integer and several closely related topics. Many of the results summarized here are closely connected with or are derived by means of various functional relations, sometimes associated with functions constructed for such use and at other times having to do with those which give rise incidentally to such applications. As in the case of other important chapters, its content is too vast for summarizing.

The next five chapters have to do with minor matters: miscellaneous theorems of divisibility, greatest common divisor, least common multiple; criteria for divisibility by a given number; factor tables, lists of primes; methods of factoring; Fermat numbers $F_n = 2^{2^n} + 1$. These chapters are necessary to completeness; more than that, some of the matter recorded still has its fascinating appeal.

In chapters XVI and XVII (pp. 381–411) we have an account of the factors of $a^n \pm b^n$, recurring series and Lucas's functions. The more or less fragmentary results here gathered together are of interest because of their intimate connection with the factorization of large numbers and the identification of large primes. Here the extent of our knowledge is so clearly marked by a definite line of cleavage between the known and the unknown, and the former is so clearly a mere fragment of the latter, that we contemplate our present state of knowledge with much the same feeling as that of the fabled Tantalus. The general theory of recurrent sequences of integers deserves a more penetrating analysis than it has yet received. It appears to be destined to bring out deep-lying connections among important parts of the theory of divisibility and primality. One who would go into this field for research could readily orient himself in it by means of Dickson's *History* and the two volumes (especially the second) of Bachmann's *Niedere Zahlentheorie.*

In Chapter XVIII (pp. 413–440), on the theory of prime numbers, we have in one respect, a departure from the plan followed in the others. That part of the subject covered by Landau's *Verteilung der Primzahlen* is passed over with a reference to its treatment in that work, particularly in the historical sections. Such other matter as pertains to the theory of primes is adequately treated in this chapter. To the various proofs that an infinitude of primes exists may be added the following: Since for a positive integer k we have $2^{2^n} \equiv -1$ or $(2^{2^n})^{2k} \equiv (-1)^{2k}$ or $2^{2^{n+k}} + 1 \equiv 2 \bmod 2^{2^n} + 1$ it follows (as is well known) that no two numbers of the form $2^{2^n} + 1$ have a common prime factor. Hence there are at least as many primes as numbers of this form and hence an infinite number.

One should consult the index for references to other tests for primes than those given in this chapter.

Chapter XIX (pp. 441–451) deals with a very interesting range of material under the heading: inversion of functions, Möbius's function $\mu(n)$, numerical integrals and derivatives. Important facts relating to inversion appear in other chapters and may be found through use of the index. Some of the inversion properties are very remarkable. A further systematic study of inversion would probably lead to novel results of interest. The relation $F(m) = \Sigma f(d)$, where d ranges over all the divisors of m, may be expressed by means of a Stieltjes integral in the form

$$F(m) = \int_0^m f(t)d_t\rho(m, t),$$

when $\rho(m, t)$ denotes the number of divisors of m not greater than t and $f(t)$ is a continuous function of t taking integral values when t is integral. The number-theoretic solution expressing f in terms of F leads to a solution in a special case of the foregoing Stieltjes integral equation. Other inversion problems may likewise be expressed by means of a numerical Stieltjes integral equation of the above or similar type. Is it likely that the inversion of numerical functions has already yielded all the results which would be brought to light by a systematic study of the inversion of numerical Stieltjes integrals formed in a manner now obvious, if these should be studied systematically under the guidance afforded by the existing extensive theory of ordinary integral equations?

The volume closes with thirteen pages on properties of the digits of numbers, at the same time one of the most elementary and one of the least interesting chapters of the book.

Not a few conjectured or empirical theorems are mentioned at various places throughout the work. These have interest as suggesting unanswered questions and as pointing out apparently simple matters which we are not at present able to treat satisfactorily. In this connection it is interesting to quote from Gauss: "The most beautiful theorems of the higher arithmetic have this characteristic that they are easily discovered by induction while their proof lies concealed and can be brought out only by a deeply penetrating investigation." And he adds that precisely this constitutes one of the greatest charms in number theory investigation.

Of the conjectured (unproved but not disproved) theorems in the present volume we call attention to the following:

1. If $2^n - 1$ is a prime p then $2^p - 1$ is a prime (pp. 22 and 24). This theorem, if true and proved, would yield an infinite sequence of identified primes; no such sequence is known at present.

2. If n is a prime of the form $24x + 11$ [$24x + 23$] and if $2^n - 1$ is composite, the least factor is of the form $24y + 23$ [$48y + 47$] (pp. 29, 30, 31).

3. If p is any number and a any divisor of $2^p - 1$, $a = 8m \pm 1$ not being of the form $2^n - 1$, then $2^a - 1$ is composite (p. 30).

4. Let $s(n) \equiv s^1(n)$ denote the sum of the divisors of n less than n and form $s^k(n) = s\{s^{k-1}(n)\}$. The set of numbers n, $s(n)$, $s^2(n)$, $\cdots$ may form a periodic cycle. When they do not it is conjectured that the set contains a prime and hence terminates (pp. 48–50).

5. Every prime $an^2 + p$ has a as a primitive root if a is a primitive root of p and p is a prime between $a/2$ and a (p. 186).

6. Let N be the greatest common divisor of all integers represented by a polynomial $f(x)$ with integral coefficients without a common factor. It is conjectured that $f(x)/N$ represents an infinitude of primes when $f(x)$ is irreducible (p. 333).

7. Every number of the following sequence is a prime (p. 376):

$$2 + 1, \quad 2^2 + 1, \quad 2^{2^2} + 1, \quad 2^{2^{2^2}} + 1, \quad \cdots.$$

8. Dickson asked if $a_i n + b_i$ $(i = 1, \cdots, m)$ represent an infinitude of sets of primes when certain named necessary conditions are satisfied (p. 417).

9. Every even integer is a sum of two primes (pp. 421–424). On pp. 424 and 425 are several related theorems: every odd number is the sum of an odd square and the double of a prime of the form $4n + 1$; every prime $4n - 1$ is the sum of a prime $4m + 1$ and the double of a prime $4h + 1$ [from which it would follow readily that every integer is a sum of four squares]; every even number is the difference of two (consecutive) primes in an infinitude of ways; every odd number greater than 3 is of each of the forms $p_1 + 2p_2$, $p_1 - 2p_2$, $2p_1 - p_2$, where p_1 and p_2 are primes; every multiple of 6 is the difference of two primes of the form $6n + 1$.

10. No three successive primes are in arithmetical progression unless one of them is 3 (p. 425).

11. Is there an unlimited number of sets of five consecutive odd integers of which four are prime (p. 426)?

12. If $n > 1$ there is at least one prime in each of the intervals $n(n - 1)$ to n^2, n^2 to $n(n + 1)$ (p. 435).

13. At least four primes lie between the squares of two consecutive primes each greater than 3 (p. 436).

14. The $(2m + 1)$th prime in order of magnitude (unity being counted as a prime) can be composed by addition and subtraction of all the smaller primes each taken once; the $(2m)$th prime can be composed similarly, except that the next earlier prime is doubled (p. 436).

It would be of interest to have some of these conjectured theorems verified for a considerable range of values; it is a service which may well be rendered by an amateur in number theory.

I have verified for primes less than 1300 the conjecture that a prime of the form $4n + 3$ is the sum of a prime $4m + 1$ and the double of a prime $4h + 1$; the number of representations is sometimes large and seems to have a tendency to increase with increasing primes $4n + 3$. I have also verified up to 1300 that primes of the form $16n + 5$ [$36n + 7$] may be represented in the form $4p + q$ [$6p + q$] where p and q are primes of the form $4k + 1$.

A work of such extent and importance as the volume under review cannot be adequately described in a short article. To give a summary is impossible, since its 500 pages are themselves a summary in a very compact but lucid style of the results of many hundred memoirs. It is a piece of work for which one cannot find a parallel in the whole of scientific history. In completeness it goes far beyond anything before attempted for a wide range of material; and the work is done with a skill and accuracy which excite admiration. All persons interested in the theory of numbers are laid under a deep debt of gratitude to Professor Dickson.

R. D. CARMICHAEL.

NOTES.

Teubner announces the publication (1918) of a *Lehrbuch der Differential- und Integralrechnung und ihre Anwendungen* in two volumes, by R. Fricke.

The *Journal of the Michigan Schoolmasters' Club* (Ann Arbor, 1919) contains the proceedings of the fifty-third meeting, May 28–29, 1918. These include the following papers: "Group recitations in mathematics" by L. D. Wines, 82–86; "The value of measurements to high school mathematics" by S. A. Courtis, 87–94.

The following new English books are announced: H. Lamb, *Infinitesimal Calculus*, third edition (Cambridge University Press)—L. Silberstein, *Elements of Vector Algebra* (Longmans)—F. Cajori, *History of the Theories of Limits and Fluxions from Newton to Woodhouse* (Open Court)—H. E. J. Curzon, *Elementary Mathematics*, I and II.

W. Ahrens has published (Berlin, 1918) a new book on mathematical recreations, entitled *Altes und Neues aus der Unterhaltungsmathematik*. It is said by a reviewer in the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* to contain several new topics in this field, and to be written in a very interesting style.

Oldenburg of Munich has published (1918) the eighth edition of *Einführung in die Mathematische Behandlung der Naturwissenschaften*, by W. NERNST and A. SCHOENFLIES, which was intended primarily for chemists. It will be recalled that *The Elements of the Differential and Integral Calculus* by J. W. A. YOUNG and C. E. LINEBARGER (New York, Appleton, 1900) was based upon the second edition of this book (1898) which first appeared in 1895.

The forty-fourth volume (Jahrgang 1913) of *Jahrbuch über die Fortschritte der Mathematik* was completed in 1918 and contained 86 + 1210 pages,—slightly more than the previous volume—The first number of the eightieth volume of the *Mathematische Annalen* was published last August.

A third edition of *American Men of Science, A Biographical Directory*, edited by J. McK. CATTELL, is in course of preparation. The appeal for information

in *Science*, July 25, 1919, states that there should be included in the book the names of all those in North America who have made contributions to the natural and exact sciences.

Archivio de Storia della Scienza (Roma, Nardecchia) is the title of a new quarterly periodical (Aldo Mieli, editor), the first number of which appeared last March. One of its articles, (pages 39–47) "Per una storia delle mathematiche nel secolo XIX" by G. Loria, seems to have been inspired by F. Cajori's address, on "Plans for a history of mathematics in the nineteenth century," in 1918, as retiring president of the Mathematical Association of America.

The Rice Institute Pamphlets, volume 1, no. 2, May, 1915, contains English translations of two lectures delivered at the opening of the Rice Institute in 1912: (1) "Henri Poincaré" by Vito Volterra, 133–162 (portrait of Poincaré) (2) "Molecular theories and mathematics" by Emile Borel, 163–193. Reference has been already made in this Monthly (*1917*, 250) to other lectures by Volterra and Borel published in *The Pamphlets*. All of these lectures are reprinted from volumes 2 and 3 of *The Book of the Opening of The Rice Institute*, Houston, Texas.

As announced in our September issue, H. G. Zeuthen, professor emeritus of the University of Copenhagen and of the Ecole Polytechnique of Copenhagen, died on his eightieth birthday. Among the books of this Nestor of Danish mathematicians are: *Die Lehre von den Kegelschnitten in Altertum* (German edition 1886), *Geschichte der Mathematik im Altertum und Mittelalter* (German edition 1896; briefer treatment, 1912), *Geschichte der Mathematik im XVI. and XVII. Jahrhundert* (German edition, 1903), and *Lehrbuch der abzählenden Methoden der Geometrie* (1912) a development of his *Encyklopädie* article (1905, French edition, 1915). As co-editor with Heiberg there had been issued two of the projected twelve volumes of *Mémoires scientifiques de Paul Tannery*.

Portraits and sketches of Zeuthen may be found in *Matematisk Tidsskrift*, A, February, 1919, in *Acta Mathematica, 1882–1912, Table générale* (Upsala. 1913), and in E. F. S. Lund's *Danske malede portraeter*, vol. 7 (Copenhagen, 1900).

ARTICLES IN CURRENT PERIODICALS.

American Journal of Mathematics, volume 41, no. 3, July, 1919: "Invariants of differential geometry by the use of vector forms" by C. D. Rice, 165–182; "On certain saltus equations" by H. Blumberg, 183–190; "Investigations on the plane quartic" by Teresa Cohen, 191–211; "On surfaces containing two pencils of cubic curves" by C. H. Sisam, 212–224; "Modular invariants of a quadratic form of a prime power modulus" by J. E. McAtee, 225–242.

Annals of Mathematics, 2d series, volume 20, no. 4, June, 1919: "Relations between abstract group properties and substitution groups" by G. A. Miller, 229–231; "The complete quadrilateral" by J. W. Clawson, 232–261; "Triply conjugate systems with equal point invariants" by L. P. Eisenhart, 262–273; "On a system of linear partial differential equations of the hyperbolic type" by T. H. Gronwall, 274–278; "Some properties of circles and related conics" by J. H. Weaver, 279–280; "Integrals in an infinite number of dimensions" by A. J. Daniell,

281–288; "On the differentiability of the solution of a differential equation with respect to a parameter" by J. F. Ritt, 289–291; "Note on the derivatives with respect to a parameter of the solutions of a system of differential equations" by T. H. Gronwall, 292–296; "On Quaternions and their generalization and the history of the eight-square theorem. Addenda" by L. E. Dickson, 297.

Atlantic Monthly, volume 124, no. 3, September, 1919: "The Scandal of Euclid, a Freudian analysis" by S. Strunsky, 332–337.

Bulletin of the American Mathematical Society, volume 25, no. 10, July, 1919: "The April meeting of the American Mathematical Society," by F. N. Cole, 433–448; "Report on the theory of the geometry of numbers" by H. F. Blichfeldt, 449–453; "Applications of the geometry of numbers to algebraic numbers" by L. E. Dickson, 453–455; "Products of skew-symmetric matrices," by A. A. Bennett, 455–458; "On the first factor of the class number of a cyclotomic field" by H. S. Vandiver, 458–460; "Corrections and note to the Cambridge Colloquium of September, 1916" by G. C. Evans, 461–463; "Circle and sphere geometry" by H. S. White, 464–467 [Review of Coolidge's *Treatise on the Circle and Sphere* (Oxford, 1916)]; Review by C. N. Moore of Karpinski, Benedict and Calhoun's *Unified Mathematics* (New York, 1918), and Pringsheim's *Vorlesungen über Zahlen- und Funktionenlehre*, Band 1 (Leipzig, 1916), 467–470; Review by L. W. Dowling of Fazzari's *I numeri reali e l'equazione esponenziale* $a^x = b$ *per le scuole medie superiori* (Palermo, 1918), 470–471; Review by F. M. Morgan of Carey's *Infinitesimal Calculus* (London, 1917), 472–473; Review by E. W. Brown of *Annuaire du Bureau des longitudes pour l'an 1919* (Paris, 1918), 473; "Notes," 474–480; "New Publications," 481–483; "Twenty-eighth annual list of papers," 484–491; "Index," 492–499.

Educational Review, volume 58, no. 3, October, 1919: "Reopening of the Ecole Normale Supérieure" by A. Cohn, 181–200 [The opening exercises occurred March 23, 1919].

Harvard Graduates Magazine, volume 27, June, 1919: "Edward Charles Pickering" by J. H. Metcalf, 516–520 and Portrait; "More Reminiscences of '66" by George Batchelor [Pages 531–532—Benjamin Peirce: "In a tablet in the Exposition in Paris at the foot of a list of twenty of the greatest mathematicians of the last two thousand years appeared the name of Benjamin Peirce, our senior professor of mathematics. In '66 he was giving instruction which nobody understood, to a select audience of students who were trying to become mathematicians. He would begin by asking questions and attending to the answers, and then as some new truth flashed upon his mind, he would forget his class and cover the blackboard with algebraic symbols.

"At this time, Thomas Hill was president of the college, a great mathematician with no practical ability as an administrator. Before coming to Cambridge, he was minister of the Unitarian Church in Waltham. It is recorded by Robert S. Rantoul that when Peirce caught sight of some new truth which he could not reduce to writing, he would hasten to Waltham and describe to Hill what was disturbing his mind. He was the only man in America who could understand Peirce, and having 'no such original inspirations to trouble him, could better express in words, the new proposition when at last he understood it,' 'and towards morning send him home to Cambridge with his problem stated on paper in his pocket, and his mind at rest.'

"In a parlor in Boston, I heard Peirce in his old age deliver his last address. With great positiveness, he affirmed his belief in immortalility. He said, 'All my life I have studied the stars at long range. I expect to study them at short range. I may ride on the tail of a comet.' He said also, 'I have conceived the idea of an algebra of universal thought, but I am no longer able to work it out, and must leave it to those who come after me.' Probably no living man knows what he meant by that proposition.

"Of students that he instructed for many years in college, he said, 'We have seen many young men of brilliant parts, of whom we expected great things as scientific investigators; but they never fulfilled the promise of their youth. After long deliberation, I have come to the conclusion that the reason they failed was because they had bad hearts. The student of science must be a truth-seeker, and a truth teller, and the man whose heart is bad cannot be either.' "

Jahresbericht der deutschen Mathematiker Vereinigung, volume 27, nos. 9–12, September–December, 1918: Determinanten und symmetrische Funktionen" by C. Kosta, 161–165; "Ausgezeichnete Elemente projektiver Gebilde, die ineinander liegen, und Folgerungen für die Homologien" by R. Sturm, 166–175; "Herstellung von Polaren" by R. Sturm, 175–178; "Die Mehrdeutigkeit von Integralen" by M. Pasch, 178–179; "Ueber die Bedingung der Integrierbarkeit" by M. Pasch, 179–181; "Ueber die Anordnung von vier Punkten einer Geraden" by L. v. Schrutka, 182–184; "Ueber die Vertauschbarkeit der Differentiationsfolge" by H. Hahn,

184–188; "Inwieweit kann Vandermonde als Vorgänger von Gauss bezüglich der algebraischen Auflösung der Kreisteilungsgleichung $x^n = 1$ angesehen werden?" by A. Loewy, 189–195; "Gaston Darboux" by A. Voss, 196–217; "Festrede zum 20 Stiftungstage der Göttinger Vereinigung zur Förderung der angewandten Physik und Mathematik" by F. Klein, 217–228 (full page portrait of Klein); "Die Forderung der Entscheidbarkeit," 228–232; "Ueber die Erweiterung des Grenzbegriffs" by M. Pasch, 232–234; "Ueber Kurven gleichmässigster Krümmung" by W. Blaschke, 234–236; "Bemerkung zu: Ueber Kurven gleichmässigster Krümmung" by R. v. Mises, 236; "Zu Felix Kleins goldenem Doktorjubiläum am 12. Dezember, 1918", *63–64*; Review by Bieberbach of Gauss's *Werke*, Band 10_1 (Leipzig, 1917), *66–68*; Review by F. Bernstein of Ziehen's *Das Verhältnis der Logik zur Mengenlehre* (Berlin, 1917), *66–68*.

Journal of the Indian Mathematical Society, volume 11, no. 2, April, 1919: The Indian Mathematical Society and its Founder, Mr. V. Ramaswami Aiyar (Portrait frontispiece), 41–45 ["Our Society has just completed the twelfth year of its existence"]; "Conics with vanishing Θ, Θ'" by R. Vythynathaswamy, 46–49; "Bernoulli's polynomials and Fourier's series" by G. R. Ranganathan, 50–56; "Legitimacy of ordinary complex numbers" by G. A. Miller, 57–59; "Astronomical notes," 59–61; Problems and solutions, 62–80.

Mathematical Gazette, volume 9, no. 140, May, 1919: "Bordered antilogarithms, Trigonometric logarithms to every two minutes. Natural functions on three pages. Tables of exact squares" by G. H. Bryan with the assistance of C. Williams, T. G. Greak, Violet James, and others, 333–352 [Quotation from "Explanation of the tables": "Although books of mathematical tables have appeared by the dozen in recent years, most of them show little or no evidence of originality in the matter of arrangement. One of the most common characteristics of such tables is that much space is taken up with unnecessary repetition, as when both logarithms and antilogarithms, squares and square roots, or sines and cosines of all circles *sic* from 0° to 90° are separately tabulated, while the information supplied in other directions is defective, as when logarithms of reciprocals are omitted and tables of squares fail to give the correct values of the squares of integers.

"The Bordered Antilogarithm Table was published in the *Mathematical Gazette* for December 1915, and several correspondents expressed a wish that other tables should also be drawn up embodying improvements in arrangement proposed by me. This has now been done, and I have to thank those associated with my name on the titles for the whole work of making the necessary tabulations. It is interesting to note that the method of tabulation introduced in the Bordered Antilogarithm Table has already been adopted in a set of tables published for use on board the training ship *Conway*."]; Review, by G. Greenhill, of A. Gray's *Gyrostatics and Rotational Motion* (London, 1918), 353–356; "Books received, contents of Journals, etc.," 357–360.

Mathematics Teacher, volume 11, no. 4, June, 1919: "Tests of mathematical ability—their scope and significance" by Agnes L. Rogers, 145–164 ["bibliography," pp. 163–164]; "Some suggestions for courses in mathematics for non-college preparatory students," 165–171 [report presented at the spring meeting of the Association of Teachers of Mathematics in New England, May 3, 1919—Chairman of Committee: C. E. Paddock]; "A statistical study in correlation of efficiency in secondary mathematics and efficiency in other high school branches" by Nelle L. Ingels, 172–176; "Some angles of the right triangle" by A. L. Booth, 177–181 ("note" by E. R. Smith); "A psychological basis for a system of education with applications to mathematics" by W. P. Webber, 182–195; "Philosophy and non-euclidean geometry" by F. A. Foraker, 196–198; "Mathematical tests:—their relation to the mathematics teacher" by J. H. Minnick, 199–205; "Book Reviews," "Notes and News," 206–209.

Nature, volume 103, June 12, 1919: "The age of the stars" by H. Shapley, 284—June 19: "Question relating to prime numbers" by A. Mallock, 305 [Quotations: "It is well known that no algebraical formula can represent prime numbers only, and that primes can only be found by trials (which may be facilitated by algebraical processes). If the mth prime number, counting from unity be denoted by n, and if n is plotted in terms of m, it will be found that n is approximately represented by a formula of the type Am^p ($A = 3\cdot 15$, $p = 1.133$, are close to the values of the constants). . . . Are there any investigations which give a reason for the tendency of n to approach a definite function of m, or as to the ultimate value of dn/dm when m increases without limit?"]—July 10: "Question relating to prime numbers" by G. N. Watson, 364–365 [Quotation: "An approximate expression for n is $m \log_e m$."]—July 17: "American astronomy", 394 [reprinted in *Science*, August 22.]

Proceedings of the National Academy of Sciences, volumes 5, no. 6, June, 1919:

"On the most general class L of Fréchet, in which the Heine-Borel-Lebesgue theorem holds true" by R. L. Moore, 206–210; "On a certain class of rational ruled surfaces" by A. Emch, 222–224—No. 7, July: "On the twist in comformal mapping" by T. H. Gronwall, 248–250 ["Note II on conformal mapping under aid of Grant no. 207 from the Bache Fund"]; "Groups involving only two operators which are squares" by G. A. Miller, 272–274; "Real hypersurfaces contained in Abelian varieties" by S. Lefschetz, 296–298.

Proceedings of the Royal Society, London, series A, volume 95, June, 1919: "Bertram Hopkinson (with portrait)" by J. A. E., xxvi–xxxvi. [Born 1874; killed in a flying accident August, 1918. When only 29 years of age he was appointed to the chair of mechanism and applied mechanics in his alma mater, the University of Cambridge. "Many will mourn him as a trusted friend, but only those who knew something of what he did in the war can have a right idea of the magnitude of the nation's loss."]

Quarterly Journal of Pure and Applied Mathematics, volume 48, no. 3, June, 1919: Calculation of eighteen more, fifty in all, Eulerian numbers from central differences of zero" by S. A. Joffe, 193–271; "A new solution of Waring's problem" by G. H. Hardy and J. E. Littlewood, 272–288.

School and Society, volume 10, July 19, 1919: "Educational research and statistics; practical uses of an algebra standard scale" by F. R. Cawl, 88–90. [Use of Hotz's *First Year Algebra Scales*, Teachers College, Columbia University, Contribution No. 90, 1918.]

School Review, volume 27, no. 6, June, 1919: "Concrete geometry in thc Junior High School" by W. H. Fletcher, 441–457.

Science, new series, volume 49, June 20, 1919: "The airplane in surveying and mapping" by E. L. Jones, 572–582—June 27: "Revista matemática Hispano-Americana" by G. A. Miller, 608–609—Volume 50, July 4, "'Working up' in a swing" by A. T. Jones, 20–21 [First paragraph: "A child sitting or standing in a swing can 'work up' until he is swinging through a considerable distance. How is it possible for him, without touching his feet to the ground, to increase the extent of his swinging? As I do not recall ever seeing any discussion of this matter, the following note may not be out of place."]—July 11: "The discovery of calculus", by A. S. Hathaway ,41–43 [First sentence: "The writer desires to call attention to certain disclosures here pointed out for the first time, whose conclusions are decisive in the matter of the celebrated controversy between Newton and Leibniz, regarding the discovery of calculus."]—July 18: "The history of science and the American Association for the Advancement of Science" by F. E. Brasch, 66–68; "Working up in a swing" by V. Karapetoff and P. E. Klopsteg, 70–71 [References for the treatment of Mr. Jones's problem are given to: (1) E. J. Routh's *Dynamics of a System of Rigid Bodies*, vol. 1, art. 287 (examples of living beings, no. 6); (2) *Zeitschrift für physikalischen und chemischen Unterricht*, vol. 16, 1913, p. 16, and vol. 17, 1914, p. 27.]

Scientia, volume 25, no. 5, May, 1919: "Le matematiche in Ispagna, ieri ed oggi. Parte prima: Dal secolo XVI alla metà del XIX" by G. Loria, 353–359 [French translation, supplément, 79–85]; Review by M. Davidson of Whitehead's *The Organisation of Thought* (London, 1917) and *Science and the Nation*, essays edited by Seward (Cambridge, 1917), 410–412.—No. 6, June: "Le matematiche in Ispagna, ieri ed oggi. Parte IIa: I matematici moderni" by G. Loria, 441–449 [French translation, supplément, 99–108]; Review by P. E. B. Jourdain of Byerly's *An Introduction to the Use of generalized Coordinates in Mechanics and Physics* (Boston, 1916), 499–500; Review by G. Scorza of Forsyth's *Lectures introductory to the Theory of Functions of two Complex Variables* (Cambridge, 1914) and Goursat's *A Course of Mathematical Analysis*, vol. 2, parts 1–2, translated by Hedrick and Dunkel (Boston, 1916–1917), 500–502; Review by A. Mieli of *L'ottica di Euclide* edited by Orio (Milano, 1918), 502–503. [In his second article on Spanish mathematics, in discussing advances of the science in the last third of the nineteenth century, Loria sketches the career of Josè Echegaray, known throughout the world, not as a mathematician, but as the popular author of dramatic productions for which a Nobel prize was awarded. He was born between 1832 and 1836 and died in 1916. He is the author of half a dozen mathematical works.] —Volume 26, no. 1, July: "Le matematiche in Portogallo; ciò che furono, ciò che sono" by G. Loria, 1–9 (French translation, supplément, 1–9); Reviews by G. Scorza of Blichfeldt's *Finite Collineation Groups* (Chicago, 1917), H. T. Hudson's *Ruler and Compasses* (London, 1916) and H. Hancock's *Elliptic Integrals* (New York, 1917), 47–49.—No. 2, August: "Origines et developpement de l'algèbre" by L. C. Karpinski, 84–101; "Il problema dei tre corpi" by R. Marcolongo, 102–112, supplément, 17–27; Review by G. Scorza of Bôcher's *An introduction to the study of integral equations* (2nd edition, Cambridge, 1914), Dickson's *Linear algebras* (Cambridge, 1914),

and, Hardy and Riesz's *The general theory of Dirichlet's series* (Cambridge, 1915), 147-150; Review by A. C. D. Crommelin of Plummer's *An introductory treatise on dynamical astronomy* (Cambridge, 1918), 150-152.

The Tôhoku Mathematical Journal, volume 15, nos. 3-4, April, 1919: "Note on Laguerre transformations," 227-231; "On Steiner's problem of closure" by K. Yanagihara, 232-235; "On curves with monotonous curvature" by T. Hayashi, 236-239; "Note on determinants whose matrix is that of an orthogonant increased or diminished by matrix unity," by T. Muir, 240-245; "On the sign and magnitude of the coefficients in the Fourier series, the sine series and the cosine series" by K. Ogura, 246-260; "Tractories in the irreversible field of force on a surface" by K. Ogura, 261-277; "A theorem on power series" by Y. Okada, 278-279; "A theorem on limits" by Y. Okada, 280-283; "An extension of a theorem of Scheefer's" by Y. Uchida, 284; "Ueber eine Ungleichung für bestimmte Integrale" by M. Jujiwara, 285-288; "Casey's Theorem in Japanese Mathematics" by T. Hayashi, 289-296 (in Japanese); "On Mr. Ône's theorem" by K. Ôishi, 297-299; "Some theorems on limits" by S. Narumi, 300-313; "Einige Sätze den Grenz wert betreffend" by T. Kubota, 314-322; "Ueber summierbare Reihen und Integrale" by M. Fujiwara, 323-329; "On the indeterminate form ∞/∞" by T. Hayashi, 330-336 (in Japanese).

Transactions of the Royal Society of Canada, volume 12, series III, Mathematical, physical and chemical sciences, December, 1918, and March, 1919: "Concerning the integrals of Leleuvre" by C. T. Sullivan, 171-184; "Rational plane unharmonic cubics" by A. M. Harding, 185-194.

AMERICAN DOCTORAL DISSERTATIONS.

R. W. Brink, "Some integral tests for the convergence and divergence of infinite series." *Published under the title:* "A new integral test for the convergence and divergence of infinite series," *Transactions of the American Mathematical Society*, 1918, volume 19, pp. 186-204. (Harvard, 1916.)

L. R. Ford, "On rational approximations to an irrational complex number," *Transactions of the American Mathematical Society*, 1918, volume 19, pp. 1-42. (Harvard, 1917.)

M. T. Hu, "Linear integro-differential equations with a boundary condition," *Transactions of the Amerieau Mathematical Society*, 1918, volume 19, pp. 363-407. (Harvard, 1917.)

W. E. Milne, "On the degree of convergence of Birkhoff's series." *Portions published as:* "On certain asymptotic expressions in the theory of linear differential equations," *Proceedings of the National Academy*, 1916, volume 2, pp. 543-545; "Note on asymptotic expressions in the theory of linear differential equations," *Bulletin of the American Mathematical Society*, 1917, volume 23, pp. 166-169. (Harvard, 1915.)

C. E. Wilder, "Problems in the theory of ordinary linear differential equations with auxiliary conditions at more than two points." *Portions published as:* "Expansion problems of ordinary linear differential equations with auxiliary conditions at more than two points," *Transactions of the American Mathematical Society*, 1917, volume 18, pp. 415-442. (Harvard, 1915.)

UNDERGRADUATE MATHEMATICS CLUBS.

Edited by U. G. Mitchell, University of Kansas, Lawrence.

CLUB ACTIVITIES.

The Mathematics Club of Albion College, Albion, Mich. [*1918*, 354-357.]

No meetings of this club were held during the period of the S. A. T. C. The first meeting of the year 1918-19 was held on January 21, 1919, and meetings were held regularly every other week thereafter, or as nearly according to this program as possible. Ten men and twelve women constituted the membership of the club for the year. Lieutenants W. Clarke Dean '21, and Murrey Fox '19,

and Captain Jesse Campbell '19 returned to the club after having been in war service.

Officers for the year 1918–19 were: President, Carla Kennedy '20; vice-president, Esther Pearl '20; secretary-treasurer, Donald Alexander '20; program committee: the vice-president, the secretary-treasurer and Floyd Harper '20.

The usual program for each meeting is as follows: (1) roll-call, each member responding to his name by a very brief discussion of an assigned topic; (2) short talk of the evening, limited to ten minutes; (3) long talk, occupying 20 to 30 minutes; (4) general discussion of any of the topics of the evening; (5) critic's report (the critic is usually a student).

Programs for 1918–19 are indicated below [(1) roll-call, (2) short talk, (3) long talk]. The short talks for the year were biographical.

January 21, 1919: Organization and election of officers.

February 4: (1) "A mathematical current event;" (2) "Thales" by Thelma Sharp '21; (3) President's address, "Mathematics applied" by Carla Kennedy '20.

February 18: (1) "A mathematical symbol, its meaning and origin;" (2) "Pythagoras" by Gertrude Pratt '21; (3) "Development and use of determinants" by Floyd Harper '20.

March 11: (1) "Special curves, equation, history and use;" (2) "Life of Euclid" by Christine Niemann '21; (3) "My experiences as a teacher of high-school mathematics" by Margaret Courtright '19.

March 25: (1) "An original mathematical jingle." (2) "Archimedes" by Donald Herrick '21; (3) "Mathematics applied to war" by Lieutenant Clark Dean '21.

April 22: (1) "Some interesting application of mathematics:" (2) "Napier" by Edgar Smith '21; (3) "Mathematics and anti-mathematics" by Vera Junkin '19.

May 6: (1) "Some desirable characteristics that I observed in my high-school teacher of mathematics;" (2) "Descartes" by Esther Pearl '20; (3) "Indeterminate forms in trigonometry" by Esther Pearl '20.

May 20: (1) None; (2) None; (3) "The relation of the teacher to the community" by Elizabeth Gordon '21.
General discussion: "How can a teacher best enter into the community life" led by Miss Gordon.

June 3: Social evening and election of officers. The officers elected for 1919–20 are: President, Esther Pearl '20; vice-president, Joyce Hadaway '20; secretary-treasurer, Almyra K. Priest '20; member of program committee, Elizabeth Gordon '21.

TOPICS FOR CLUB PROGRAMS.

16. Codes and Ciphers.

The devising of codes and ciphers and of methods for solving them is closely allied to mathematics. In fact, since every system of secret writing must be

based upon a certain fundamental set of symbols or sounds, all problems of deciphering such messages are essentially problems in permutations and combinations. The interest in the subject for students of mathematics is doubtless as much in the challenge to mathematical mastery as to the fascination of mystery.

The art of occult writing dates back to the ancient Greeks. The Spartans are credited with having sent messages by tattooing them upon the bodies of slaves. Sometimes, for greater secrecy, a slave whose scalp had been shaved and tattooed was not allowed to depart until his hair had grown long enough to conceal completely the message. When the slave had reached his destination the shaving of his scalp again revealed the message. The first record of secret writing on parchment, however, is probably that given by Plutarch in his life of Lysander where he gives[1] the following description of the manner of transmitting to that commander the order for his return:

> This scroll is made up thus: when the Ephors send an admiral or general on his way, they take two round pieces of wood, both exactly of a length and thickness, and cut even to one another; they keep one themselves and the other they give to the person they send forth; and these pieces of wood they call *scytales*. When, therefore, they have occasion to communicate any secret or important matter, making a scroll of parchment long and narrow like a leathern thong, they roll it about their own staff of wood, leaving no space void between, but covering the surface of the staff with the scroll all over. When they have done this, they take off the scroll, and send it to the general without the wood. He, when he has received it, can read nothing of the writing, because the words and letters are all broken up; but taking his own staff, he winds the slip of the scroll about it, so that this folding, restoring all the parts into the same order that they were in before, and putting what comes first into connection with what follows, brings the whole consecutive contents to view round the outside. And this scroll is called a *staff*, after the name of the wood, as a thing measured is by the name of the measure.

From Lysander's time (about 400 B.C.) to the present, great use has been made of secret writing by government officials in various connections of which the military has been the most important. A considerable number of special works on the subject have been written[2] but with the possible exception of Klüber's work, which seems to be the most widely known, these are not very likely to be available to most American students. The most available source for a brief discussion of the development of ciphers is likely to be the articles "Cryptography" by John Eglinton Bailey in recent editions of the *Encyclopædia*

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[1] *Plutarch's Lives*, the translation called Dryden's, corrected from the Greek and revised by A. H. Clough, in 5 vols. (Boston, Little, Brown & Co., 1905), Vol. III, pp. 125–126. A parallel rendering of the same passage is found in John and William Langhorne's translation (New York, 1858), p. 314.

[2] Some of those most frequently cited are:

Trithemius, John, *Polygraphia*, cited as published in 1500 and in various later editions.

Portae, J. B., *De furtivis literarum notis*, Neapoli, 1563, and 1602, Londoni, 1591; another edition: *De occultis literarum notis*, Montisbeligardi, 1593, Argentorati (Strassburg), 1603 and 1606. Discusses different modes of secret writing, of which one hundred and eighty are explained, and a method proposed by which they can be multiplied ad infinitum.

Schott, Kasper, *De Magia Universali*, Würzburg, 1676.

Klüber, J. L., *Kryptographik*, Tübingen, 1809.

Fleissner, *Handbuch der Kryptographik*, Vienna, 1881.

Delastelle, F., *Cryptographie*, Paris, 1902.

Brittanica and similar articles in several other encyclopædias. A classification of ciphers and a considerable number of interesting examples are given by Ball in his *Mathematical Recreations and Essays*.[1] But the most interesting illustrations and historically important examples are contained in various articles[2] printed in periodical magazines to be found in nearly every university or large public library.

Ciphers and codes are both based upon substitutions, the distinction being in the unit chosen as the basis of the substitution. A cipher is based upon the substitution of symbols for letters and a code is based upon the substitution of words or symbols for words. For example the two lines

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
+	∧	%	*	=	/	$	÷	π	?	d	h	4	k	b	v	6	8	0	(	)	√	l	a	1	t

could be made the basis of a cipher by agreeing that each letter in the upper line should always be represented by the character immediately beneath it in the second line. In this simple cipher the message "The Americans are coming" would read "(÷= +4=8π%+k0 +8= %b4πk$". Such a cipher could be improved by the further agreement that any character not appearing in the second of the two lines forming its basis should be considered as a "dead" or "null" character. By the use of these "null" characters as space fillers the division into words, which is of great aid in deciphering, could be avoided. However, any considerable message written in so simple a cipher as the above would be read in a very few minutes by an expert, since the normal relative frequency of occurrence[3] of letters in English is:

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[1] Fifth ed. (London, 1911), pp. 395–423; fourth ed. (1905), pp. 293–322.

[2] Of American articles appearing before the recent great war the following may be cited:

Century Magazine, Vol. 63, pp. 83–92, "Secret Writing, the Ciphers of the Ancients and some of those in Modern Use," by John H. Haswell.

Harper's Monthly, Vol. 97, pp. 105–109, "A Rebel Cipher Dispatch" by David Homer Bates. The author was manager and cipher-operator for the U. S. Military Telegraph Corps at Washington from 1861 to 1865. He describes ciphers used on both sides during the Civil War. Much other interesting information is given by the same author in an article entitled "Lincoln in the Telegraph Office," *Century Magazine*, Vol. 74, pp. 290–306.

North American Review, Vol. 128, 315–325, "Cryptography in Politics," by John R. G. Hassard. Gives examples of various ciphers and discusses some of the cipher telegrams connected with the presidential election of 1876. Other information concerning these cipher telegrams is given in the *Nation*, Vol. 27, p. 234, p. 250 and Vol. 28, p. 112.

Cosmopolitan, Vol. 36, pp. 475–478, 584–590, 715–718, three excellent articles on "Cryptography" by George Wilkes. Gives many examples of interesting and historical ciphers.

Bookman, Vol. 28, pp. 450–451, "Poe and Secret Writing," by Firman Dredd.

[3] The table given is from an article "Enciphering and Deciphering Codes" by Frank Morgan, of Fort Leavenworth, *Scientific American*, Vol. cxiii, No. 8 (August 21, 1915), p. 159. After giving an example of solving a much more difficult cipher message Mr. Morgan says: "We may say here that all practicable military ciphers are so marked by characteristic arrangement of the letters that no reasonably expert cipher man will be long in doubt as to the general method of enciphering. . . . A single short message would give trouble but military messages generally have fifty or more words."

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
20	4	8	11	33	6	5	16	17	1	2	10	8	19	21	6	$\frac{1}{4}$	18	17	23	8	3	5	1	5	1.

As an illustration of code,[1] suppose that a German spy and his correspondents agree upon the following substitutions:

a = the
French ship = market
sailed from New York = price
sailed from Boston = quotation
to-day = is
for Marseilles = any even number
for Bordeaux = any number with a fraction.

The intelligence that "A French ship sailed from New York to-day for Marseilles" in this code becomes the innocent business message "The market price is 110." By a slight change of wording, "The market quotation is $110\frac{1}{2}$" carries the information that "A French ship sailed from Boston to-day for Bordeaux."

Curiously enough, ciphers, which have been so widely used for secret writing and which were invented for that purpose, are not as securely secret as codes, which owe their development to their usefulness and economy in cabling commercial messages. Ciphers will, however, continue to be used where only *temporary* secrecy is desired, because, as Mr. Strother says, "cipher messages can be written and translated (by one's correspondent) without any equipment, like a code book, and much more rapidly than code. Thus, if a general in the field wishes to send a message ordering a colonel to advance in two hours, he sends it in cipher, because it would take the enemy more than two hours to decipher the message even if he intercepted it immediately, and because after the two hours have elapsed the information in the message would be of no value to him."

The recent great war not only led to renewed interest[2] in secret writing but also to the development of new forms. German spies made large use of geometric figures, drawings, sketches and pictures[3] as well as ciphers, codes and enciphered

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[1] Cf. Strother, French, "German Codes and Ciphers," *World's Work*, June, 1918, p. 144.

[2] The following articles have appeared in the *Scientific American* during the war:

Brewton, William W., "The Science of the Cipher and an Explanation of Bacon's Undecipherable System," *Scientific American Supplement*, Vol. LXXIX, No. 2059 (June 19, 1915), pp. 394–395.

Edwards, E. C., "Cipher Codes and their Uses," *Sci. Am.*, Vol. CXIII, No. 1 (July 3 1915), p. 9.

Morgan, Frank, "Enciphering and Deciphering Codes," *Sci. Am.*, Vol. CXIII, No. 8 (August 21, 1915), p. 159.

Paddock, Ira J., "Cipher Codes Simplified," *Sci. Am.*, Vol. CXIII, No. 13 (Sept. 25, 1915), p. 271.

Woodworth, H. S., "A Simple Cipher Code," *Sci. Am.*, Vol. CXIII, No. 14 (Oct. 2, 1915), p. 291. Described a code adaptable to use with a slide rule.

Berkel, Ernest, "A Telegraphic Cryptogram," *Sci. Am.*, Vol. CXIII, No. 24 (Dec. 11, 1915), p. 519. Gives a convenient method for using figures in telegraphic messages.

[3] See in this connection the following:

Post, Melville Davisson, "German War Ciphers," *Everybody's Magazine*, June, 1918, pp. 28–34. Gives examples of diagrams and sketches. According to Mr. Post, the victory at the Marne was probably due, in part, to the amazing good fortune

codes. In spite of the fact that the Allies controlled all overseas mails and cables, the German Government was apparently able to communicate at will, for a time at least, with its embassies and agents everywhere.

It would seem as if no article on this subject should omit a reference to the work of John Wallis, who has been called by D. E. Smith[1] "one of the world's greatest decipherers of cryptic writing," or to Francis Bacon's famous biliteral cipher[2] and Ignatius Donnelly's remarkable attempt[3] to prove, by means of a cipher, that Bacon was the author of Shakespeare's works.

Readers of Edgar Allan Poe's "Gold Bug" are probably aware of his belief that every cipher could be resolved and his challenge[4] to his readers to submit any which he could not decipher.

If any of our readers have never read Conan Doyle's "Adventures of the Dancing Man" a mention of it in this connection may lead to the enjoyment of its delightful mystery.

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel and Otto Dunkel.

Send all communications about problems to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

2788. Proposed by WARREN WEAVER, University of Wisconsin.

Out of a freshman class of one thousand, various quiz sections are chosen by chance. After a semester's work A is found to stand 12th in a section of 18 students, and B is found to stand 18th

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of the French war office in obtaining, by an extraordinary accident, which can not now be made public, the key to the German radiograph cipher code. "General Joffre knew the German orders wherever the radiograph signal was used in the advance on Paris. And it is possible that the knowledge of the radiograph code enabled the American Government to decipher the messages sent from the great wireless station at Nauen to Count von Luxburg, and so to make public the vast system of German intrigue that has amazed the world" (p. 29).

Strother, French, "Fighting Germany's Spies," *World's Work*, June, 1918, pp. 134–153. Gives descriptions and examples of the "Playfair" cipher (used by the British army in the field and pronounced by Mr. Strother "the cleverest transposition cipher ever devised"), the "Chess Board" cipher and the Bolo Pasha code.

Literary Digest, "A Graphic Spy Code," Vol. 56, p. 38, and "Keeping Government Cipher-Codes Secret," Vol. 51, pp. 546–548.

[1] Cf., Smith, D. E., "John Wallis as a Cryptographer" *Bulletin of the American Mathematical Society*, Vol. 24, No. 2 (Nov. 1917) pp. 82–96.

[2] *The Works of Francis Bacon, Baron of Verulam, Viscount of St. Alban, and Lord High Chancellor of England*, collected and edited by James Spedding, Robt. Leslie Ellis and Douglas Denon Heath (London, 1879), Vol. I, pp. 659–661. In a note on p. 843 Spedding credits the idea of a biliteral cipher, which Bacon seems to claim as his own, to John Baptist Porta (cf. *l.c.* above) although employed by him in a different manner.

[3] Donnelly, Ignatius, "The Great Cryptogram," R. S. Peale & Co., Chicago, N. Y., London, 1888.

[4] *The Works of Edgar Allan Poe*, with an introduction and a memoir by Richard Henry Stoddard (A. C. Armstrong & Son, N. Y., 1884), Vol. I, pp. 431–451. This essay on "Cryptography" contains, according to the editor, all that Poe had to say of importance on the subject. The challenge, which was first printed in *Graham's Magazine* for July, 1841, is reprinted on p. 442.

In Stedman and Woodberry's edition of Poe's works (Chicago, 1896) the essay on "Cryptography" appears in volume 9, pp. 260–278.

in a section of 27 students. Show that, had the whole class of one thousand taken the work in one large section, the most probable result is that B would have finished the semester's work six places ahead of A, and that the expectation of B's lead on A is 11.

2789. Proposed by KURT LAVES, University of Chicago.

Given a quadrilateral $ABCD$ for which $AC + BC < AD + BD$ to construct, by means of the ruler and compass only, the pair of tangents from D to the hyperbola (ellipse) for which A and B are the foci and C a point on the hyperbola (ellipse).

2790. Proposed by J. W. LASLEY, JR., University of North Carolina.

How shall we buy 12 eggs for 80 cents, if eggs sell as follows: hen eggs at 5 cents each, duck eggs at 7 cents each, and turkey eggs at 8 cents each, provided we buy some of each?

2791. A cup of wine is suspended over a cup of equal capacity full of water; through a small hole in the bottom, the wine drips into the water, and the mixture drips out at the same rate. When the wine cup is empty, what part of the contents of the lower cup is water? [Proposed by Charles Gilpin, Jr., Philadelphia, as Problem 287 in *The Mathematical Visitor*, January, 1881, volume 1, page 193. No solution was published in the *Visitor*.]

2792. Proposed by B. J. BROWN, Kansas City.

Solve the differential equation,

$$x^2(1 - x)\frac{d^2y}{dx^2} + 2x(2 - x)\frac{dy}{dx} + 2(1 + x)y = x^2.$$

461 (Algebra) [June, 1916]. Proposed by E. T. BELL, University of Washington.

(1) Two events have probabilities p, q respectively. The events may be either (i) mutually independent; or (ii) mutually exclusive. Assign meanings to the symbol p^q, in terms of the two events where p^q is written for $p \times p \times \cdots \times p$, ($q$ factors p), in cases (i), (ii), and $p \times p$ has the customary meaning (as a probability).

(2) What relations, if any, other than (i) and (ii) can exist between two events? Upon what postulates is the answer to this based?

463A (Geometry) [May. 1915]. Proposed by B. J. BROWN, Kansas City.

If μ and ν are the parameters of the two confocal conics through any point on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

show that $\mu + \nu + a^2 + c^2 = 0$, along a central circular section.

470 (Geometry) [September, 1915]. Proposed by R. E. MORITZ, University of Washington.

Prove that

$$\theta = (\lambda + (q/p)\mu)\pi, \qquad (\lambda = 1, 2, 3, \cdots, q - 1;\ \mu = 0, 1, 2, \cdots, p - 1),$$

and

$$\theta = (2\lambda - 1)\pi/2 + (q/p)(4\mu \pm 1)\pi/2, \qquad (\lambda = 1, 2, 3, \cdots, (q - 1)/2;\ \mu = 0, 1, 2, \cdots, p - 1),$$

determine the same set of points on the curve $\rho = a \cos (q/p)\theta$, where p and q are two odd integers without a common factor, and a is any constant.

499 (Geometry) [November, 1916]. Proposed by NATHAN ALTSHILLER, University of Oklahoma.

Find the surfaces all the plane sections of which are circles.

501 (Geometry) [November, 1916]. Proposed by R. P. BAKER, University of Iowa.

Find the minimum amount of lumber one inch thick required to pack a gross of spheres three inches in diameter in a rectangular box.

SOLUTIONS OF PROBLEMS.

430 (Algebra) [March, 1915]. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the following equations algebraically and graphically:

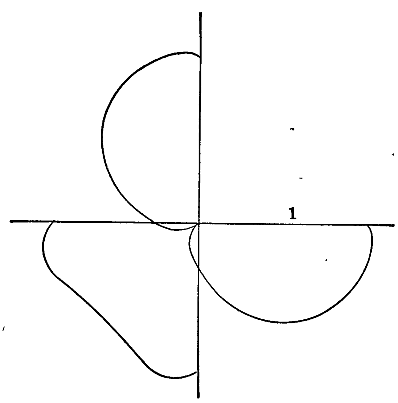
$$x^y + y^x = xy, \qquad x^x + y^y = x + y.$$

SOLUTION BY A. A. BENNETT, University of Texas.

There is no pair of real values, with the doubtful exception of (0, 0), which furnishes a simultaneous solution to the two equations given.

The equations being transcendental no "algebraic" solution is attempted. The discussion here is analytical, supplemented by a graph.

In examining the second of the above equations, it is convenient to distinguish three types of real values, viz., (A), the type in which the variable has any positive real value; in both (B) and (C) are considered only negative and rational numbers when written as irreducible fractions; in (B) the numerator of such fractions is even, while in (C) it is odd. Irrational negative values or rational negative values which when reduced to lowest terms have even denominators cannot appear among real solutions.



The following graph is liable to misinterpretation and will be qualified.

In the first quadrant there is a single point, (1, 1).

In the second quadrant, the graph is discontinuous, x being of type (C) and y of type (A). An infinite number of points certainly exist satisfying the equation.

In the fourth quadrant, x is of type (A) and y of type (C). Above remarks apply by symmetry.

In the third quadrant there are three branches to consider: (i) an apparent continuation of the branch of the second quadrant, (ii) an apparent continuation of the branch of the fourth quadrant, (iii) a separated branch containing $(-1, -1)$. Untractable questions of irrationality make doubtful the existence of any actual solutions in the third quadrant other than $(-1, -1)$. If any exist, they are on the branches above noted and are of the following types. On (i) x is of type (C) and y of type (B), on (ii) x is of type (B), and y of type (C); on (iii) x and y are each of type (C).

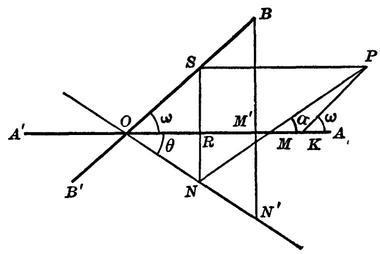
A real evaluation of $f(x, y) \equiv x^y + y^x - xy$ is only possible for both x and y rational, so far as points of the above graph are concerned. A study of $f(x, y)$ on the points of the graph shows it, even if existent, very far from zero, except as the origin is approached on branches (i) and (ii) in the third quadrant.

519 (Geometry) [September, 1917]. Proposed by OTTO DUNKEL, Washington University.

Given the conjugate axes $A'OA$ and $B'OB$ of an ellipse, points of the curve may be constructed as follows: Drop the perpendicular BM' to OA and produce it to N' so that $BN' = AO$. Draw a straight line through O and N'. Upon a straight edge, say that of a slip of paper, the points N, M, and P are marked so that $NM = N'M'$ and $MP = M'B$. Place the straight edge so that N falls on ON' and M on OM' and mark the position of P. This gives a point of the ellipse and by sliding the straight edge into new positions other points may be rapidly obtained. If the axes are perpendicular this gives the familiar trammel construction. Prove the correctness of this construction.

I. SOLUTION BY F. H. SAFFORD, University of Pennsylvania.

Let the axes of coördinates be taken along OA and OB, calling OA and OB, a and b, respectively. Then for P, x is OK and y is KP. From the given conditions, $MP = M'B = b \sin \omega$; hence, from the triangle MKP, (if $\alpha = \angle KMP$),



(1) $y = KP = MP \sin\alpha/\sin\omega = b\sin\alpha.$

Also $NM = N'M' = a - b\sin\omega$, and from the triangle ONM, $OM = NM\ (\sin\alpha\cot\theta + \cos\alpha)$. From triangle $ON'M'$, $\cot\theta = OM'/N'M' = b\cos\omega/NM \therefore OM = b\sin\alpha\cos\omega + NM\cos\alpha = a\cos\alpha - b\sin(\omega - \alpha)$. From triangle MKP, $MK = MP\sin(\omega - \alpha)/\sin\omega = b\sin(\omega - \alpha)$. Hence,

(2) $x = OM + MK = a\cos\alpha.$

Thus (1) and (2) are the desired (parametric) equations of the locus of P.

II. Remarks by the Proposer.

The part of the above proof following (1) can be shortened somewhat thus: Draw the straight line NRS perpendicular to OA, and join P to S. Then $NM/NP = N'M'/N'B = NR/NS$ and hence RM is parallel to SP and also $x = OK = SP$. It now follows that $x = PN\cos\alpha = a\cos\alpha$.

The problem may also be treated geometrically as is done in Rouché and Comberousse, *Traité de Géométrie*, deuxième partie, 8e éd., 1912, pp. 341–345.

332 (Mechanics) [October, 1916]. Proposed by E. E. MOOTS, University of Arizona.

A correct wording of this problem is given in 490 (Geometry) [May, 1916], a solution of which, by A. M. Harding, was published in February, 1917.

198 (Number Theory) [November, 1913; June, 1919]. Proposed by the late ARTEMAS MARTIN.

Prove that every even number is the sum of two prime numbers.

Note by R. C. Archibald, Brown University.

This is Goldbach's empirical theorem and the conjecture appears in a letter to Euler dated June 7, 1742 (*Corresp. Math. Phys.*, ed. Fuss, Vol. 1, 1843, p. 127). The first published statement of the theorem was by E. Waring in his *Meditationes Algebraicæ*, 1770, p. 217. E. Haussner verified the law for numbers up to 10000 (*Jahresbericht der Deutschen Math. Verein.*, Vol. 5, 1896, 62–66), and E. Maillet proved that every even number $\leqq 350000$ (or 10^6 or $9\cdot10^6$) is, in default by at most 6 (or 8 or 14), the sum of two primes (*L'Intermédiaire des mathématiciens*, volume 12, 1905, p. 108). These notes are taken from L. E. Dickson's *History of the Theory of Numbers*, volume 1, 1918, where the complete history of the theorem may be found on pages 421–425. No proof of this theorem has yet been discovered.

201 (Number Theory) [December, 1913; June, 1919]. Proposed by E. T. BELL, University of Washington.

Eisenstein proposed (*Crelle*, t. 27, p. 282) as the simplest of several problems: "In the expansion of

$$\frac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1,$$

where p is prime, to show that the coefficients of the various powers of z are all divisible by p."

Solution by R. C. Archibald, Brown University.

$$\frac{1 + z + z^2 + \cdots + z^{p-1}}{(1 - z)^{p-1}} - 1 = \frac{1 - z^p}{(1 - z)^p} - 1,$$

$$= \left(pz - \frac{p(p-1)}{1\cdot2}z^2 + \cdots + \frac{p(p-1)}{1\cdot2}z^{p-2} - pz^{p-1}\right)(1 - z)^{-p}.$$

Multiplying the first factor of this product by the second factor, expanded, we have the desired result. For, in each factor the coefficients are integers, and p is contained in every coefficient of the first factor.

Also solved by P. J. da Cunha, A. Pelletier, and Elijah Swift.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Dr. W. V. N. GARRETSON has been appointed assistant professor of mathematics in Rutgers College.

Dr. R. L. CHARLES has been promoted to an associate professorship of mathematics in Lehigh University.

Assistant professor J. D. BOND, of the mathematics department of the Agricultural and Mechanical College of Texas, has been granted a year's leave of absence and will study at the University of Michigan where he will hold a university fellowship.

Mr. E. T. BROWNE has accepted a fellowship in mathematics at the University of Chicago for the current academic year.

Mr. WARREN WEAVER has returned to Throop College, Pasadena, Cal., after spending the past year as instructor in mathematics at the University of Wisconsin.

Lieutenant FREDRICK WOOD has returned from service overseas to the University of Wisconsin as instructor in mathematics.

Lieutenant F. L. BROWN, of the meteorological service, has been appointed instructor in mathematics at Northwestern University.

Dr. ARNOLD DRESDEN has resumed duties as assistant professor of mathematics at the University of Wisconsin, after a year of service with the Red Cross in France.

During the war Dr. E. J. MOULTON, associate professor of mathematics at Northwestern University, had charge of the Dearborn Observatory and the astronomy department of the University, in addition to his regular duties in the mathematics department.

Dr. C. R. DINES, formerly assistant professor of mathematics at Dartmouth College, has been in the Federal Reserve Bank of Chicago for the past year.

At Dartmouth College, Professor J. W. YOUNG is on leave of absence, for the current academic year. He is residing at Hanover, N. H., working on the National Committee on Mathematical Requirements, of which he is chairman and treasurer. Professor R. D. BEETLE has been appointed chairman of the mathematics department, in the absence of Professor YOUNG. Assistant Pro-

fessor F. M. MORGAN has been appointed Freshman Class Officer of the College. Professor L. L. SILVERMAN has resigned to enter a banking house in Boston.

At Northwestern University, Dr. H. B. CURTIS, formerly of Barnard College, has accepted a position as instructor in mathematics.

Miss M. E. DANIELLS has been promoted from an instructorship to an assistant professorship of mathematics at Iowa State College.

Professor F. F. DECKER, of the department of mathematics of Syracuse University, has been made director of the evening session of that institution.

M. E. GRABER, professor of mathematics at Heidelberg University, has been appointed professor of physics at Morningside College, Sioux City, Iowa.

Dr. T. H. GRONWALL has been appointed mathematics and dynamics expert on the technical staff of the ordnance office, of the War Department, Washington. The University of Texas has granted leave of absence to Professor A. A. BENNETT to carry on work in this same office during 1919–20.

E. V. HUNTINGTON, associate professor of mathematics in Harvard University, has been promoted to a full professorship of mechanics. His teaching activities will be divided as heretofore between the Division of Mathematics and the Division of Engineering.

Professor LAURENCE HADLEY of Earlham College has resigned to accept a position as associate professor of mathematics at Purdue University.

Mr. M. F. JOHNSON, late of Michigan Agricultural College, has been appointed instructor of mathematics at the University of Michigan.

Dr. E. A. KIRCHER has resigned his appointment as instructor of mathematics at Yale University to enter business.

Professor G. H. LING, dean of the Faculty of Arts of the University of Saskatchewan, has been appointed acting-president of the University during a three months vacation of the president.

Assistant Professor C. N. MILLS, of South Dakota State College, has been elected professor of mathematics at Heidelberg University, Tiffin, Ohio.

F. W. PARSONS of Ohio Northern University has been made assistant professor of mathematics and engineering in Whittier College.

Dr. T. M. PUTNAM, professor of mathematics and dean of the undergraduate division in the University of California, has been appointed acting dean of the college of letters and science in the place of the late Professor H. M. STEPHENS.

Assistant Professor W. W. RANKIN, of the University of North Carolina, has been granted leave of absence to serve for the year 1919–20 as instructor in mathematics at Columbia University.

At the University of Wisconsin Dr. EUGENE TAYLOR has been appointed assistant professor of mathematics and Miss MARY A. COLPITTS instructor in mathematics.

F. C. TOUTON, formerly of the St. Joseph (Mo.) Junior College, received the degree of doctor of philosophy last June from Columbia University and has been appointed supervisor of high schools for the state of Wisconsin.

Assistant Professor M. O. TRIPP, of the University of Maine, has been promoted to an associate professorship of mathematics.

Dr. E. S. ALLEN, of the University of Michigan, has been appointed assistant professor of mathematics at West Virginia University.

Dr HENRY BLUMBERG, recently associate in mathematics at the University of Illinois, has been promoted to an assistant professorship of mathematics.

Mr. D. C. KAZARINOFF, recently of Carleton College, has been appointed instructor in Mathematics at the University of Michigan.

Mr. A. D. CAMPBELL remained as instructor in mathematics at Cornell University instead of accepting the appointment to Yale University announced in the MONTHLY for June.

V. J. BOUSSINESQ, professor of calculus of probabilities and mathematical physics at the University of Paris, has retired as honorary professor after twenty-three years of service in the university. He was born in 1842.

A. M. G. FLOQUET, professor of mathematical analysis at the University of Nancy has been retired as honorary professor. He was born in 1849.

Professor H. S. WHITE, of Vassar College, spent the summer at the Bureau of Standards, Washington.

Dr. J. W. GLOVER, professor of mathematics and insurance at the University of Michigan, spent the month of September in New York City, serving as Acting President of the Teachers Insurance and Annuity Association of America. Professor Glover is a member of the Board of trustees and acted for Dr. H. S. PRITCHETT, who is spending the summer in the West.

The *Bulletin of the American Mathematical Society* announces that the Royal Institute of Venice has awarded the Querini-Stampalia prize to Professor G. D. BIRKHOFF, of Harvard University, for his papers "The restricted problem of three bodies" (*Rendiconti del Circolo Matematico di Palermo*, volume 39, 1915), and "Dynamical systems with two degrees of Freedom" (*Transactions of the American Mathematical Society*, volume 18, 1917). The prize had been offered for some important advances in the theory of periodic solutions of differential equations. Its value is 3,000 crowns.

During the past summer Assistant Professor LOUISA M. WEBSTER of the department of mathematics of Hunter College conducted a summer high school for girls. She originated this self-supporting school eleven years ago and has served throughout as its director. Instruction was given this summer to nearly fourteen hundred girls. The work is so coordinated with that of the New York City high schools that these have all agreed to accept its ratings. A similar school for boys is projected for next year.

At the University of Minnesota, a committee of twenty members of the Faculty of the Arts College was organized last year, with Professor R. M. BARTON of the mathematics department as chairman, for the purpose of meeting all students in personal conferences. These conferences serve to evince a healthy interest in the scholarship of the pupils, provide for their registration, and, particularly for the freshmen and sophomores, have a distinctly moral bearing. The results of some forty-three hundred conferences during the second semester were found gratifying; the students were appreciative of this personal attention and were through this agency able largely to clarify the notions of the purposes of their university study. This is a new manifestation of the increasing tendency of instructors over the country to meet students outside of the classroom in an advisory capacity, a plan that has been found of great value for departmental purposes by the staff of the department of mathematics in a few institutions.

Degrees of Doctor of Philosophy in mathematics were conferred in the academic year 1918–1919 as follows, with theses as indicated:

UNIVERSITY OF CHICAGO: GEORGE HOFFMAN CRESSE: "On the class number of binary quadratic forms."

HARVARD UNIVERSITY: LOUIS BRAND: "I. On linear equations with an infinite number of variables. II. On infinite systems of linear integral equations. III. Flexual deflections and statistically indeterminate beams." CLARENCE NEWTON REYNOLDS: "On the zeros of solutions of linear differential equations." CHAN-CHAN TSOO: "The geometry of a Non-Euclidean line-sphere transformation."

The degree of Doctor of Philosophy in education was conferred, at Columbia University, upon F. C. TOUTON, whose dissertation was on, "The solution of an original exercise in geometry."

The twenty-sixth summer meeting of the American Mathematical Society was held at the University of Michigan, September 2–4, in conjunction with meetings of the American Astronomical Society and the Mathematical Association of America. Eighty-one members were in attendance, the largest number on record. A report of a joint session of the three organizations on Thursday afternoon and of the joint dinner is given elsewhere in this number of the MONTHLY. Thirty-three papers were presented at the other session, titles and abstracts of which are given in the November number of the *Bulletin of the American Mathematical Society*. Professors BEMAN, CURTISS, SNYDER and BLISS presided at the various sessions and Professors KARPINSKI and E. J. MOULTON acted as secretary.

VOLUME XXVI NOVEMBER, 1919 SUPPLEMENT

THE AMERICAN MATHEMATICAL MONTHLY

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FOR THE ACADEMIC YEAR 1919-1920

LANCASTER, PA., AND PROVIDENCE, R. I.
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1919

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For a list of all officers and Council members for preceding years see page 40.

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Note. This list gives data for the members of the Association as taken from the register issued May, 1918, unless changes in position, mailing address, etc., have been reported to the secretary. When members are known to be occupied in other lines than mathematics, this fact is indicated. The upper index preceding some of the names shows the year of the Association in which these members were elected to membership, *e.g.*, the upper index 1 signifies election to membership during 1916, 2 during 1917, etc.; the absence of such an index signifies charter membership, *i.e.*, an enrollment within the organization period January 1 to April 1, 1916.

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[3]Arms, Dr. R. A. Instr., Univ. of Pennsylvania. *608 High St., Pottstown, Pa.*
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JACKSONVILLE. J. E. Sanders.

GEORGIA. (17)

ATHENS. Pond, S. P. Sanford, R. D. Stephens.
ATLANTA. F. Field, Morton, Skiles, D. M. Smith, Stamy.
COLUMBUS. Powell.
DECATUR. A. I. Young.
DEMOREST. Rogers.
EMORY UNIVERSITY. Peed, Rumble.
MACON. Holder.
PHILOMATH. Barrow.
ROME. Hightower.
SURRENCY. Brindle.

IDAHO. (2)

CALDWELL. J. M. Rankin.
MOSCOW. H. H. Conwell.

ILLINOIS. (90)

ABINGDON. Whitted.
CARTHAGE. I. L. Miller.
CHAMPAIGN. Crathorne, Rosenbach, Townsend.
CHAPIN. C. C. Carter.
CHARLESTON. E. H. Taylor.
CHICAGO. Abrams, E. M. Allen, Bixby, Bliss, Butler, Cobb, Dickson, Dines, Escott, W. J. Ferguson, Foberg, Hassler, Jarrett, J. H. Jones, Kanies, Kinney, Krathwohl, Laves, A. C. Lunn, M. Macdonald, MacMillan, Miser, E. H. Moore, F. R. Moulton, G. W. Myers, Nicolet, Nyberg, C. I. Palmer, Pettersen, Phalen, Schottenfels, Schweitzer, Slaught, S. S. Stahl, Van der Vries, Weld, Wilczynski, Winter, V. L. Wright, J. W. A. Young.
DECATUR. Risley.
EVANSTON. D. F. Campbell, Curtiss, Doll, Kerr, Logsdon, E. J. Moulton, Newell, Roman, C. E. Wilder, R. E. Wilson.
FREEPORT. Mensenkamp.
GALESBURG. Sellew.
JACKSONVILLE. M. Anderson, G. H. Scott.
JOLIET. E. L. Thompson.
LAKE FOREST. McNeill.
LA SALLE. Carus.
LEBANON. Stokes.
LISLE. Ondrak.
MACOMB. Ginnings.
MONMOUTH. Winbigler.
MOUNT MORRIS. Shively.
ORLEANS. Holmes.
PEORIA. Comstock.
ROCKFORD. Jacobs, B. I. Miller.
ROCK ISLAND. Cederberg.
TAYLORVILLE. Dappert.
URBANA. Blumberg, Camp, Carmichael, Coble, Edington, Emch, Kempner, Kustermann, Lytle, G. A. Miller, Shaw, Steimley, Wahlin.
WASHINGTON. Thielbar.

INDIANA. (28)

BLOOMINGTON. Davisson, Hanna, Hennel, Rothrock, K. P. Williams, H. E. Wolfe.
COLUMBIA CITY. Knisely.
CRAWFORDSVILLE. Cragwall, P. W. Hill.
GOSHEN. Lehman.
HANOVER. Laurence.
INDIANAPOLIS. Beckett, E. N. Johnson.
LAFAYETTE. Zehring
MONTPELIER. Clevenger.
NORTH MANCHESTER. Dotterer.
NOTRE DAME. Caparo.
SOUTH BEND. Motts.
WEST LAFAYETTE. Bates, Doan, Graves, Hadley, Hazard, Hodge, Kenyon, C. K. Robbins, Spitzer, R. B. Stone.

IOWA. (38)

AMES. Blakslee, Chaney, Colpitts, Daniells, Farnum, Kiefer, Madson, McKelvey, Pattengill, Roberts, Sage, Stanton.
CEDAR FALLS. Condit, Wester.
CEDAR RAPIDS. Coffin.
DES MOINES. Corey, Neff.
DUBUQUE. Zimmerman.
FAIRFIELD. Schmiedel.
FAYETTE. Simonson.
GRINNELL. Albert, McClenon, Rusk.
INDIANOLA. C. W. Emmons.
IOWA CITY. R. P. Baker, J. W. Campbell, Chittenden, Reilly, Rietz, Weida, W. H. Wilson.
MOUNT VERNON. McGaw.
MUSCATINE. Stein.
OSKALOOSA. S. M. Hadley.
SIOUX CITY. Graber, Van Horne.
TABOR. E. H. Thomas.
WELLMAN. Kreth.

KANSAS. (32)

BALDWIN. Garrett.
EMPORIA. Hyde, Lindquist.
KANSAS CITY. L. T. Dougherty, Helwig, Woodyard.
LAWRENCE. Ashton, Black, Gouwens, Jordan, Lefschetz, Marm, U. G. Mitchell, C. A. Nelson, Stouffer, J. J. Wheeler.
LINDSBORG. H. N. Olson.
MANHATTAN. W. H. Andrews, H. E. Porter, Remick, Stratton, A. E. White.
NEWTON. Richert.
OTTAWA. Bouse.
PITTSBURG. Broadlick, Shirk.
STERLING. T. Bell.
TOPEKA. Harshbarger, Newson.
WICHITA. Dueker, Hoare, W. O. Mendenhall.

KENTUCKY. (9)

BEREA. Kean.
DANVILLE. Crooks.
GEORGETOWN. C. H. Richardson.
LEXINGTON. Boyd, J. M. Davis, Downing, Rees, G. W. Smith.
LOUISVILLE. Marshall.

LOUISIANA. (5)

BATON ROUGE. Nichols, S. T. Sanders.
NEW ORLEANS. Cater, Dinwiddie, Spencer.

MAINE. (12)

BANGOR. Robinson.
BRUNSWICK. Moody.
KENNEBUNK. Goodrich.
LEWISTON. Ramsdell.
ORONO. Aley, Anning, J. N. Hart, Tripp.
WATERVILLE. Ashcraft, B. E. Carter, Trefethen.
WOODLAND. A. S. Adams.

MARYLAND. (32)

ABERDEEN PROVING GROUND. Galajikian, Somers.
ANNAPOLIS. Bingley, Bramble, J. A. Bullard, Capron, Clements, Cresse, Dederick, Dillingham, Eppes, A. Hall, Hemke, W. W. Johnson, Mayer, Robert, Root, Shenton, J. Tyler.
BALTIMORE. Bacon, Cohen, Harry, Hulburt, Lewis, F. Morley, Murnaghan, Reed, Eugene R. Smith, Thomsen.
FREDERICK. L. O. Brown.
PORT DEPOSIT. E. C. Cook.
ROLAND PARK. Marrow.

MASSACHUSETTS. (78)

AMHERST. Esty, F. C. Moore, Olds.
BOSTON. Beatley, Brigham, Bruce, G. W. Evans, Gaylord, Gould, King, Osborne.
CAMBRIDGE. Bailey, Barnett, Birkhoff, Bouton, Bradley, B. H. Brown, Coolidge, H. N. Davis, H. T. Davis, N. F. Davis, W. S. Franklin, E. V. Huntington, Kellogg, Kennelly, Lipka, C. L. E. Moore, Osgood, Phillips, Rutledge, H. W. Tyler, E. B. Wilson, F. S. Woods, Zeldin.
EVERETT. Bryant.
HAVERHILL. Card.
HOLYOKE. Moriarty.
LEE. M. M. Smith.
LOWELL. Lupien.
MOUNT HERMON. Wagar.
NEWBURYPORT. Steck.
NORTHAMPTON. S. R. Benedict, Munroe, Rambo.
PITTSFIELD. Washburne.
SOUTH HADLEY. Doak, Hazlett, Le Stourgeon, E. N. Martin, S. E. Smith.
SPRINGFIELD. Clark.
TUFTS COLLEGE. Mergendahl, Ransom, H. Rice, L. H. Rice.
WATERTOWN. Haigler, C. A. Hobbs, Kenison, Ward.
WELLESLEY. Chandler, Copeland, A. M. Howe, Merrill, G. E. Preston, Clara E. Smith, Stark, Vivian, E. R. Worthington, M. M. Young.
WEST NEWBURY. Carleton.
WILLIAMSTOWN. Agard, Hardy.
WORCESTER. Butterfield, R. K. Morley, Rafferty, A. G. Webster, A. H. Wheeler, F. B. Williams.

MICHIGAN. (33)

ALBION. Sleight.
ANN ARBOR. Beman, Bradshaw, Denton, P. Field, Ford, Glover, Hildebrandt, Hussey, M. F. Johnson, Karpinski, Markley, A. L. Nelson, R. B. Robbins, Running, Welton, Ziwet.
DETROIT. Baldwin, Thome.
EAST LANSING. L. C. Emmons, Plant, Speeker, Steward.
FENTON. Tryon.
GRAND RAPIDS. Rinck.
HILLSDALE. Herron.
HOLLAND. Lampen.
HOUGHTON. Grant.
KALAMAZOO. Everett, C. B. Williams.
MARQUETTE. Spooner.
YPSILANTI. Erickson, Lyman.

MINNESOTA. (22)

COLLEGEVILLE. Hansen.
DULUTH. Sr. Brigetta.
FERGUS FALLS. Wollan.
HERON LAKE. L. E. Lunn.
MINNEAPOLIS. R. M. Barton, Bauer, Beal, Brooke, Bussey, Dalaker, D. Jackson, Mathews, Newkirk, Peterson, Reeve, Shumway, Thorp, Underhill.
NORTHFIELD. M. B. White.

MINNESOTA (*continued*)

ST. JOSEPH. Sr. Magna.
ST. PAUL. R. A. Johnson.
WINONA. Quigley.

MISSISSIPPI. (2)

GULFPORT. Chapman.
UNIVERSITY. Hume.

MISSOURI. (43)

CAPE GIRARDEAU. B. F. Johnson, Payne.
COLUMBIA. E. Allen, Callaway, Hedrick, L. Ingold, W. D. A. Westfall.
FULTON. Sweazey.
KANSAS CITY. A. C. Andrews, B. J. Brown, Kent, Luby.
KIRKSVILLE. Cosby, Epperson, Jamison, Zeigel.
LA GRANGE. Selves.
LIBERTY. Fleet.
MEXICO. Caldwell.
PARKVILLE. R. A. Wells.
ST. JOSEPH. Z. Ferguson.
SPRINGFIELD. Finkel, E. Gibson.
ST. LOUIS. Ammerman, Borgmeyer, Brennan, Calman, A. Davis, Dunkel, Hinrichs, Huff, Musselman, Nauer, Rider, Roever, Robertson, Schwartz, Shannon, E. Stephens, Weeks.
TARKIO. Jenison.
WARRENSBURG. Scarborough, Urban.

MONTANA. (3)

GLASGOW. Calderwood.
MISSOULA. Carey, Lennes.

NEBRASKA. (12)

BETHANY. Fitzpatrick.
CERESCO. L. C. Walker.
CRETE. J. N. Bennett.
FREMONT. E. M. Stahl.
LINCOLN. Babbitt, Brenke, Candy, Chatburn, Gaba.
OMAHA. Frankish.
PERU. Howie.
YORK. Feemster.

NEVADA. (1)

RENO. Haseman.

NEW HAMPSHIRE. (10)

DURHAM. Garabedian, Slobin.
EXETER. Sweet.
HANOVER. Beetle, Bill, Mathewson, F. M. Morgan, Silverman, J. W. Young.
MANCHESTER. Hopkins.

NEW JERSEY. (27)

ATLANTIC CITY. Kline.
EAST ORANGE. Koch.
HACKENSACK. Sullivan.
HOBOKEN. Gunther.
JERSEY CITY. Berger.
LAWRENCEVILLE. Durell.
MONTCLAIR. M. I. Cook, A. B. Turner.
NEWARK. Duncan, Koch.
NEW BRUNSWICK. Garretson, R. Morris, Titsworth.
NORTH BERGEN. Mallory.
PATERSON. Caster.
PLAINFIELD. R. W. Lord.
PRINCETON. E. P. Adams, Eisenhart, Fine, P. Franklin, H. D. Thompson, Veblen, Willson.
RIDGEWOOD. Phelps.
SUMMIT. Webb.
TRENTON. Colliton, Seymour.

NEW MEXICO. (4)

ALBUQUERQUE. Barnhart.
EAST LAS VEGAS. Rodgers.
ROSWELL. Pearson.
SOCORRO. Reece.

NEW YORK. (109)

ALBANY. Birchenough, G. M. Conwell.
AURORA. Van Benschoten, Willis.
BROOKLYN. Bergstresser, W. J. Berry, Bowden, Locke, Schuyler, Tanzola, G. F. Wilder.
BUFFALO. L. F. Ott.
CLINTON. H. S. Brown, Carruth, Ferry, Fitch.
CORNING. Foster.
CORNWALL-ON-HUDSON. H. R. Dougherty.
EAST ELMHURST. Hanson.
FLUSHING. Oglesby.
GARDEN CITY. J. G. Coffin.
GENEVA. Durfee.
GLOVERSVILLE. B. G. Westfall.
HAMILTON. A. W. Smith, J. M. Taylor.
ITHACA. A. D. Campbell, Carver, Gillespie, Hurwitz, McMahon, Owens, Ranum, Tanner.
KENKA PARK. Norton.
MOUNT VERNON. Breckenridge.
NEW YORK. Auerbach, Blair, Brewster, G. A. Campbell, Chamberlain, Dennett, C. H. Douglas, Eckersley, Edmondson, Fiske, Fite, Frankel, Goertz, Grad, Graham, Grove, Hawkes, Henderson, Himwich, Hirsch, Hodgdon, Hoppe, Joffe, Kasner, Langellotti, Langman, Latham, Lehmann, Linehan, Maclay, MacNeish, Merriman, H. B. Mitchell, Molina, Paaswell, Pedersen, Penn, Pooler, Reddick, Requa, F. G. Reynolds, Saurel, Schmall, Siceloff, Simons, D. E. Smith, R. R. Smith, Spearing, Van Nuys, Waldo, E. Walker, L. M. Webster, Wechsler, E. E. Whitford, W. O. Wiley.
POUGHKEEPSIE. Cowley, Cummings, M. E. Wells.
ROCHESTER. W. Betz, Gale, Mirick, Watkeys.
SCHENECTADY. Vedder.
SILVER BAY-ON-LAKE GEORGE. A. H. Huntington.
SYRACUSE. W. G. Bullard, Decker, Lindsey, Secy. Pi Mu Epsilon Frat., Roe, W. E. Taylor.
TROY. Lundin.
WEST POINT. C. P. Echols.
WHITE PLAINS. J. Allen.
YONKERS. Hubert.

NORTH CAROLINA. (14)

CHAPEL HILL. W. Cain, A. W. Hobbs, Lasley.
CLAREMONT. Georges.
DAVIDSON. J. L. Douglas.
DURHAM. Patterson, M. R. Richardson.
ELON COLLEGE. Amick.
GREENSBORO. G. W. Mendenhall, Pegram.
GUILFORD COLLEGE. Brinton.
JAMESTOWN. Ragsdale.
RALEIGH. Barney.
WILMINGTON. H. B. Smith.

NORTH DAKOTA. (2)

JAMESTOWN. T. W. Jackson.
UNIVERSITY. Hitchcock.

OHIO. (69)

ADA. W. S. Beckwith.
ALLIANCE. Trott.
ASHTABULA. J. N. Cain.
BEREA. Dustheimer.
BOWLING GREEN. Overman.
BLUFFTON. Hirschler.
ATHENS. Borger.
CINCINNATI. Brand, Hancock, Kindle, C. N. Moore, E. S. Smith, Yowell.
CLEVELAND. E. R. Beckwith, Deming, Focke, W. W. Johnson, Palmié, Pfahl, Pitcher, Pritchard, Shamos, Showman, Simon, C. F. Thomas, D. T. Wilson.
COLUMBUS. C. L. Arnold, Bareis, Bohannan, Hoover, Kuhn, Morningstar, C. C. Morris, A. F. Preston, J. B. Preston, Rasor, Rickard, Singer, Swartzel, Wildermuth.
DAYTON. Hoffmann.
DEFIANCE. A. G. Caris.
DELAWARE. Armstrong, Austin.
GAMBIER. R. B. Allen.
GRANVILLE. Peckham, F. B. Wiley.

OHIO (*continued*)

HILLIARD. J. H. Weaver.
HIRAM. E. H. Clarke.
OBERLIN. Anderegg, Cairns, Carr, Davidson, Sinclair.
OXFORD. W. Anderson, Baudin, Glazier, McCain, Urner.
ROSS. Haldeman.
TIFFIN. C. N. Mills.
TOLEDO. Brandeberry.
KENT. Faught.
MARIETTA. Horn.
WILBERFORCE. Woodard.
WILMINGTON. E. W. Martin, Spinks.
WOOSTER. R. P. Thomas, Yanney.

OKLAHOMA. (9)

NORMAN. Altshiller, Duval, Gossard, Reaves.
SHAWNEE. W. T. Short.
STILLWATER. Gundersen.
TAHLEQUAH. Hackler.
TULSA. Holtwick.
WEATHERFORD. McCormick.

OREGON. (7)

EUGENE. DeCou, Milne.
FOREST GROVE. West.
PORTLAND. Griffin, Merriss, A. R. Williams.
RAINIER. Thayer.

PENNSYLVANIA. (71)

ALLENTOWN. Bauman.
BEAVER FALLS. Colwell.
BETHLEHEM. Charles, P. A. Lambert, Rau, J. B. Reynolds.
BRYN MAWR. A. Pell, A. J. Pell, C. Scott, Torrey, B. M. Turner.
CARLISLE. Landis.
COLLEGEVILLE. Clawson.
CONSHOHOCKEN. Sensenig.
DEVON. J. A. Clarke.
EASTON. W. S. Hall, W. M. Smith, Snyder.
GETTYSBURG. Granville.
GLEN MILLS. Gummere.
GLENSIDE. E. H. Worthington.
GROVE CITY. Ramsey.
HAVERFORD. Reid, A. H. Wilson.
LEWISBURG. H. S. Everett.
LINCOLN UNIVERSITY. W. L. Wright.
LANSDOWNE. Chambers, Glenn.
MEADVILLE. Akers.
MECHANICSBURG. E. B. Miller.
MYERSTOWN. Kiess.
NORRISTOWN. Gehman.
PARADISE. Eshleman.
HARRISBURG. Whited.
PHILADELPHIA. Ballantine, K. D. Brown, Burley, Crawley, Dill, H. B. Evans, Fisher, Haines, Linton, Minnick, R. L. Moore, Partridge, Rittenhouse, Safford, Walsh, Weyl.
PITTSBURGH. Baird, Bishop, Bland, Foraker, James, Riggs, Taber, Webber.
SEWICKLEY. Connelly.
SOUTH BETHLEHEM. MacNutt.
STATE COLLEGE. J. E. Davis, Gravatt, Rowe, Sallade, C. G. Simpson, Edwin R. Smith, Sousley.
SWARTHMORE. Marriott, J. A. Miller.
WASHINGTON. Atchison, Bert.
WILKINSBURG. Hugins.

PHILIPPINE ISLANDS. (1)

MANILA. V. Mills.

RHODE ISLAND. (10)

EDEN PARK. C. R. Adams.
NEWPORT. Peaslee.
PROVIDENCE. Archibald, T. H. Brown, R. W. Burgess, Chace, Currier, H. P. Manning, Perry, R. G. D. Richardson.

SOUTH CAROLINA. (4)

CHARLESTON. R. G. Thomas.
COLUMBIA. Coleman.
GREENVILLE. Earle.
SALUDA. Ramage.

SOUTH DAKOTA. (5)

BROOKINGS. G. L. Brown.
HURON. H. S. Myers.
RAPID CITY. McLaury.
REDFIELD. P. A. Field.
VERMILION. McKinney.

TENNESSEE. (7)

KNOXVILLE. H. E. Buchanan.
MARYVILLE. Knapp.
MEMPHIS. H. M. Manning.
MURFREESBORO. R. B. Wood.
NASHVILLE. S. I. Jones, Ott.
SEWANEE. S. M. Barton.

TEXAS. (21)

AUSTIN. H. Y. Benedict, A. A. Bennett, Dodd, Ettlinger, I. I. Nelson, M. B. Porter.
BROWNSVILLE. de la Garza.
BRYAN. McAlister.
CANYON. L. G. Allen.
COLLEGE STATION. Bond.
DALLAS. E. H. Jones, Mahoney.
FORT WORTH. Alexander.
GALVESTON. Underwood.
GEORGETOWN. Wunder.
HOUSTON. Daniell, Lovett.
SAN ANGELO. Hagelstein.
SAN ANTONIO. B. R. Allen, Roach.
STEPHENVILLE. J. L. Riley.

UTAH. (4)

LOGAN. Saxer.
SALT LAKE CITY. J. L. Gibson, Pehrson, Unseld.

VERMONT. (4)

BURLINGTON. Swift, E. Thomas.
ESSEX JUNCTION. Donahue.
MIDDLEBURY. Perkins.

VIRGINIA. (18)

ABINGDON. Aldrich.
ASHLAND. T. McN. Simpson.
BLACKSBURG. O'Shaughnessy, J. E. Williams.
CLIFTON STATION. O. Stone.
HAMPDEN-SIDNEY. J. S. Miller.
HOLLINS. Dickinson.
LEXINGTON. L. W. Smith, C. W. Watts.
LYNCHBURG. Larew.
MONTEREY. Colaw.
RICHMOND. Duke.
RICHMOND COLLEGE. Gaines.
SALEM. Carpenter.
SWEET BRIAR. Morenus.
UNIVERSITY. W. H. Echols, Luck.
WOODBURY FOREST. W. L. Lord.

WASHINGTON. (14)

PULLMAN. Hix, Isaacs.
SEATTLE. E. T. Bell, Boothroyd, Coffrey, Gavett, Moritz, Neikirk, Raynor, Stanwick.
TACOMA. Hanawalt.
WAITSBURG. Hays.
WALLA WALLA. Bratton, Eells.

WEST VIRGINIA. (6)

BETHANY. Balch, Hess.
BUCKHANNON. C. E. White.
MORGANTOWN. Eiesland, Hodgson.
WHEELING. Githens.

WISCONSIN. (27)

BELOIT. W. A. Hamilton, Haynes, Suffa.
JANESVILLE. Cortes E. Smith.
LA CROSSE. Adkins.
MADISON. F. E. Allen, Dowling, Dresden, W. W. Hart, Lane, H. L. Olson, Slichter, H. L. Smith, Touton.
MILTON. A. E. Whitford.
MILWAUKEE. K. S. Arnold, Boren, Ericson, Frumveller, F. Wood, Yeaton.
PLATTEVILLE. Warner, W. H. Williams.
RIPON. Woodmansee.
RIVER FALLS. McMillan.
SINSINAWA. Sr. Dobbin.
SUPERIOR. C. W. Smith.

WYOMING. (3)

LARAMIE. Fitterer, Ridgaway, Stromquist.

FOREIGN MEMBERS. (Other than Canada.)

ARGENTINA. (1)

BUENOS AIRES. Broggi.

CHINA. (7)

CHANGSHA. Leavens.
PEKING. Chang Shen-Fu, Heinz, Hsia.
SHANGHAI. Patten, E. L. Sanford.
TANGSHAN. Yen.

ENGLAND. (2)

CAMBRIDGE. Richmond.
LONDON. Greenhill.

INDIA. (2)

CALCUTTA. Bose.
AMRELI. Pandya.

ITALY. (3)

BOLOGNA. Enriques, Pincherle.
PISA. Bianchi.

KOREA. (1)

PYENGYANG. Parker.

PORTUGAL. (1)

LISBON. da Cunha.

SOUTH AFRICA. (1)

RONDEBOSCH. Muir.

TURKEY. (1)

CONSTANTINOPLE. Mourad.

RECAPITULATION OF MEMBERSHIP.

Individual members October 1, 1919	1,100	
Institutional members October 1, 1919	87	
Total membership October 1, 1919		1,187
Total membership January 1, 1918		1,140

CHARTER MEMBERSHIP.

Individual charter members	1,045	
Institutional charter members	52	
Total charter membership		1,097
Net gain in individual members	55	
Net gain in institutional members	35	
Total net gain over charter membership		90
Total net gain since January 1, 1918		47

Constitution and By-Laws of the Mathematical Association of America

Article I—Name and Purpose.

1. This organization shall be known as The Mathematical Association of America.

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field.

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association; such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

Article III—Officers.

1. The officers of this Association shall be a President, two Vice-Presidents, a Secretary-Treasurer and twelve additional members of an Executive Council, together with a Committee of three on Publications, who shall be *ex-officio* members of the Council.

2. The President and Vice-Presidents shall be elected annually for a term of one year, and four members of the Council shall be elected annually for a term of three years. They shall be eligible for reëlection, but not for more than two consecutive terms. The Secretary-Treasurer, and the Committee on Publications, consisting of the Manager, the Editor, and one other member, shall be appointed by the Council.

3. The Council shall transact the official business of the Association and shall report its actions at the annual meeting of the Association and in the official journal. Any proposed action of the Council which makes or alters a question of policy shall be published in the official journal before final action has been taken, so that members of the Association may make known to the Council their individual views.

4. The Council shall have authority to fill vacancies *ad interim*.

Article IV—Meetings.

1. The annual meeting of the Association shall be held at such time and place as the Council may direct.

2. The Council shall have power to call other meetings of the Association whenever it may be deemed expedient.

Article V—Sections.

1. Any group of members of this Association may petition the Council for authority to organize a Section of the Association for the purpose of holding local meetings. The Council shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such sections.

Article VI—Official Journal.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Council shall have full control of the publication and sale of the official journal.

Article VII—Dues.

1. An individual member of the Association shall pay an initiation fee of two dollars at the time of his election.

2. The annual dues of an individual member shall be three dollars, including a subscription to the official journal.

3. The annual dues of an institutional member shall be five dollars, including two subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1, of any year, shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

Article VIII—Amendments

This Constitution may be amended at any annual meeting of the Association by a two-thirds vote of those present and voting, provided that such amendment shall have been printed in the official journal at least one month before the date of such meeting.

BY-LAWS.

1. *Election of Members.* Election to membership shall be by vote of the Council upon written application from the individual or institution seeking admission.

Those who were admitted to membership before April 1, 1916, constitute the list of charter members.

2. *Nomination and Election of Officers.* Two months before the date of the annual meeting, all members shall be given an opportunity to nominate by mail a candidate for each office for the ensuing year. One month before the annual meeting, the Council shall announce two candidates for each office, one being the person who received the highest vote in the nominations and the other being selected by the Council from among the several nominees next in order.

The election shall be by mail or in person and shall close on the day of the annual meeting.

3. *Committees.* The official journal shall be under the general management of the Committee on Publications. There shall also be appointed by the Council a Board of Associate Editors who shall give assistance in connection with the official journal under the direction of the Committee on Publications.

The Council may appoint any other committees and delegate to them such power as may, in its judgment, seem desirable.

4. *Price of Publications.* The Council shall fix the price of the official journal, and of any other publications of the Association to non-members, but in no case shall the journal be sold for less than the annual dues of individual members, as specified in Article VII of the Constitution.

5. *Amendments.* These By-Laws may be amended at any annual meeting under the same conditions as specified in Article VIII of the Constitution.

PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION.

Presidents.

E. R. Hedrick	1916
Florian Cajori	1917
E. V. Huntington	1918
H. E. Slaught	1919

Vice-Presidents.

E. V. Huntington	1916
G. A. Miller	1916
D. N. Lehmer	1917, 1918
Oswald Veblen	1917
J. W. Young	1918
R. G. D. Richardson	1919
H. L. Rietz	1919

Secretary-Treasurer.

(Appointed by the Council after 1918.)

W. D. Cairns	1916–

Committee on Publications.

(Appointed by the Council.)

H. E. Slaught	1916–
R. D. Carmichael	1916–1918
W. H. Bussey	1916–1918
R. C. Archibald	1919–
W. A. Hurwitz	1919–

Elected Members of the Council.

D. N. Lehmer	1916
R. E. Moritz	1916–
K. D. Swartzel	1916
Oswald Veblen	1916
R. C. Archibald	1916–1917
Florian Cajori	1916, 1918–
M. B. Porter	1916–1917
J. W. Young	1916–1917
B. F. Finkel	1916–
E. H. Moore	1916–
J. N. Van der Vries	1916–1918
Alexander Ziwet	1916–1918
E. R. Hedrick	1917–
Helen A. Merrill	1917–
D. E. Smith	1917–
Elizabeth B. Cowley	1918–
G. A. Miller	1918–
E. J. Wilczynski	1918–
L. P. Eisenhart	1919–
E. V. Huntington	1917, 1919–

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Compiled by Helen A. Merrill and Department Editors.

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QUESTIONS AND DISCUSSIONS—SUBJECTS.

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RECENT PUBLICATIONS—NOTES.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, R. C. ARCHIBALD, Brown University, Providence, R. I.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 27 King Street, Oberlin, Ohio.

MATHEMATICS AND STATISTICS, WITH AN ELEMENTARY ACCOUNT OF THE CORRELATION COEFFICIENT AND THE CORRELATION RATIO.[1]

By EDWARD V. HUNTINGTON, Harvard University.

1. Introduction.—The first president of the Association, Professor E. R. Hedrick, in his retiring address in 1917, dwelt at length on the important position which this Association should take in relation to the large and growing field of applied mathematics. The Association should accept as perhaps its primary obligation the duty of interpreting the results of pure mathematics to the workers in the field of applied mathematics. This does not mean the "degradation of pure mathematics to utilitarian purposes." It means rather the *search for identity of essential form among apparently diverse problems.* When once the essential form of a problem has been recognized, the marvelously compact and sure analysis of formal mathematics either supplies directly the solution or illuminates the nature of the difficulty. This search for identity of form among the diversities of practical problems is then the task of the interpreter—a task which demands on the one hand a quick sympathy with the needs of the practical sciences, and on the other hand an unswerving loyalty to the rigorous ideals of the purely formal doctrines. The experience of the war has only served to emphasize the ever-growing need of such interpreters and the importance of the work of codification of problems which only they can perform.

2. The Importance of Mathematics in Modern Statistics. "Biometrika."—With this brief word of introduction, I desire to bring to the attention of the Association the opportunities for such interpretative service presented in a comparatively new field of mathematics, namely, the field of mathematical statistics.

Mathematicians as such seem to me to have been slow to enter this field. Of the professional mathematicians in this country only about a dozen have thought it worth while to join the American Statistical Association (one of the oldest learned societies in the United States, founded in 1839 and now having over 800 members). Of the published papers read before the American Mathematical Society during the last five years, only three or four have had any relation to statistics. The very terminology of modern statistical method is unfamiliar to the great majority of professional mathematicians. Few of us, I fancy, have ever heard of the "tetrachoric functions," "homoscedastic linear regressions," or "mesokurtic skew distributions." Most of the development of the science has been left to the economists, the actuaries, the biologists, the psychologists, and, more recently, the pedagogues. The result has been a wide scattering of the

[1] Retiring Address of the President of the Mathematical Association of America, read at the summer meeting, Ann Arbor, Michigan, at a joint session of the Mathematical Association of America, the American Mathematical Society, and the American Astronomical Society, Sept. 4, 1919. Complete references to papers cited only by title will be found in § 14.

literature of statistical theory; many theoretical results have been first developed in articles having miscellaneous titles like "Family likeness in stature," "The trend of the stock market," or "The reliability of spelling scales"; any unification of effort was clearly lacking.

A distinct epoch in the development of mathematical statistics was marked by the founding of *Biometrika* in 1901, by Karl Pearson, W. F. R. Weldon, and Francis Galton, for the express purpose of bringing together the biologist and mathematician into a partnership of mutual helpfulness. This journal has become a veritable storehouse of rapidly advancing statistical theory, not only of interest to the student of evolution, but also of fundamental importance to the statistician in every field of science; and it needs only a cursory glance through its pages to show what an essential part mathematics has played in this development.

Among the mathematical topics which have proved useful in statistics may be mentioned the following, selected quite at random: the theory of probability in all its phases; determinants; conic sections and quadric surfaces (especially conjugate diameters); hypergeometric series; the Gamma function; Bernoulli's numbers; Stirling's Theorem for factorials; all kinds of interpolation- and quadrature-formulas; hyperbolic functions and properties of the catenary; reversion of series; differential equations of various types; and multiple integrals in n-dimensional space. Such mathematical subjects as these are part of the everyday equipment of the biometricians of the Pearson school. Any adventuresome biologist who tries to apply some technical statistical method without an adequate knowledge of its mathematical foundation is likely to call down on his head the righteous indignation of a ready and vigorous Pearsonian pen, which will ruthlessly expose his ignorance!

And yet the cry is always for more mathematics, and ever more. Many a vital problem in heredity is still unsolved solely because the difficulties of the mathematical analysis have not yet been surmounted. In a recent number, Pearson wrote:

> It is greatly to be desired that the "trigonometry" of higher dimensional plane space should be fully worked out, for all our relations between multiple correlation and partial correlation coefficients of n variates are properties of the "angles," "edges," and "perpendiculars" of sphero-polyhedra in multiple space. It would be a fine task for an adequately equipped pure mathematician to write a treatise on "spherical polyhedrometry"; he need not fear that his results would be without practical application, for they embrace the whole range of problems from anatomy to medicine, and from medicine to sociology and ultimately to the doctrine of evolution. (*Biometrika*, **11**, 1916, p. 237.)

Who shall say that modern statistics is not a worthy field for mathematical endeavor?

In the field of statistical method and theory, the most characteristic single problem is the problem of *correlation*. The establishment of the existence or non-existence of correlation between two things is the final goal of most statistical work. Of the several mathematical measures of correlation which have been proposed, the *correlation coefficient* and the *correlation ratio* are perhaps the most

fundamental. It may not be inappropriate, therefore, to devote the remainder of this address to an elementary exposition of the meaning of these two quantities, not with the thought of adding anything essentially new to the theory, but in the hope of providing a convenient starting point from which some of us who are not already familiar with the subject of statistics may begin our study of its modern developments.

3. The Central Problem of Statistics. Correlation between Two Functions, $x(i)$, $y(i)$.—In the problem of correlation, what is sought for is some measure of agreement or disagreement between *two series of paired quantities*, $x_1, x_2, x_3, \cdots x_n$ and $y_1, y_2, y_3, \cdots y_n$, where the pair x_1, y_1 are supposed to belong to an "individual" numbered 1, the pair x_2, y_2 to an "individual" numbered 2, and so on; the total number of individuals, n, being called the total "population."

For example, x and y may be the height and weight of individual men; or x may be the rainfall and y the price of wheat for individual years; and so on.

In order to state the problem geometrically, we may plot the x's and y's as ordinates against the individual-numbers i as abscissas, as in Fig. 1; what we seek is then some measure of agreement or disagreement between the *two curves or functions*

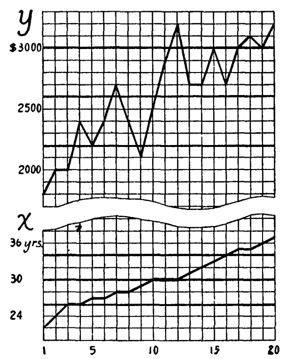

FIG. 1. x = age, y = income, of a certain group of men (n = 20). Means: $\bar{x}$ = 30 years, $\bar{y}$ = 2600 dollars. Standard deviations: σ_x = 4.025 years, σ_y = 417.1 dollars. Correlation coefficient: r = .855.

$$x = f(i), \qquad y = \varphi(i)$$

over a range of values $a \leqq i \leqq b$ (the function being here defined for only a finite number of values of i).

4. Notation: x, y; ξ, η; X, Y.—In order to reduce these two curves to a comparable basis, it is convenient to take two preliminary steps, one concerning the base-line, or origin, the other concerning the scale, or unit of measurement.

As to the *base-line*, we agree to refer each of the given series to the (arithmetical) mean of that series as origin, the means being given by

$$\bar{x} = \frac{1}{n} \sum (x_i), \qquad \bar{y} = \frac{1}{n} \sum (y_i).$$

In other words, instead of plotting the given x's and y's, we plot the "deviations from the means," ξ and η, where

$$\xi_i = x_i - \bar{x}, \qquad \eta_i = y_i - \bar{y}.$$

As to the *scale*, we agree to take as the unit of measurement for each series the "standard deviation" of that series, namely

$$\sigma_x = \sqrt{\frac{1}{n} \sum (\xi_i^2)}, \qquad \sigma_y = \sqrt{\frac{1}{n} \sum (\eta_i^2)},$$

where the "standard deviation," σ_x (or σ_y), may be interpreted as the radius of gyration of the x-curve (or y-curve) about a horizontal axis through its mean height. In other words, instead of plotting the x's and y's or the ξ's and η's in their original units (kilograms, dollars, degrees, or what not), we plot the "ratios" X and Y (as in Fig. 2), where

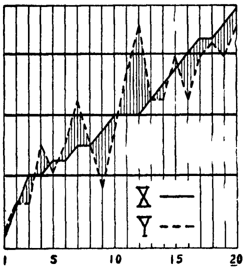

FIG. 2. $X = \frac{x - \bar{x}}{\sigma_x}$, $Y = \frac{y - \bar{y}}{\sigma_y}$.

$$X_i = \xi_i/\sigma_x, \qquad Y_i = \eta_i/\sigma_y.$$

These X's and Y's will be pure numbers, independent of the original units, and having the following properties:

$$\frac{1}{n}\sum (X_i) = 0, \qquad \frac{1}{n}\sum (Y_i) = 0;$$

$$\frac{1}{n}\sum (X_i^2) = 1, \qquad \frac{1}{n}\sum (Y_i^2) = 1.$$

That is, the arithmetical mean of X and the arithmetical mean of Y are both zero, while their standard deviations (or radii of gyration) are both unity.

For the purpose of studying correlation, the reduced curves formed by plotting X_i and Y_i against i will be more convenient than the original curves formed by plotting x_i and y_i against i; and any criterion of agreement or disagreement between the X's and Y's may be accepted as a valid criterion of agreement or disagreement between the x's and y's.

Note.—The term "standard deviation" was introduced by Pearson in 1894.

It should be noted that in case x (or y) is a constant, X (or Y) will be indeterminate. This case, which presents no special interest, will be excluded from consideration in what follows.

5. The Correlation Coefficient r. Definition and Properties.—The most widely accepted measure of correlation between two given series or curves x and y, is the Pearsonian (or "product-moment") *coefficient of correlation*, r, which may be defined in terms of the reduced quantities X and Y by the simple formula

$$r = \frac{1}{n}\sum (X_i Y_i).$$

That is, r is the mean product of corresponding pairs of values of X and Y. By aid of the simple transformations

$$r = 1 - \frac{1}{2}\frac{1}{n}\sum [(X_i - Y_i)^2], \qquad r = -1 + \frac{1}{2}\frac{1}{n}\sum [(X_i + Y_i)^2],$$

it is easily shown that r cannot exceed 1 in absolute value. The case $r = +1$, called the case of *perfect positive correlation*, will occur when and only when the X and Y curves coincide. The case $r = -1$, called *perfect negative correlation*,

will occur when and only when the Y curve is the reflection of the X curve in the axis of i. Either case will occur when and only when the η's are directly proportional to the ξ's; or, what amounts to the same thing, when and only when the original y's and x's are connected by a linear equation

$$\frac{y_i - \bar{y}}{\sigma_y} = \pm \frac{x_i - \bar{x}}{\sigma_x}.$$

We see, therefore, that in the Pearsonian sense, *perfect correlation (positive or negative) between two sets of quantities x and y means nothing more nor less than the existence of a linear algebraic equation connecting those quantities.* Indeed, a better name for the coefficient of correlation might be the "coefficient of linear relationship."

In general, the given sets of values will not be linearly related, and the value of r will be less than 1.

6. Relation of r to the Method of Least Squares.—The significance of the coefficient of correlation, r, may be brought out further by the following considerations. In the expression

$$r = 1 - \frac{1}{2}\frac{1}{n}\sum [(X_i - Y_i)^2],$$

the quantity

$$\Delta = \frac{1}{2}\frac{1}{n}\sum [(X_i - Y_i)^2]$$

may be regarded as a measure of the *total discrepancy,* in the sense of the method of least squares, between the curves X and Y. Clearly, when the discrepancy Δ is zero, or the curves coincide, the correlation r is perfect ($r = 1$); and as the discrepancy increases the correlation decreases. Hence r is seen to be a suitable measure of the degree of approach to coincidence of the two curves X and Y.

We note that as r varies from $+1$ to -1, Δ will vary from 0 to 2, the value $r = 0$ corresponding to $\Delta = 1$.

A further connection with the method of least squares will be noted below (§ 9).

7. Equivalent Formulas for r. The Case of a Continuous Variable.—Equivalent and more familiar formulas for r are

$$r = \frac{\sum(\xi\eta)}{n\sigma_x\sigma_y}, \qquad \text{or} \qquad r = \frac{\sum(\xi\eta)}{\sqrt{\sum(\xi^2)}\sqrt{\sum(\eta^2)}}.$$

The importance of the product-factor $\Sigma(\xi\eta)$ was evident in the work of Bravais in 1846, and was re-discovered and applied by Galton in 1886–1888; but the complete expression for r, in the form just stated, was first given by Pearson in 1896. [Compare also L. March (1905).]

For purposes of numerical computation, the following formula is to be preferred in practice (J. A. Harris, *Amer. Naturalist,* Vol. 44, p. 693–699, 1910; L. L. Thurstone, *Psychological Bulletin,* Vol. 14, pp. 28–32, 1917):

$$r = \frac{\frac{1}{n}\sum(xy) - \bar{x}\bar{y}}{\sqrt{\frac{1}{n}\sum(x^2) - \bar{x}^2}\sqrt{\frac{1}{n}\sum(y^2) - \bar{y}^2}},$$

where, in the denominator,

$$\sigma_x = \sqrt{\frac{1}{n}\sum(x^2) - \bar{x}^2} \quad \text{and} \quad \sigma_y = \sqrt{\frac{1}{n}\sum(y^2) - \bar{y}^2}$$

are the standard deviations of the x's and y's.

We note in passing that if the n individual-numbers, i, are replaced by a continuous variable, t, the formulas above become modified as follows (the summations being replaced by integrations): Given $x = f(t)$ and $y = \varphi(t)$ for the range $a \leqq t \leqq b$. Let

$$\bar{x} = \frac{1}{b-a}\int_a^b x dt, \qquad \bar{y} = \frac{1}{b-a}\int_a^b y dt; \qquad \xi = x - \bar{x}, \qquad \eta = y - \bar{y};$$

$$\sigma_x^2 = \frac{1}{b-a}\int_a^b \xi^2 dt, \qquad \sigma_y^2 = \frac{1}{b-a}\int_a^b \eta^2 dt; \qquad X = \frac{\xi}{\sigma_x}, \qquad Y = \frac{\eta}{\sigma_y}.$$

Then

$$\frac{1}{b-a}\int_a^b X dt = 0, \quad \frac{1}{b-a}\int_a^b Y dt = 0; \quad \frac{1}{b-a}\int_a^b X^2 dt = 1, \quad \frac{1}{b-a}\int_a^b Y^2 dt = 1;$$

and

$$r = \frac{1}{b-a}\int_a^b XY dt = \frac{\int_a^b \xi\eta dt}{(b-a)\sigma_x\sigma_y} = \frac{\frac{1}{b-a}\int_a^b xy dt - \bar{x}\bar{y}}{\sqrt{\frac{1}{b-a}\int_a^b x^2 dt - \bar{x}^2}\sqrt{\frac{1}{b-a}\int_a^b y^2 dt - \bar{y}^2}}.$$

8. The Correlation Graph, or Scatter-Diagram.—Up to this time we have thought of the values of x_i and y_i as functions of i, and have plotted them as two separate curves on the axis of i as a base. It is often more convenient to think of y as a function of x, and to plot the graph of y against x in the ordinary way. The result will not, however, be an ordinary graph, since y will not, in general, be a single-valued function of x. To any value of x may correspond many values of y, any one of which may be repeated more than once. A typical graph of y as a function of x will therefore appear as in Fig. 3, called a *correlation graph* (Galton, 1888), in which every dot (x, y) represents a pair of values of x and y belonging to some individual, and the total number of dots is equal to the total "population," n. (Multiple dots would be indicated in the figure by clusters.)

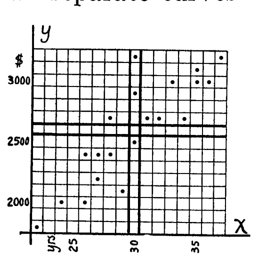

FIG. 3. Correlation graph.

In practice, such a graph is usually divided into squares of suitable size, and the number of dots falling within each square is indicated by a numeral written within that square. This gives a "correlation table" for x and y. Or, each such numeral may be replaced by a z-ordinate of corresponding height, erected at the middle of the square, and a sort of tent-cloth spread over the tops of these ordinates. This gives a "correlation surface," the height of which at any point of the x, y plane shows the density of the distribution of the dots at that point. For our present purpose, however, the simple correlation graph or "scatter-diagram," is all that we shall need.

9. Regression Lines and Coefficients of Regression.—Let us now seek a linear expression that shall give us most accurately the value of y corresponding to any given x.

In the special case when $r = 1$ we know (§ 5) that y is simply a linear function of x, and the equation

$$y - \bar{y} = \frac{\sigma_y}{\sigma_x}(x - \bar{x}) \qquad \text{or} \qquad Y = X$$

will give y exactly in terms of x (or Y in terms of X). In this special case all the dots in the correlation graph will lie on this line. In the general case (r not equal to 1), we may write an approximate equation

$$y' - \bar{y} = \lambda\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \qquad \text{or} \qquad Y' = \lambda X$$

and seek to determine the arbitrary factor λ so that the values of y' (or Y') obtained from this equation shall be equal as nearly as possible to the true values of y (or Y). That is, we seek to determine λ so that the "least-squares" total error,

$$e_y^2 = \frac{1}{n}\sum[(y' - y)^2] \qquad \text{or} \qquad E_y^2 = \frac{e_y^2}{\sigma_y^2} = \frac{1}{n}\sum[(Y' - Y)^2]$$

shall be a minimum. An easy transformation (using the simpler notation) gives

$$E_y^2 = 1 - r^2 + (\lambda - r)^2,$$

which will clearly be a minimum when $\lambda = r$. Hence the "best" straight line to give y in terms of x will be

$$y' - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \qquad \text{or} \qquad Y' = rX,$$

while the amount of total error involved in using y' for y (or Y' for Y) is

$$e_y^2 = \sigma_y'^2(1 - r^2) \qquad \text{or} \qquad E_y^2 = 1 - r^2.$$

Clearly, if $r = 1$, this error is zero; and if $r = 0$, the error takes its maximum value. Hence again we see that the value of r is a suitable criterion of the approach to linear relationship of the variables x and y.

The straight line just obtained is called the *line of regression* of y on x, and the factor $r(\sigma_y/\sigma_x)$ is called the *coefficient of regression of y on x.*

Similarly, the straight line

$$x' - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) \qquad \text{or} \qquad X' = rY$$

will be the "best" straight line for giving x in terms of y, the "total error" being

$$e_x^2 = \sigma_x^2(1 - r^2) \qquad \text{or} \qquad E_x^2 = 1 - r^2.$$

This line is called the *line of regression of x on y*, and the factor $r(\sigma_x/\sigma_y)$ is called the *coefficient of regression of x on y.*

The two lines of regression will not coincide unless $r = \pm 1$.

It will be observed that *the coefficient of correlation, r, is the geometrical mean of the two coefficients of regression.* This fact is the starting point for the theory of "partial correlation coefficients" for any number of variables (Yule, 1897 and 1907), into which we cannot enter here.[1]

10. Curves of the Means, or Regression Curves.—The correlation graph, with its scattered dots, may, for some purposes, be simplified as follows.

Thinking of y as a function of x, let us replace each column by a representative dot located at the mean of that column. We thus obtain what is called the *curve of the means of the columns*, or the *regression curve of y on x* (Pearson, 1896).

Or again, thinking of x as a function of y, we may replace each row by a

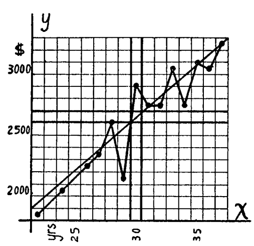

FIG. 4. Curve of the means of the columns, and line of regression of y on x. Coefficient of regression (y on x) = .886; correlation ratio (y on x) = η_{yx} = .941.

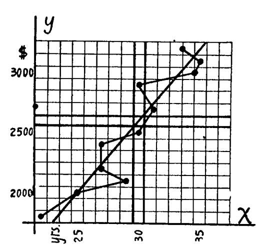

FIG. 5. Curve of the means of the rows, and line of regression of x on y. Coefficient of regression (x on y) = .825; correlation ratio (x on y) = η_{xy} = .916.

representative dot located at the mean of that row, thus obtaining the *curve of the means of the rows* or the *regression curve of x on y.* (See Figures 4 and 5).

If the curves of the means happen to be straight lines, we have the case of *linear regression*, which was the earliest, and for a time the only case considered (Galton, 1888; Pearson, 1895; Pearson, 1905). *In this case it can be readily shown that the curves of the means, or curves of regression, will coincide with the lines of regression defined in* § 9.

[1] An excellent example of the application of partial correlation coefficients in practical statistics may be found in W. F. Ogburn's "Analysis of the standards of living in the District of Columbia in 1916:" *Quart. Pub. Amer. Statistical Association*, vol. 16, June, 1919, pp. 374–389.

In order to prove this most clearly, for the case, say, of the column-means, let us adopt the following notation, expressed, for convenience, in terms of X and Y (§ 4):

X = the abscissa of any column
n_X = the number of dots in the column
Y_X = the ordinate of any dot in the column
$\bar{Y}_X$ = the mean of the column; that is,

$$\bar{Y}_X = \frac{1}{n_X} S_X(Y_X),$$

where S_X denotes summation from dot to dot within the column (X = constant).

These $\bar{Y}_X$'s are the ordinates of the curve of the means in question. If this curve is a straight line, then we must have

$$\bar{Y}_X = \mu + \lambda X,$$

where μ and λ are some constants. But, summing from dot to dot over the whole table,

$$\frac{1}{n}\sum(\bar{Y}_X) = \frac{1}{n}\sum(\mu) + \frac{1}{n}\sum(\lambda X)$$

or[1]

$$\frac{1}{n}\sum(Y) = \mu + \lambda\frac{1}{n}\sum(X),$$

whence

$$\mu = 0.$$

Again, multiplying each side by X, and summing as before,

$$\frac{1}{n}\sum(X\bar{Y}_X) = \lambda\frac{1}{n}\sum(X^2)$$

or[2]

$$\frac{1}{n}\sum(XY) = \lambda(1),$$

whence

$$\lambda = r.$$

Hence, if the curve is a straight line at all, it must be the line

$$\bar{Y}_X = rX$$

which is precisely the regression line of y on x.

[1] Thus, if S_A denotes summation from column to column, while S_X denotes summation from dot to dot within the column X, then

$$\frac{1}{n}\Sigma(\bar{Y}_X) = \frac{1}{n}S_A[S_X(\bar{Y}_X)] = \frac{1}{n}S_A[S_X(Y)] = \frac{1}{n}\Sigma(Y) = 0.$$

[2] Thus,

$$\frac{1}{n}\Sigma(X\bar{Y}_X) = \frac{1}{n}S_A[S_X(X\bar{Y}_X)] = \frac{1}{n}S_A[XS_X(\bar{Y}_X)] = \frac{1}{n}S_A[XS_X(Y)] = \frac{1}{n}S_A[S_X(XY)] + \frac{1}{n}\Sigma(XY) = r.$$

11. The Correlation Ratio, η_{yx}. Definition and Properties.—Whether the curve of the means of the columns is linear or not, the whole system of dots in a correlation graph may be regarded as forming a band of more or less irregular width along this curve; and the *narrower this band, the more nearly will y be a single-valued function of x.*

On the other hand, the dots may be regarded as forming a band along the curve of the means of the rows; and the narrower this band, the more nearly will x be a single-valued function of y.

Considerations of this sort led Pearson in 1905 to introduce two new measures of correlation, η_{yx} and η_{xy}, called the *correlation ratio of y on x* and the *correlation ratio of x on y* respectively. If one speaks simply of *the* correlation ratio, η, one usually means the correlation ratio of y on x. The definition and properties of this quantity may be explained as follows.

In order to secure a measure of the width of the band of dots about the curve of the means of the columns, we begin by letting

d_X = the standard deviation of a column, or the radius of gyration of the column about its mean; that is,

$$d_X = \sqrt{\frac{1}{n_X} S_X(Y_X - \bar{Y}_X)^2},$$

where S_X denotes summation from dot to dot within the column (§ 10). These d_X's will measure the "scatter" of the dots in each column about the mean of that column.

What we then need, as a measure of the total "scatter" of the dots about the curve of the means, is some kind of average of the d_X's for all the columns.

The most obvious average to take for this purpose would be either the *mean* of the d_X's, namely,

$$D' = \frac{1}{A} S_A(d_X),$$

where S_A denotes summation from column to column, and A is the number of columns, or else the *standard deviation* of the d_X's, namely,

$$D'' = \sqrt{\frac{1}{A} S_A(d_X^2)}.$$

It is found more convenient, however, to take a sort of *pseudo-standard deviation,* namely,

$$D = \sqrt{\frac{1}{n} S_A(n_X d_X^2)}$$

in which each d_X^2 is "weighted" by the number of dots in that column, and the sum divided, consequently, by the total number of dots, n, instead of by the number of columns, A.[1]

[1] Here D' is the mean height and D'' the radius of gyration of the "scedastic curve" formed by plotting d_X against X. Both these averages were considered by Pearson (1905, p. 10), but immediately abandoned in favor of D.

This quantity D may be transformed without difficulty as follows:[1]

$$D^2 = \frac{1}{n} S_A[S_X(Y_X - \overline{Y}_X)^2] = 1 - \frac{1}{n} S_A(n_X \overline{Y}_X{}^2).$$

The *correlation ratio of y on x*, is then defined as

$$\eta_{yx} = \sqrt{\frac{1}{n} S_A(n_X \overline{Y}_X{}^2)}$$

where S_A denotes, as before, summation from column to column.

This quantity η_{yx} clearly lies between 0 and 1; and the relation $\eta_{yx}{}^2 = 1 - D^2$ shows that the case $\eta_{yx} = 1$ *will occur when and only when the band of dots narrows down into coincidence with the curve of the means of the columns; that is, when and only when y is a single-valued function of x.* The narrower the band, the more nearly will η_{yx} approach 1.

In terms of the original variables, x and y, the definition becomes

$$\eta_{yx} = \frac{\sqrt{\frac{1}{n} S_A(n_x \bar{y}_x{}^2) - \bar{y}^2}}{\sqrt{\frac{1}{n} \sum (y^2) - \bar{y}^2}}$$

which is the most convenient form for computation.

The *correlation ratio of x on y*, namely η_{xy}, is obtained by simply interchanging x and y and replacing S_A by S_B (to denote summation from row to row).

It should be especially noted that the significance of the two correlation ratios is concerned with the single-valuedness of the connection between x and y, rather than with any particular form of connecting equation. Indeed, they might well be called the "coefficients of single-valued relationship."

For the case of more than two variables, definitions of "partial" and "multiple" correlation ratios have been given by Pearson (1915).

12. Relation between η_{yx} and r.—An important relation between the correlation ratio, η_{yx}, and the correlation coefficient, r, may be brought out by a further study of the curve of the means of the columns (the regression curve of y on x).

If the curve of the means happens to be a straight line, we have the case of *linear regression* (§ 10).

---

[1] Thus:

$$\begin{aligned} D^2 &= \frac{1}{n} S_A[S_X(Y_X - \overline{Y}_X)^2] \\ &= \frac{1}{n} S_A[S_X(Y_X{}^2)] - 2\frac{1}{n} S_A[\overline{Y}_X S_X(Y_X)] + \frac{1}{n} S_A[S_X(\overline{Y}_X{}^2)] \\ &= \frac{1}{n} \Sigma(Y^2) - 2\frac{1}{n} S_A(\overline{Y}_X n_X \overline{Y}_X) + \frac{1}{n} S_A(n_X \overline{Y}_X{}^2) \\ &= 1 - \frac{1}{n} S_A(n_X \overline{Y}_X{}^2). \end{aligned}$$

If the curve of the means is not a straight line, we may seek the straight line which fits the curve with a minimum total discrepancy.

Let $Y' = \mu + \lambda X$ be the required straight line, where μ and λ are to be determined so as to make the total discrepancy between the line and the curve a minimum.

As the expression for the total discrepancy, it would be most natural to take the "least-squares" error

$$\frac{1}{A} S_A[(\mu + \lambda X - \overline{Y}_X)^2],$$

where S_A denotes summation from column to column, and A = the number of columns. It proves more convenient, however, to take a "pseudo-least-squares" expression

$$E^2 = \frac{1}{n} S_A[n_X(\mu + \lambda X - \overline{Y}_X)^2],$$

in which the square of the difference in each column is "weighted" by the number of dots in that column before summing. This quantity E^2 may be transformed, by a straightforward process,[1] into

$$E^2 = \eta_{yx}^2 - r^2 + \mu^2 + (\lambda - r)^2,$$

where η_{yx} has the value defined above (§ 11).

Evidently, to make E a minimum we must put $\mu = 0$ and $\lambda = r$. Hence the line which best fits the curve of the means of the columns (in the pseudo-least-squares sense) is simply the line of regression (§ 9):

$$Y' = rX \qquad \text{or} \qquad y' - \bar{y} = r \frac{\sigma_y}{\sigma_x}(x - \bar{x}).$$

The total discrepancy between the line and the curve is

$$E^2 = \eta_{yx}^2 - r^2 \qquad \text{or} \qquad e^2 = \sigma_y^2(\eta_{yx}^2 - r^2).$$

From this last equation we see that *the correlation ratio, η_{yx}, will be equal to the correlation coefficient, r, when and only when the regression of y on x is linear*; that is, when and only when the curve of the means of the columns reduces to a straight line.

[1] Noting that

$$\frac{1}{n} S_A(n_X X) = \frac{1}{n} \Sigma(X) = 0,$$

$$\frac{1}{n} S_A(n_X X^2) = \frac{1}{n} \Sigma(X^2) = 1,$$

$$\frac{1}{n} S_A(n_X \overline{Y}_X) = \frac{1}{n} S_A[S_X(Y_X)] = \frac{1}{n} \Sigma(Y) = 0,$$

and

$$\frac{1}{n} S_A(X n_X \overline{Y}_X) = \frac{1}{n} S_A[X S_X(Y_X)] = \frac{1}{n} S_A[S_X(X Y_X) = \frac{1}{n} \Sigma(XY) = r.$$

Similarly, $\eta_{xy} = r$ when and only when the regression of x on y is linear.

Thus, in the case of linear regression, it makes no difference whether we use the correlation ratio or the correlation coefficient. In the case of non-linear regression, the correlation ratio is usually to be preferred.

It should be noted, in this connection, that one of the curves of the means may be a straight line and the other not, so that we may have "linear regression" for y on x and not for x on y.

13. Conclusion. Remarks on the Probable Error.—We have thus completed our sketch of the elementary theory of the correlation coefficient and the correlation ratio. Space does not permit any account of other methods of measuring correlation, such as the method of contingency, the method of correlation by ranks (including Spearman's "Foot-Rule"), the method of four-fold division, etc., which have been devised for special purposes (compare Pearson, 1907; Ritchie-Scott, 1918). Nor can we discuss the precautions that are necessary in practical computation on account of "corrections for grouping" in the correlation table.

At least a word must, however, be said in regard to the question of *probable errors*. The probable error of the correlation coefficient, r, is usually given as

$$.67449 \frac{1 - r^2}{\sqrt{n}}$$

and that of the correlation ratio, η_{yx}, as roughly,

$$.67449 \frac{1 - \eta_{yx}^2}{\sqrt{n}},$$

while the probable error of the quantity $\zeta = \eta_{yx}^2 - r^2$ (used as a test for linearity of regression) is, still more roughly,

$$.67449 \left(2 \frac{\sqrt{\zeta}}{\sqrt{n}} \right);$$

and we are continually warned that *no confidence can be placed in any of these quantities unless their value is three or four times their probable error.*

Now what is the real significance of these probable errors? They have required some extraordinarily intricate mathematics for their determination; it is hardly possible to explain their essential nature in a few words. This much, can, however, be said. The question is primarily a question of what is technically known as the "errors of random sampling." Unless the statistical material with which we are dealing is a "sample" of a larger statistical "population," no question of "probable error" will arise.

The simplest case may perhaps be stated as follows: Suppose a correlation table exists, with a population N, and an unknown correlation coefficient, r; and suppose a sample population, n, drawn from N, is found to have a correlation coefficient r_1. Another sample of the same size would doubtless have a different

coefficient, r_2. Suppose the total number of samples of this size that can be drawn from N is m, and let

$$r_1, r_2, r_3, \cdots r_m$$

be the correlation coefficients for these m samples. Further, let $\bar{r}$ be the mean of these r's, which we may take to be equal to the true value r, and let ρ be their standard deviation:

$$\rho = \sqrt{\frac{1}{m}\sum(r - \bar{r})^2}.$$

Then *if the r's are assumed to be distributed according to the normal law of error*, it can be shown that half of them will lie between $\bar{r} + 0.6745\rho$ and $\bar{r} - 0.6745\rho$, and the other half outside those limits. When we say, then, that an observed value, r_k, has a "probable error" of $p = .6745\rho$, or that the true value of r is as likely to lie within as without the limits

$$r_k + p \qquad \text{and} \qquad r_k - p,$$

we mean simply that *if we actually tested all the mathematically possible samples, half the results would lie within these limits and half without*; *the important point being that we do not know to which half our particular sample may belong.* Any statement as to the size of a probable error is always a statement of our ignorance; but even this ignorance may give wise information, for while all ignorance is deplorable, some ignorance is more deplorable than others. As a general rule, the smaller the size of the sample, the greater the depth of our ignorance.

14. References.—The full titles of the articles referred to above by date only, are here appended. The best bibliography of the original memoirs—to which every serious student of statistics must eventually refer—is given in Yule's *Introduction to the Theory of Statistics*, 3d edit., 1916.

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MEETING OF THE MINNESOTA SECTION.

The annual spring meeting of the Minnesota Section was held on Saturday, May 31, 1919, at Science Hall, Hamline University. There were thirty-one present including the following members of the Association: R. M. Barton, W. O. Beal, Edla G. Berger, W. H. Bussey, H. H. Dalaker, C. H. Gingrich, R. A. Johnson, Sister Mary John (institutional representative), R. M. Mathews, C. A. V. Peterson, Jessie G. Quigley, W. D. Reeve, Ella A. M. Thorp, C. H. Yeaton.

The program consisted of chairman's remarks by C. H. Gingrich, Carleton College, a paper by W. F. Swann, of the department of physics, University of Minnesota, a book review by W. O. Beal, of the department of astronomy, University of Minnesota, a memorial tribute to Father William Earnshaw Etzel by Father Moynihan, President of St. Thomas College, a discussion of college entrance requirements for mathematics opened by W. D. Reeve, College of Education, University of Minnesota, popular talks on mathematics by Jessie G. Quigley, College of St. Teresa, and five papers upon mathematical subjects by R. M. Mathews, Duluth Central High School, W. H. Kirchner of the college of engineering, University of Minnesota, R. A. Johnson, Hamline University, D. C. Kazarinoff, Carleton College, and W. H. Bussey, University of Minnesota.

The program was unusually well selected and presented and all papers proved of interest to the audience. Mr. Reeve, in opening the discussion of college entrance requirements in mathematics, urged the use of mental tests in addition to the method of certification and presented a considerable body of data which has grown out of his experience in this work.

Mr. Mathews's paper on "The graphics of intersections of line and conic" appears elsewhere in this issue of the Monthly.

Mr. Kirchner followed his discussion of last year upon the desirability of introducing Descriptive Geometry as a course in college mathematics, with a short paper upon "The co-point of three planes" in which he set up the conditions for the common point of three planes from the point of view of descriptive geometry and interpreted these in terms of the corresponding conditions from the view points of analytic geometry and algebra.

Dr. Johnson's paper dealt with the well-known theorem usually ascribed to

Fontené: The necessary and sufficient condition that the pedal circle of a point be tangent to the nine-point circle in that the point and its isogonal conjugate be collinear with the circumcenter. An alternative method of finding, with ruler and compasses, the intersections of a line with a conic given by five points was incidentally developed.

Mr. Kazarinoff's paper on "Some properties of focal conics of quadrics" appears elsewhere in this issue of the MONTHLY as "Dupin's Theorem."

President Moynihan of the College of St. Thomas related the events of the life of Father William Earnshaw Etzel in a very interesting manner, giving an insight into the motives and accomplishments of Father Etzel at once keen and sympathetic. Father Etzel was a charter member of the Section, an officer and an ardent helpful worker whose presence is greatly missed.

Mr. Swann's paper on "The diurnal variation of the atmospheric potential-gradient" is in abstract as follows:

Various phenomena in cosmical physics lend support to the view that the upper atmosphere is highly conducting as compared with the air near the earth. Thus, for example, considerations based on the ionization of the upper atmosphere by the sun's ultra-violet light show that this agency is capable of accounting for such a high conductivity that the resistance of a column of the upper air equal in length to the earth's quadrant would be no more than that of a column 3 cms. long and of the same cross section at the earth's surface. Thus, as far as this agency is concerned, the upper atmosphere may, for electrostatic purposes, be looked upon as a perfect conductor. Although the ultra violet light affects only the sunlit portions of the atmosphere, we have reason to believe that the corpuscular radiation responsible for the aurora is more evenly distributed.

If two concentric spheres be maintained at a difference of potential, the field at the surface of the inner sphere will, from symmetry, be the same at all points. If a dent be made in the outer sphere, the distance between the spheres at this point will be decreased, and since each sphere is at the same potential at all points, the *field* at the portion of the surface of the inner sphere under the dent will be stronger than the field elsewhere. This crude illustration suggests the view that if the upper atmosphere is in a highly conducting state, but the high conductivity on the sunlit side extends to lower levels than on the opposite side, the potential gradient should be higher in the former case than in the latter.

Here then would be an influence playing a part in the determination of the diurnal variation of the potential gradient, and such as to predict a maximum by day and a minimum by night, which condition is found, in a general way, to prevail. In its simplest aspects it would predict a diurnal variation of the air-earth conduction current-density of the same type as the potential-gradient. In practice it is found that the potential-gradient and conductivity at the earth's surface vary in opposite ways in such a manner as to secure greater constancy in the conduction current-density than in either of its constituent factors. However, it is by no means universally true that the conduction current-density remains even approximately constant; it is thus of interest to trace out in some-

what greater detail the consequences of the view outlined above, by working out an example where the change of conductivity with altitude is not absolutely sudden, as in the crude illustration cited, but takes place in a continuous manner.

In the problem considered, a distribution of potential-gradient over the surface of the earth is expressed in a series of Legendre coefficients in the form:

$$\left(\frac{\partial V}{\partial r}\right)_{r=a} = A_0 + A_1P_1(\mu) + \cdots + A_nP_n(\mu) + \cdots \tag{1}$$

r, θ, being the polar coördinates of a point, and θ being measured from the line joining the center of the earth to that of the sun.

It is shown that a solution of the equation of continuity, subject to $V = 0$ at the sphere of radius a (the radius of the earth), and giving the distribution (1) over this sphere is given by:

$V =$ any function of ψ,

where,

$$\psi = -\frac{\sigma_0A_0a^2}{r} + \frac{\sigma_0A_1}{3}\left(r - \frac{a^3}{r^2}\right)P_1(\mu) + \cdots + \frac{\sigma_0A_n}{2n+1}\left[\frac{r^n}{a^{n-1}} - \frac{a^{n+2}}{r^{n+1}}\right]P_n(\mu) + \cdots,$$

provided that σ the conductivity is given by:

$$\sigma = \left(\frac{dV}{d\psi}\right)^{-1}, \text{ while}$$

σ_0 is the conductivity at the surface of the earth. As a particular example, the case

$$\left(\frac{\partial V}{\partial r}\right)_{r=a} = X_0 + X_0\alpha(1 + \cos\theta) + \cdots \tag{2}$$

is considered, α being a constant which, for illustration, is taken as 0.25. This expression corresponds to a decrease in $(\partial V/\partial r)_{r=a}$ from $\theta = 0$ to $\theta = \pi$. Values of σ and $\partial V/\partial r$ given respectively by:

$$\sigma = \sigma_0 e^{\beta\left[1-\left\{\frac{a}{r} - \frac{\alpha}{3(1+\alpha)}\left(\frac{r}{a} - \frac{a^2}{r^2}\right)\cos\theta\right\}\right]} \tag{3}$$

and

$$\frac{\partial V}{\partial r} = X_0\left[(1+\alpha)\frac{a^2}{r^2} + \frac{\alpha}{3}\left(1 + \frac{2a^3}{r^3}\right)\cos\theta\right]e^{-\beta\left[1-\left\{\frac{a}{r} - \frac{\alpha}{3(1+\alpha)}\left(\frac{r}{a} - \frac{a^2}{r^2}\right)\cos\theta\right\}\right]} \tag{4}$$

satisfy the necessary conditions for this case, β being constant. It is shown that if β is chosen so as to make σ increase by a factor of 30 at an altitude of 10 kilometers, the conductivity at such an altitude as is given by $r = 2a$, for example, is practically infinite for all values of θ; and, the distribution of $\partial V/\partial r$ in the region $r < 2a$ is sensibly the same as would have resulted from any other expression for σ which was sensibly the same as that chosen in the region $r < 2a$, even though, as r increased in value, this other value of σ gradually departed from the form (3) in such a way as to avoid the peculiarities associated with that form for large values of r.

It will be observed that the form (3) is such that σ increases more rapidly with r when $\theta = 0$ than for any other value of θ, and it is thus of the type to represent qualitatively the influence of the sun's ultra-violet light. The constant α may be taken as representing the intensity of the sun's effect. If there were no ultra-violet light, α would be zero, and (3) and (4) would indicate equal rates of increase of σ and of decrease of $\partial V/\partial r$ with altitude for all values of θ. Moreover, the potential-gradient would be constant over the earth's surface, as shown by (2).

The views outlined in the paper are applicable to the case of the solar eclipse, and are in line with experiment in predicting a diminution of the potential-gradient as a result of removal of the sun's rays.

Mr. Bussey's paper on "Methods of finding the square root of a number by means of a computing machine" gave two methods. (1) If a is an approximate value of the square root of a number $N = (a + b)^2$, the number $\frac{1}{2}[(N/a) + a]$ is a better approximation. This follows from the fact that

$$\tfrac{1}{2}[(N/a) + a] = (a + b) + b^2/2a$$

which is nearly equal to $(a + b)$ if b is small in comparison with a. The divisions involved in this method can be performed with ease on a computing machine. The method is equivalent to Newton's method of approximation. (2) The square root of a number less than 10,000 can be found by repeated subtraction by means of the following theorems:

(i) $a^2 =$ the sum of a consecutive odd numbers beginning with unity.

(ii) $2abt + b^2 =$ the sum of b consecutive odd numbers beginning with $2\,at + 1$.

Example:

$$\left.\begin{array}{r} 529-100 = 429 \\ 429-300 = 129 \end{array}\right\}\ \text{Two subtractions; therefore the first figure of the square root is 2.}$$

$$\left.\begin{array}{r} 129-\ 41 = 88 \\ 88-\ 43 = 45 \\ 45-\ 45 = 0 \end{array}\right\}\ \text{Three subtractions; therefore the second figure of the square root is 3. Answer, 23.}$$

The method can be extended so as to apply to numbers greater than 10,000 and to numbers which are decimal fractions.

Mr. Beal's paper reviewed *An introductory treatise on dynamical astronomy*, by H. C. Plummer (see this MONTHLY, June, 1919, pages 253–254).

Miss Quigley gave numerous illustrations of success she has attained by popular lectures in interesting students in mathematics and its study.

The program was followed by an excellent dinner at which the Section was guest to Hamline University. For the ensuing year R. A. JOHNSON was elected chairman, R. M. BARTON, secretary-treasurer, C. H. GINGRICH, C. H. YEATON, R. M. MATHEWS, executive committee.

It was voted that resolutions upon the death of Father Etzel be drawn up and placed in the minutes of the Section and that a copy of the same be sent to St. Thomas College.

R. M. BARTON, *Secretary-Treasurer.*

THE WORK OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS.

Since the publication of the statement concerning the work of the National Committee on Mathematical Requirements in the September 1919 number of the MONTHLY, the following events have occurred.

The office of the Chairman of the Committee has been located in the Musgrove Building, Hanover, New Hampshire. Mr. J. A. Foberg's office is at 3829 North Tripp Avenue, Chicago, Illinois.

The following members have been added to the Committee:

Mr. A. C. OLNEY, Commissioner of Secondary Schools, Sacramento, California.

Mr. P. H. UNDERWOOD, Ball High School, Galveston, Texas.

Miss EULA A. WEEKS, Cleveland High School, St. Louis, Missouri.

Mr. WALTER DOWNEY, of the English High School, Boston, Massachusetts, has been appointed by the New England Association of Teachers of Mathematics to take the place of Mr. G. W. EVANS, who is spending a year's leave in China.

A report on "The Reorganization of the Introductory Course in Secondary School Mathematics" by G. W. Evans has been received, and is being revised with a view to making it the basis of discussion on the part of teachers organizations throughout the country. A report by A. R. Crathorne on "Change of Mind between High School and College as to Life Work" has also been received; it is expected that this report will be published soon. Professor Crathorne's work on "A Critical Study of the Correlation Method as Applied to Grades" is progressing rapidly. An extended report on "Experimental Schools and Courses" by Mr. Raleigh Schorling and one by Mr. J. A. Foberg on "Mathematics for the Junior High School" are in preparation, as is also a preliminary report on the "Revision of College Entrance Requirements" by the Chairman. A comprehensive bibliography of the recent literature on the subject of the teaching of mathematics is being made under the direction of Professor David Eugene Smith.

The following further items regarding the work of the Committee may be of interest.

A letter addressed to some 30 publishing houses asking their coöperation in the work of the Committee has received a very generous response. As a result, the Committee now has a collection of almost 200 modern textbooks relating to secondary and elementary college mathematics.

The efforts of the Committee to make contact with organizations of teachers of mathematics throughout the country is also beginning to yield results. The directory of such organizations now includes about 40. Requests for speakers on the part of such organizations at meetings during October and November have been received from the Indiana State Teachers Association, Northwestern Illinois Teachers Association, New Jersey Association of Mathematics Teachers,

Iowa Association of Mathematics Teachers, New York State Teachers Association, Texas State Teachers Association, the High School Conference of the University of Illinois, the Association of Teachers of Mathematics in the Middle States and Maryland and the North Dakota Educational Association.

An effort is being made to get the work of the Committee before the public through the general educational periodicals of the country. Letters have been sent out to over 50 such periodicals. Those which have been heard from have given a very cordial response.

The Committee plans further to promote the formation of new organizations of teachers, where such organizations are needed and do not exist at the present time. These organizations may be sectional, covering a considerable area, or they may consist merely of local clubs which can meet at frequent intervals for the discussion and study of the problems of the Committee. It is hoped that such clubs can be organized in all the larger cities where they do not already exist.

It plans also to establish contact directly with individual teachers. The committee feels that this is necessary in addition to their work through organizations in order to induce such individuals to become active and in order to make the work through organizations effective. Plans for establishing this contact with individuals on a large scale are under consideration.

Organizations can be of assistance by sending to the Committee a statement of the name of the organization, its officers for the coming year, the time and place of its meetings and information regarding proposed programs. If any organization has within the last ten years issued any reports on topics connected with the work of the Committee, copies of such reports should, if available, be sent both to Mr. Young and Mr. Foberg. If this is impossible, a statement regarding the character and place of publication of any such reports would be welcome.

Individuals can be of assistance

1. By keeping the Committee informed of matters of interest that come to their notice;
2. By suggesting ways in which the Committee can be helpful;
3. By sending to the committee in duplicate reprints of any articles they publish on subjects connected with the Committee's work;
4. By furthering the work of the Committee among their colleagues, organizing discussions, etc.

It is not too much to say that the existence of this Committee with its present resources gives the teachers of mathematics, both individually and through their organizations, a unique opportunity to do really constructive work of the highest importance in the direction of reform. They can surely be counted on to make the most of this opportunity.

[*For later information see page 462.*]

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

In the first discussion this month, Mr. Kazarinoff gives simple demonstrations of several theorems on conic sections in space, suggested by the properties of focal conics of a system of confocal quadrics. Without access to original sources or other adequate references, the author was led to these theorems by direct study of Figs. 1 and 2 of the article. Though Mr. Kazarinoff's theorems lack entire novelty it is hoped that the directness and simplicity of his demonstrations may aid in making them more widely known and admired; and that students of conics may thereby be incited to look into the generalizations of Chasles and Plücker.

The second discussion is of interest as affording an instance of the need of a theorem in mathematics for the purpose of answering a particular question in an applied science. In connection with a condition for stability of thermodynamic equilibrium of a fluid phase of a two-component body Professor Trevor is led to a property of homogeneous functions. The result is itself of interest, and suggests the possibility of generalizations. Probably the denominator $x_3 + x_4$ might be altered to any linear function of the variables with like results. Do similar theorems exist for more general types of transformation? What are the corresponding facts for other isobaric functions than homogeneous? Can the facts in the present and other cases be deduced in any simpler way?

Direct interpretations of imaginary intersections of geometric loci, by operations on the figure itself, have frequently aroused interest. Several such treatments exist for determining complex roots of quadratic and cubic equations. Mr. Mathews gives in the third discussion a unified treatment, by which a number of such questions can be resolved by the application of essentially a single idea—his *virtual image* of a circle with regard to a straight line.

I. DUPIN'S THEOREM.[1]

By D. C. KAZARINOFF, University of Michigan.

It is well known that the consideration of the family of confocal quadrics

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1 \quad (a^2>b^2>c^2), \tag{F}$$

leads to the three focal conics:[2]

[1] Read before the Minnesota Section of the Mathematical Association of America, May 31, 1919.

[2] C. Smith, *An elementary treatise on Solid Geometry*, 10th ed., London, 1905, pp. 144, 145.

Ellipse:
$$\frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} = 1, \qquad z = 0, \tag{E}$$

Hyperbola:
$$\frac{x^2}{a^2 - b^2} + \frac{z^2}{c^2 - b^2} = 1, \qquad y = 0, \tag{H}$$

Imaginary conic:
$$\frac{y^2}{b^2 - a^2} + \frac{z^2}{c^2 - a^2} = 1, \qquad x = 0. \tag{I}$$

Furthermore the following property is well known:[1]

The locus of the vertices of the right circular cones which envelop the surfaces (F) *consists of the focal conics* (E), (H), *and* (I).[2]

Since the part of the plane $z = 0$ interior to (E) and the part of the plane $y = 0$ which does not contain the center and is bounded by (H) may both be regarded as limiting cases of surfaces (F),[3] the following particular property may be derived:

Property (A) (see Fig. 1): *Given any ellipse* (E) *and any hyperbola* (H) *in perpendicular planes, so placed that the foci of* (H) *are the vertices of* (E), *and vice versa. Then the locus of the vertices of right circular cones on the elliptic base* (E) *is the hyperbola* (H), *and vice versa.*[4]

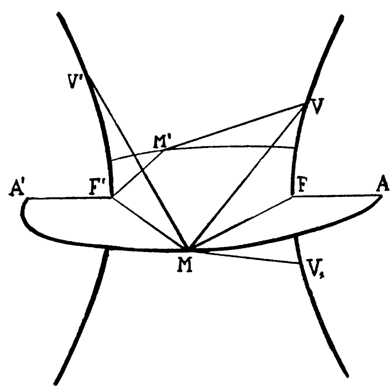

FIG. 1.

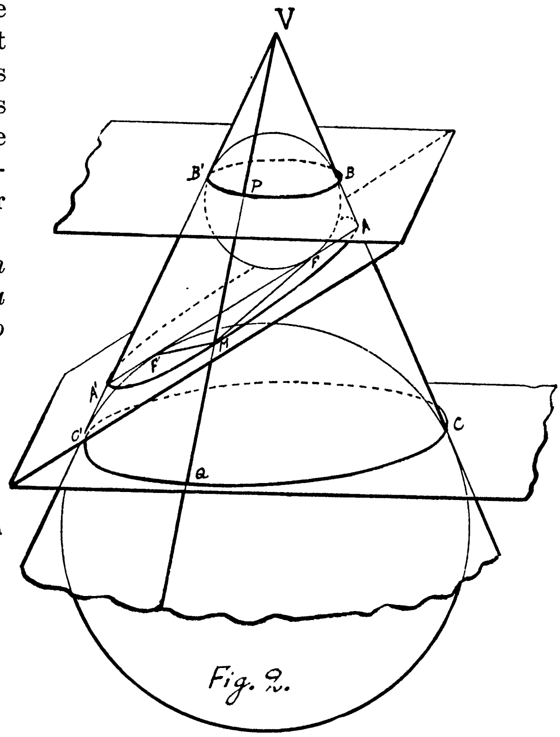

FIG. 2.

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[1] C. Smith, *idem*, pp. 153–155. See also G. Salmon, *A treatise on conic sections*, 3d ed., London, 1855, p. 303.

[2] This result is due to Steiner, *Journal für die reine und angewandte Mathematik*, vol. 1 (1826), pp. 47 ff.—EDITOR-IN-CHIEF.

[3] C. Smith, *idem*, pp. 144, 145.

[4] This result is due to C. Dupin, *Correspondance sur l'école polytechnique*, vol. 2 (Jan., 1813), p. 424; in the *Correspondance* for April, 1804, Hachette attributed the result to Dupin.—EDITOR-IN-CHIEF.

My purpose is to present here an elementary proof of the property (A) and to deduce from it some other reciprocal properties of (E) and H.

Proof of property (A) (see Fig. 2):

Let V be the vertex of a right circular cone on the elliptic base $AA'M$ of which the vertices and the foci are respectively A, A' and F, F'. The points of contact of two spheres inscribed in the cone and touching the plane $AA'M$ will be the foci F, F'.[1]

The plane VAA' passes through the centers of the spheres and is therefore perpendicular to the plane $AA'M$.

Furthermore,

$$VA' - VA = B'A' - BA = A'F - AF = FF'.$$

Hence V is a point on an hyperbola with vertices and foci respectively at F, F' and A, A' in a plane perpendicular to the plane $AA'M$.[2] Since it can be proved in a similar way that from every point of the above hyperbola the ellipse $AA'M$ will be projected by a right circular cone, we have (Fig. 1):

The locus of the point V is an hyperbola with the same relative position to the ellipse $AA'M$ as (H) has with respect to (E).

Mutatis mutandis we can prove the second part of the property (A).

From Fig. 2 we have also:

$$MV + MF' = MV + MQ = VQ = \text{constant}, \tag{b}$$

$$MV - MF = MV - MP = VP = \text{constant}. \tag{c}$$

Property (B) (See Fig. 1):

V, V' being any two fixed points on different branches of the hyperbola (H), and M any point on the ellipse (E),

$$MV + MV' = \text{constant}.$$

Proof:

$$MV + MF' = \text{constant},$$

$$MV' + MF = \text{constant},$$

$$MF' + MF = AA' = \text{constant}.$$

Hence

$$MV + MV' = \text{constant}.$$

Property (C):

V, V_1 being any two fixed points on the same branch of (H), and M any point on the ellipse (E),

$$MV - MV_1 = \text{constant}.$$

Proof:

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[1] G. Salmon, *idem*, footnote on p. 305.

[2] This property is due to G. P. Dandelin, *Nouv. Mém. Acad. Bruxelles*, volume 2 (1822), p. 172. —Editor-in-Chief.

$$MV - MF = \text{constant}.$$

$$MV_1 - MF = \text{constant}.$$

Hence

$$MV - MV_1 = \text{constant}. \qquad \text{Q. E. D.}$$

Property (D):

M, M′ being any two fixed points on the ellipse (E), and V any point on the hyperbola (H),[1]

$$VM - VM' = \text{constant}.$$

Proof:

$$MV + MF' = M'V + M'F' = \text{constant}.$$

Hence

$$VM - VM' = \text{constant}. \qquad \text{Q. E. D.}$$

It can be proved without difficulty that the locus of the points having the property (B) is the hyperbola (H), as well as that the locus of the points having the property (C) is the ellipse (E).

II. A Property of Homogeneous Functions.

By J. E. Trevor, Cornell University.

When a body constituted of two independent component substances and subject to no mechanical or thermal separation of its parts is in a state of thermodynamic equilibrium, the body may exhibit distinct liquid or aëriform parts. Any such part has variable mass, composition, and thermodynamic state, and is termed a fluid "phase" of the body. When x_1, x_2, x_3, x_4 denote the volume, the entropy, and the component-masses of a two-component phase, the energy of the phase is a homogeneous function $E(x_1, x_2, x_3, x_4)$ such that

$$t \cdot E(x_1, x_2, x_3, x_4) = E(tx_1, tx_2, tx_3, tx_4), \tag{1}$$

where t is any positive number.[2] The specific volume, specific entropy, and specific component-masses of the body are defined by the equations

$$y_1 = \frac{x_1}{x_3 + x_4}, \qquad y_2 = \frac{x_2}{x_3 + x_4}, \qquad y_3 = \frac{x_3}{x_3 + x_4}, \qquad y_4 = \frac{x_4}{x_3 + x_4} = 1 - y_3.$$

When $t = 1/(x_3 + x_4)$, and X is written for $x_3 + x_4$, the equation (1) becomes

$$E = X \cdot E(y_1, y_2, y_3, 1 - y_3)$$

$$= X \cdot e(y_1, y_2, y_3), \tag{2}$$

where the "specific energy" e of the phase is a function of the three variables y_1, y_2, y_3.

[1] Properties (B), (C) and (D) are due to Dupin, *Correspondance sur l'école polytechnique*, Jan., 1807, p. 218, and Jan., 1813, p. 424. For further references in this connection, and to generalizations by Chasles and Plücker, the *Encyclopédie des Sciences Mathématiques*, tome III, vol. 4, pp. 81–86, and tome III, vol. 3, pp. 51–52, may be consulted.—Editor-in-Chief.

[2] Functions which possess this restricted form of homogeneity are termed "positively homoneous" by Bolza, *Lectures on the Calculus of Variations*, Chicago, 1904, p. 119.—Editor.

If we write Taylor's expansion in the notation $\delta E = \delta^1 E + \delta^2 E + \cdots$, where $\delta^n E$ is the sum of terms of the nth order, we have that the criterion of the stability of the thermodynamic equilibrium of a fluid phase is $\delta^2 E \geq 0$, where the sign of equality holds only when the variations of the variables satisfy the conditions

$$\frac{\delta x_1}{x_1} = \frac{\delta x_2}{x_2} = \frac{\delta x_3}{x_3} = \frac{\delta x_4}{x_4}. \tag{3}$$

In the desire to express the conditions of stability in terms of derivatives of the specific energy $e(y_1, y_2, y_3)$, it becomes necessary to establish a relation between the sum of terms $\delta^2 E$ and the variations of the function e. Giving the four variables x_i independent increments δx_i, the functions E, e in (2) obtain increments δE, δe, and we have

$$\delta E = e\delta X + X\delta e + \delta X \delta e,$$

or

$$\delta^1 E + \delta^2 E + \cdots = e\delta X + X \cdot \delta^1 e + X \cdot \delta^2 e + \cdots + \delta X \cdot \delta^1 e + \delta X \cdot \delta^2 e + \cdots. \tag{4}$$

The first member of this equation is a power series in δx_1, δx_2, δx_3, δx_4, with constant coefficients. The second member is a power series in δy_1, δy_2, δy_3, δX, with constant coefficients. Now, since $x_i = Xy_i$, the variables δx_i have the values

$$\delta x_i = (X + \delta X)\delta y_i + y_i \delta X \qquad (i = 1, 2, 3, 4), \tag{5}$$

where $y_4 = 1 - y_3$ and $\delta y_4 = -\delta y_3$. If we replace the δx_i in the first member of (4) by these values, both members of (4) will be power series in δy_1, δy_2, δy_3, δX.

The first term of the first member of (4) is a sum of terms $\delta^1 E = \Sigma A_i \delta x_i$, or by (5),

$$\delta^1 E = X \cdot \Sigma A_i \delta y_i + \delta X \cdot \Sigma A_i \delta y_i + \delta X \cdot \Sigma A_i y_i. \tag{6}$$

The first, and no other, of these three expressions contains δy_i with no factor δX. No terms in δy_i alone occur in $\delta^2 E$, $\delta^3 E$, $\cdots$, since by (5) the terms occurring there are at least quadratic in the variables. So the first expression in the right-hand member of (6) is equal to the sum of the terms in δy_i in the second member of (4),

$$X \cdot \Sigma A_i \delta y_i = X \cdot \delta^1 e. \tag{7a}$$

It follows that $\delta^1 e = \Sigma A_i \cdot \delta y_i$; wherefore the second expression in (6) is

$$\delta X \cdot \Sigma A_i \delta y_i = \delta X \cdot \delta^1 e. \tag{7b}$$

The third expression in (6) is in δX. No terms in δX alone occur in $\delta^2 E$, $\delta^3 E$, $\cdots$, since these terms are at least quadratic. So the third expression in (6) is equal to the term in δX in the second member of (4),

$$\delta X \cdot \Sigma A_i y_i = e \cdot \delta X. \tag{7c}$$

On substituting these three results, (6) becomes

$$\delta^1 E = (X + \delta X)\delta^1 e + e\delta X. \tag{8}$$

On subtracting (8) from (4), member by member, there remains

$$(9) \qquad \delta^2 E + \delta^3 E + \cdots = X\cdot\delta^2 e + X\cdot\delta^3 e + \cdots + \delta X\cdot\delta^2 e + \delta X\cdot\delta^3 e + \cdots.$$

The first term of the first member of (9) is a sum of terms $\delta^2 E = \Sigma A_{ij}\delta x_i \delta x_j$, or by (5)

$$\delta^2 E = \Sigma A_{ij}[(X + \delta X)\delta y_i + y_i\delta X][(X + \delta X)\delta y_j + y_j\delta X].$$

Expanding the indicated product,

$$(10) \quad \begin{aligned} \delta^2 E = X^2\cdot\Sigma A_{ij}\delta y_i\delta y_j &+ 2X\delta X\cdot\Sigma A_{ij}\delta y_i\delta y_j + (\delta X)^2\cdot\Sigma A_{ij}\delta y_i\delta y_j \\ &+ X\delta X\cdot\Sigma A_{ij}(y_i\delta y_j + y_j\delta y_i) + (\delta X)^2\cdot\Sigma A_{ij}(y_i\delta y_j + y_j\delta y_i) \\ &+ (\delta X)^2\cdot\Sigma A_{ij}y_iy_j. \end{aligned}$$

The first, and no other, of these six expressions is in $\delta y_i\delta y_j$. No terms in this product alone occur in $\delta^3 E$, $\delta^4 E$, $\cdots$, since by (5) the terms occurring there are at least cubic in the variables. So the first expression in (10) is equal to the sum of terms in $\delta y_i\delta y_j$ in the second member of (9),

$$(11a) \qquad X^2\cdot\Sigma A_{ij}\delta y_i\delta y_j = X\cdot\delta^2 e.$$

It follows that $\delta^2 e = X\cdot\Sigma A_{ij}\delta y_i\delta y_j$; wherefore the second and third expressions in (10) are

$$(11b) \qquad 2X\delta X\cdot\Sigma A_{ij}\delta y_i\delta y_j = 2\delta X\cdot\delta^2 e,$$

$$(11c) \qquad (\delta X)^2\cdot\Sigma A_{ij}\delta y_i\delta y_j = \frac{(\delta X)^2}{X}\delta^2 e.$$

The fourth expression in (10) is in $\delta X\delta y_j$ (respectively $\delta X\delta y_i$). No terms in this product occur in $\delta^3 E$, $\delta^4 E$, $\cdots$, since these terms are at least cubic. So the fourth expression is equal to the terms in this product in the second member of (9). But such terms are absent; thus

$$(11d) \qquad X\delta X\cdot\Sigma A_{ij}(y_i\delta y_j + y_j\delta y_i) = 0.$$

It follows that $\Sigma A_{ij}(y_i\delta y_j + y_j\delta y_i) = 0$, and hence that the fifth expression vanishes,

$$(11e) \qquad (\delta X)^2\cdot\Sigma A_{ij}(y_i\delta y_j + y_j\delta y_i) = 0.$$

The sixth expression in (10) is in $(\delta X)^2$. No terms in this square occur in $\delta^3 E$, $\delta^4 E$, $\cdots$, since these terms are at least cubic. So the sixth expression is equal to the terms in $(\delta X)^2$ in the second member of (9). But such terms are absent, so that

$$(11f) \qquad (\delta X)^2\cdot\Sigma A_{ij}y_iy_j = 0.$$

On substituting these six results, (10) becomes

$$\delta^2 E = [X + 2\delta X + (\delta X)^2/X]\delta^2 e.$$

I.e., the expressions $\delta^2 e$ and $\delta^2 E$ are connected by the relation

$$\delta^2 E = \frac{(X + \delta X)^2}{X} \delta^2 e. \tag{12}$$

Since the coefficient of $\delta^2 e$ in (12) is positive, it follows that the criterion of stability $\delta^2 E(x_1, x_2, x_3, x_4) \geq 0$ and the criterion $\delta^2 e(y_1, y_2, y_3) \geq 0$ are equivalent. In both forms of the criterion the sign of equality holds when the variations of the variables satisfy the conditions (3). When these conditions are satisfied, e is constant and the equation

$$\delta E = (X + \delta X)\delta^1 e + e\delta X + \frac{(X + \delta X)^2}{X} \delta^2 e + \cdots$$

reduces to $\delta E = e\delta X$, which represents an obvious physical fact. Since the foregoing method of obtaining the results (8) and (12) is tedious, I should be glad to learn of a simpler way of finding them.

III. Graphical Constructions for Imaginary Intersections of Line and Conic.[1]

By R. M. Mathews, University of Minnesota.

Introduction.—Several methods commonly known for the graphical solution of a quadratic equation are incomplete, as they are explained only for real roots. The object of this paper is to complete these constructions and to generalize them to find graphically the intersections of an arbitrary line with any conic.

Intersection of a circle and a line.—When a line l cuts a circle O the length of the chord formed is $2\sqrt{r^2 - d^2}$, where r is the radius of the circle and d is the distance from the center O to the chord.

When the line does not cut the circle in real points, $d > r$, the sign of $r^2 - d^2$ is negative and the length of the chord is imaginary. Let us replace the given circle by a second which cuts the line in real points such that the length of the chord formed is $2\sqrt{d^2 - r^2}$. Draw OM perpendicular to the line to cut the circle at T and l at M, and extend OM to O' so that $MO' = r$ (Fig. 1); with O' as center draw a circle tangent to the given one at T. The length of the chord which it cuts on l is $2\sqrt{d^2 - r^2} = 2i\sqrt{r^2 - d^2}$.

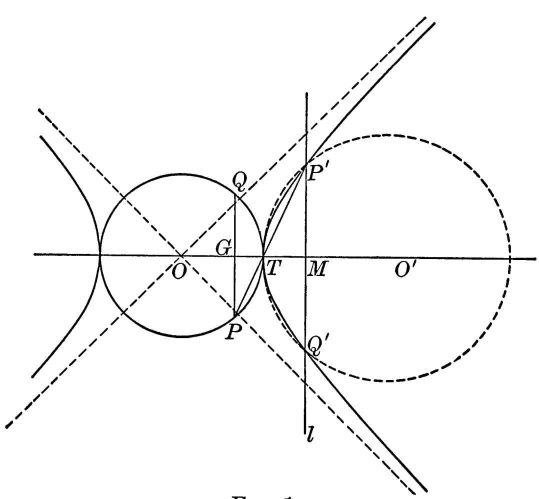

Fig. 1.

[1] Read before the Minnesota Section of the Mathematical Association of America, May 31, 1919.

If the points of intersection be P' and Q', they may represent the imaginary points in which the line cuts the given circle, and $P'Q'$ the length of the imaginary chord.

Let us call this second circle O' the *virtual image* of the given circle with regard to the line.

Applications in analysis.—(i) The graph of

$$x^2 + y^2 + 2gx + 2fy + k = 0 \tag{1}$$

is a circle with center at $(-g, -f)$ and radius $+\sqrt{g^2 + f^2 - k}$. The graph of

$$\alpha x + \beta y + \gamma = 0 \tag{2}$$

is a straight line with intercepts $(-\gamma/\alpha, 0)$, $(0, -\gamma/\beta)$. The algebraic solution of the system (1), (2) leads to a quadratic equation in x

$$ax^2 + bx + c = 0. \tag{3}$$

whence

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \tag{4}$$

The values of y are found by substitution of these x's in (2). Let P and Q be the points of intersection and M the midpoint of the chord PQ.

When the points P and Q are real we ordinarily read out their coördinates directly and so neglect to observe that in (4) the number $-b/2a$ is the abscissa of M while $\sqrt{b^2 - 4ac}/2a$ is half the difference of the abscissæ of P and Q. This guides us in reading out the solutions for the case of imaginary points of intersection. Let the virtual image of (1) cut (2) in points P' and Q', with M' the midpoint of chord. For the intersection of this image with (2), we have

$$x' = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a},$$

and accordingly the values for x are

abscissa of $M' \pm i/2 \times$ (difference of abscissæ of P' and Q'),

while the values for y are

ordinate of $M' \pm i/2 \times$ (difference of ordinates of P' and Q').

(ii) The algebraic solution of the system of equations

$$x^2 + y^2 + 2g_1x + 2f_1y + k_1 = 0, \tag{5}$$

$$x^2 + y^2 + 2g_2x + 2f_2y + k_2 = 0, \tag{6}$$

is effected by subtracting the one from the other and solving the resulting linear equation with (1). The graph of this linear equation is the radical axis of the two circles given by the equations. Thus the graphical solution consists in constructing the radical axis of the circles and finding its intersections with one of them.

(iii) A combination of the preceding results enables us to construct a length to represent the imaginary tangent from a point to a circle and to find a point whence we can read the complex coördinates of the imaginary points of tangency.

Graphical solution of a quadratic equation.—If x_1 and x_2 are the roots of the quadratic equation

$$x^2 + px + q = 0,$$

then

$$x_1 + x_2 = -p, \qquad x_1 \cdot x_2 = q.$$

Take OM on the x-axis equal to $-p/2$; on the y-axis mark two points H and K such that $OH \cdot OK = q$. Draw that circle through H, K which has its center on the line through M parallel to OY. If this circle cuts OX in x_1 and x_2, then $x_1 \cdot x_2 = OH \cdot OK = q$, and $\frac{1}{2}(x_1 + x_2) = -\frac{1}{2}p$. Thus the abscissæ x_1 and x_2 are roots of the quadratic.

When the circle fails to cut the x-axis, construct its virtual image with regard to the axis. It will cut the line in two points x, and x_2 and the roots of the quadratic are

$$OM \pm \tfrac{1}{2}i \,|\, x_1 - x_2 \,|\,.$$

The proxy of a circle.—To extend these developments to any conic we examine further the geometry of a circle and its virtual image. The point T is a center of similitude of the two circles (Fig. 1). Points on the image can be constructed by drawing any radius OX and then line XT to be cut in X' by a radius $O'T'$ parallel to OX. In particular, on OT take OG as a third proportional to OM and OT. Draw chord through G perpendicular to OT to cut circle at P; then PT cuts l at P' on l.

Secondly, each line of the pencil of parallels to l has a virtual image of the circle O and is cut by it in two points P' and Q'. The locus of P' is an equilateral hyperbola, with center at O, vertex at T, and with OT as transverse axis. Accordingly, each line parallel to l either cuts the circle in real points or cuts the hyperbola in points such that the discriminant for the solution of line and hyperbola is the negative of the discriminant for the solution of line and circle. For this reason it is proposed to call the hyperbola the *proxy* of the circle for the direction l.

Finally, the hyperbola can be plotted from the circle by taking for each line l a corresponding point G and proceeding as explained above.

Intersection of an ellipse and a line.—Consider a system of a circle, a line, the virtual image and the proxy of the circle with regard to the line. Rotate the system around a convenient diameter of the given circle, and project orthogonally upon the initial plane. The two circles become ellipses, the line a line, and the proxy equilateral hyperbola becomes a general hyperbola, because this transformation leaves the line at infinity invariant. For this reason parallel lines go into parallel lines, and similar triangles become similar triangles. Thus, all constructions that depend upon relations of similarity and proportion in the original figure are preserved, with the modification that orthogonal diameters become conjugate diameters.

The details for graphical solution of an ellipse with a line may now be recounted (Fig. 2). Across the curve draw two chords parallel to l and join their midpoints, obtaining the diameter conjugate to l. Let it cut the conic at T' and T, and cut l at M. Bisect $T'T$ for center O. Locate G on OT so that OG is a third proportional to OM and OT; through G draw a chord parallel to l and project its ends through T upon l as P' and Q'. Now the abscissæ of the points of intersection of the line and the ellipse are

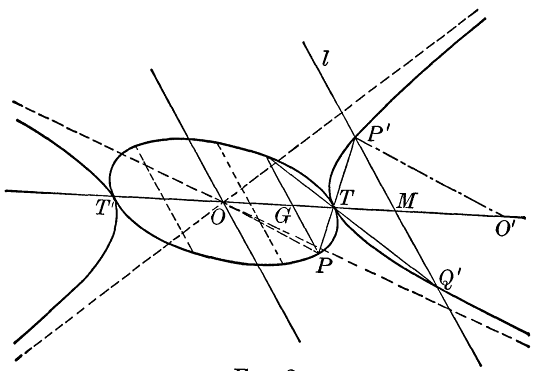

FIG. 2.

abscissa of $M \pm \frac{1}{2}i \times$ (difference of abscissæ of P' and Q'),

and the ordinates are

ordinate of $M \pm \frac{1}{2}i \times$ (difference of ordinates of P' and Q').

The center of the ellipse which is the virtual image of the given one is on OM extended to O' with $MO' = OT$. This image ellipse can be traced by drawing secants through T. If such an one, s, cuts the ellipse at R, draw $O'R'$ parallel to OR to cut s at R', a desired point. Or draw two conjugate diameters through O' to correspond to a convenient pair in the initial curve, and then plot the image with a trammel.

The locus of the points P' and Q' as line l moves parallel to itself is the hyperbola proxy for the ellipse. It can be plotted by locating P' and Q' for l in a series of positions; or we can remark that we have three points, T and one pair P', Q', and a pair of conjugate diameters, whence the curve can be constructed in a variety of ways.

Intersection of an hyperbola and a line.—We have just seen that for an ellipse and a pencil of parallels the proxy is an hyperbola. This hyperbola cuts a line of the pencil in two points such that in the analytic expression for their coördinates the discriminant is the *negative* of the discriminant for said line and the ellipse. Accordingly, those lines which the ellipse cuts in real points the hyperbola cuts in points with the corresponding complex coördinates. Thus the proxy of a hyperbola is that ellipse for which it is proxy, and all the constructions of the preceding paragraph can be applied to the case of a line and hyperbola.

Intersection of a parabola and a line.—The tangent to a parabola at a point T separates all lines parallel to it into two classes, those which cut the curve in real points and those which do not. Let P and Q be points on the curve with chord PQ parallel to the tangent at T (Fig. 3). Extend PT to P' with $P'T = PT$. Locate Q' similarly. Then $PQP'Q'$ is a parallelogram with PQ and $P'Q'$ equally distant from T. When this construction is repeated for every line PQ, parallel

to the tangent at T, the locus of P' and Q' is a parabola congruent and symmetrical to the first through T, and with the midpoints of the parallel chords collinear on a diameter of the curves. This new curve cuts in real points all lines parallel to tangent at T, that the first did not, and conversely.

To solve graphically for imaginary points of intersection of a line and parabola: draw two chords parallel to the line and join their midpoints obtaining a diameter to cut curve in T and l in M. On MT take $TO = MT$ and through O draw PQ parallel to l. Then PT cuts l in P' and QT cuts it in Q', and the coördinates of the imaginary points of intersection are read out as before.

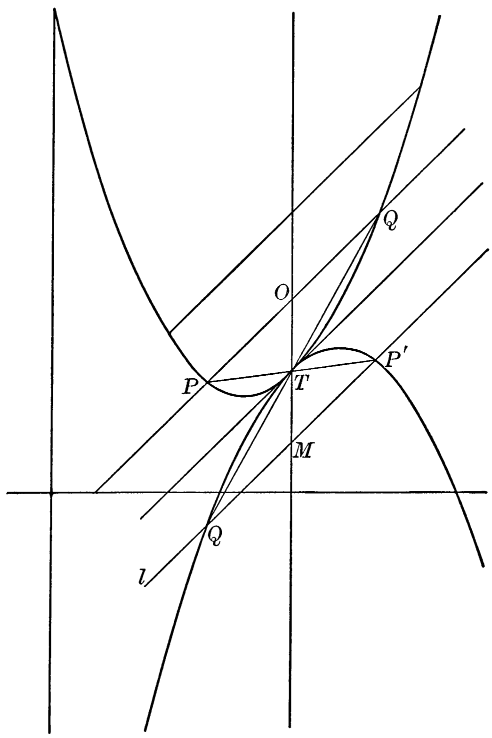

FIG. 3.

Solution of a quadratic equation.—Two commonly used graphical solutions of a quadratic equation

$$(1) \qquad x^2 + px + q = 0$$

depend on parabolas. In the first we plot the function

$$(2) \qquad y = x^2 + px + q$$

and read out the intercepts on the x-axis. The failure of this method when the roots are complex can be remedied by drawing a line parallel to the x-axis with the vertex of the curve half way between them and proceeding as in the previous paragraph.

In the second method, we remark that the function (2) can be regarded as the sum of the expressions

$$y_1 = x^2 \qquad \text{and} \qquad y_2 = -px - q.$$

Thus the parabola $y_1 = x^2$ will be the same for all quadratics and is graphed once for all. Each of the two points in which the line $y_2 = -px - q$ cuts the curve has an abscissa x which makes $y_1 = y_2$ and so is a root of (1). With the addition of the methods of the preceding section we have a construction that solves every quadratic equation graphically.

Note: The topic of this paper is discussed in J. G. Hamilton and F. Kettle, *Graphs and Imaginaries*, London, Arnold, 1904; and A. Schultze, *Graphic Algebra*, New York, Macmillan, 1908.—EDITOR-IN-CHIEF.

RECENT PUBLICATIONS.

REVIEWS.

Vital Statistics, An Introduction to the Science of Demography. By G. C. WHIPPLE, Professor of Sanitary Engineering in Harvard University. New York, Wiley, 1919. xii + 517 pages. 12mo. Price $4.00.

This interesting and instructive book was not written primarily for mathematicians, but it stresses the need of mathematics and an intelligent, discriminating use of mathematics. The preface reads: "This book is written for students who are preparing themselves to be public-health officials and for public-health officials who are willing to be students . . . it emphasizes the need of using vital statistics with truth, with imagination and with power."

Printed in very attractive form, with numerous diagrams, tables, and illustrative matter (with 529 pages, but only $\frac{3}{4}$ inch in thickness), the book should fulfill its mission; it should also make an appeal to the general public; and, indeed, to mathematicians who are interested in the social sciences.

The mathematics involved runs from arithmetic up to the theory of probability and to correlation. As an application of Bernoulli's Theorem in probability, an epidemic of typhoid is postulated for a city where milk is supplied by two dealers—page 399. If the epidemic is due to the water supply of the city, each dealer should have among his customers about the same percentage of typhoid cases—other circumstances being the same. The problem is to determine how many cases above his proportionate number a milk dealer must have in order that suspicion may properly fall upon the milk he sells.

The author condenses in 60 pages some of the elements of probability, correlation, and life insurance—trying to impart some information to students not well trained in mathematics. In this condensation, certain infelicities have made their appearance. Frequency and probability are not adequately distinguished (p. 398). The so-called "probable error" is confused with the most probable error (pp. 390, 391). The name "curve of error" is applied to a cumulative curve of error (pp. 386, 387), and the latter is improperly described at the foot of p. 390. The "expectation of life" is confused with the "most probable lifetime" on p. 422, but on pp. 427, 428, the proper distinction is made. The Combined Experience Mortality Table, mentioned on p. 431, was based upon the data of British companies instead of American companies, and the date of the Northampton Table, given on p. 431 as 1762 is given by Henry Moir as 1783. It is not clear what distinction the author had in mind on p. 410 between Pearson's coefficient of correlation and Galton's. That the graphical method of getting the coefficient of correlation on pp. 412, 413, is defective, can be seen by supposing that each wheat-price were raised ten shillings—this would not alter the coefficient of correlation, but the computation on p. 413 would be altered. On p. 411,

the figure 0.54 is said to represent "low correlation." In a probability table on p. 392, the figure 0.5872 appears instead of 0.5817, and on p. 393 the figures 0.58 and 0.77 appear instead of 0.57 and 0.78, respectively. In the solution of the problem on p. 399, relating to the milk dealers and mentioned above, hardly enough allowance has been made for natural dispersion.

No attempt was made by the reviewer to check the many tables and statistical data in the book, but in glancing at the population table on p. 158, it was noticed that the population of Cleveland, Ohio had been used also for Cincinnati, Ohio—evidently a typographical error.

Vital Statistics is arranged with a set of exercises and questions at the end of each chapter, so that it may be used as a text-book. Such a stimulus to thought can not be too highly recommended, even for books not intended primarily as texts. In these questions, numerous references to the literature of the subject are given.

There is also a short bibliography in Appendix I. Here misprints appear in the names of Willford I. King and Harald Westergaard. Another book of Westergaard's, also published by Fischer, is *Die Grundzüge der Theorie der Statistik,* a book of some interest to mathematicians. E. Davenport's *Principles of Breeding,* with Appendix by H. L. Rietz (Ginn & Co.) may be added to the list; also, Merriman's *Text-Book on the Method of Least Squares* (Wiley). A more extended bibliography, for advanced work, would have included the names of Czuber, Poincaré, Borel, Bachelier, Bruns, Blaschke, Carvallo, Markoff, Montessus de Ballore, and many of the older authors. Among periodicals, *Biometrika* might well be mentioned. Appendices II and III give model state laws for reports of sickness, births and deaths, and standard certificates of birth and death. Appendix IV is a 5-place table of logarithms. Finally, there is an Index.

The following is a condensation of the "Contents" as listed on pp. vii–xii. For mathematicians, Chapters II, XII, XIII, XIV, will be of special interest,—as showing how mathematics is applied.

Contents. Chapter I. Demography: History of statistics—Vital Statistics; The statistical method. II. Statistical Arithmetic: Collection of data; Tabulation; Precision and accuracy; Logarithms; The slide-rule; Classification and generalization; Averages; The moving average; Mechanical devices. III. Statistical Graphics: Types of diagrams; Graphical deceptions; Summation diagrams; Polar coördinates; Double coördinate paper; Logarithmic cross-section paper. IV. Enumeration and Registration: United States census; Credibility of census returns; Births, deaths, marriages, sickness. V. Population: Estimation of population; Arithmetical increase—Geometrical increase; Decreasing rate of growth—Difference between estimate and fact; Population of U. S. cities (a table); Classification of population; Errors due to use of round numbers; Other sources of error; Standards of age distribution. VI. Death-rates, Birth-rates, Fecundity. Marriage-rates; Divorce-rates; Marriage-rates, birth-rates and death-rates in Sweden; Birth; rates, death-rates, marriage-rates and divorce-rates in Massachusetts. VII. Specific Death-Rates: Age, sex, marital condition, nationality; Chronological changes; Particular diseases. VIII; Causes of Death: Nosology; International list of causes of death; Present-day classification. IX. Analysis of Death-Rates: Two methods. X. Statistics of Particular Diseases: Tuberculosis; Diphtheria; Typhoid fever; Cancer. XI. Studies of Deaths by Age Periods: Infant mortality, its chronological reduction; Pre-natal deaths; Expectation of life at different ages; Causes of infant deaths; Maternal mortality; Diseases of early childhood; Proportionate mortality; Median age of persons living. XII. Probability: Natural frequency; Coin tossing; Chance; Binomial theorem; Chance and natural phenomena; Frequency curves, including skew curves;

Frequency curves shown by summation diagrams—Deviation from the mean; Standard deviation; Coefficient of variation; Computation of coefficient from grouped data; Probable error; Doubtful observations; The probability scale; Probability cross-section paper. Another use of probability; The frequency curve as a conception. XIII. Correlation: Correlation and causality; Laws of causation; Methods of correlation; Galton's coefficient of correlation; Example of low correlation—Correlation shown graphically; Correlation table; Use of mathematical formulæ; Secondary correlation; The lag—Coefficient of correlation and the lag; Other secondary correlations; The epidemiologist's use of correlation. XIV. Life Tables: Probability of living a year; Mortality tables; Most probable life-time; "Vie probable"; Expectation of life; Comparison of the three methods; Life-tables based on living populations; Mathematical formulæ; Early history of life-tables; Recent life tables; United States Life Tables: 1910; A few comparisons. XV. A Commencement Chapter: Military statistics; Industrial diseases; Accidents; Infantile paralysis; Sanitary index.

EDWARD L. DODD.

The Sumner Line, or Line of Position as an Aid to Navigation. By GEORGE C. COMSTOCK. New York, Wiley. 6 + 70 pages. 12 mo. Price $1.25.

This is one of a dozen or more of little books on navigation that have appeared within the past two years as a result of the increased activity of the Navy and the Merchant Marine. Many teachers of the subject have apparently felt that the standard American text book on navigation, "The American Practical Navigator," originally by Nathaniel Bowditch, is, in some respects, open to criticism, and they have tried to improve upon the explanations and methods used in various parts of the book.

Professor Comstock has confined his book to problems connected with the determination of the Line of Position, the method of locating a ship at sea, which has been adopted by the U. S. Navy.

Presuming familiarity on the part of the reader with the elements of navigation, the Nautical Almanac, and such instruments as the sextant, compass, chronometer, and log, the author begins with an explanation of the sub-solar point, *i. e.*, the point on the earth directly under the sun (or any other heavenly body). He then takes up the Circle of Equal Altitude, and the location, by the methods of St. Hilaire and de Aquino, of the Line of Position or that part of the Circle of Equal Altitude which is in the vicinity of the observer's ship.

The theory of the spherical traverse tables is fully explained, and their use is illustrated not only in finding the azimuth and the "calculated" altitude, but also in solving such problems as finding the amplitude, the time when a star will cross the Prime Vertical, and the identification of an unknown star.

One of the novel features of the book is the method of finding the coördinates of the intersection of two Lines of Position. This is commonly done at sea by plotting the lines on a chart. In Bowditch and in other epitomes, methods, based on the solution of right triangles, are given for computing the coördinates of intersection when a chart is not available. Professor Comstock has resorted to analytical geometry and has expressed his results in terms which can be handled by the use of plane traverse tables. This is of interest from a mathematical point of view, but probably will not become popular with practical navigators because of the many possibilities of mistakes, especially in algebraic signs.

Another feature of the book is the revival of horizon observations. Methods of determining the longitude from sunrise and sunset observations used to be given in the old epitomes, but, on account of the uncertainty of refraction near the horizon, those methods were omitted years ago. Professor Comstock has prepared a special table of Horizon Corrections for different temperatures and pressures, which, under normal conditions at sea, ought to enable the navigator to get fairly good results from horizon sights.

Throughout the book Forms are used for the systematic arrangement of the computations, and, to accompany the book, pads of Blank Forms have been prepared for working the St. Hilaire or de Aquino methods of finding the Line of Position. On each blank is a skeleton chart for plotting the lines.

F. SLOCUM.

NOTES.

In his *Varia Socratica*,[1] pages 151–155, A. E. Taylor discusses the familiarity of Socrates with mathematical science.

A new edition of the *Encyclopedia Americana* is nearing completion and is especially noteworthy here on account of the valuable mathematical articles contained therein. The edition is expected to comprise thirty volumes and the mathematical articles were prepared by a number of different American authors.

The following memoirs on Ulisse Dini have been recently published: By L. Bianchi in *Rendiconti della r. accademia dei Lincei, scienze fisiche*, Roma, 1919, (10 pages); by F. d'Arcais in *Atti del r. istituto Veneto di Scienze, lettere edarti*, Venezia, 1919 (7 pages).

A compiler of a mathematical dictionary might find the following recent volume in the Manuali Hoepli series of some service: D. M. Mele, *Dizionario internazionale di aeronavigazione e costruzioni aeronautiche, italiano, francese, inglese, tedesco con indice delle quattro lingue in alfabeto unico.* Milano, 1919. (7 + 227 pp.).

A new work by Professor A. N. Whitehead, entitled *An Enquiry concerning the Principles of Natural Knowledge*, has been published by the Cambridge University Press 12s. 6d. The four parts of the book deal with: The traditions of science, The data of science, The method of extensive abstraction, The theory of objects. In the investigation the scientific concepts of space and time are considered as the first outcome of the simplest generalization from experience.

The interesting and attractive first two numbers of *Norsk Matematisk Tidsskrift*, the official organ of the recently organized Norwegian Mathematical Society (*1919*, 324) are heartily welcomed by the Monthly. The first number, of 32 pages, contains articles on Sylow and his scientific work, on number theory polynomials and on an elementary proof of Kepler's first and second laws, besides brief notes, problems for solution, and news of the Society (which had 375 members on February 1, 1919). As inserts are also a fine portrait of Sylow and the laws of the Society. In addition to more problems, notes concerning the Society and its members, and book reviews, the second number, of 48 pages, contains articles on Indian mathematics, application of a formula for inversion to the theory of numbers, distribution of chance errors in observation, and graphical integration. The articles are written in Norwegian, Swedish, French, and German.

ARTICLES IN CURRENT PERIODICALS.

Bollettino di Bibliografia e Storia delle Scienze Matematiche, Volume 20, No. 3, 1918: "Sulla significato della parola 'prospettiva' usata nella legge sulla conservazione dei monumenti" by G. Pittarelli, 65–76; 17 reviews, 76–94 [including reviews by G. Loria of

[1] First series (St. Andrews University Publications, No. IX), Oxford, Parker, 1911.

Miller's *Historical Introduction to Mathematical Literature* (New York, 1916), of Cajori's *William Oughtred, a great seventeenth-century teacher of mathematics* (London and Chicago, 1916), and of *Fundamental Conceptions of Modern Mathematics* by Richardson and Landis (Chicago, 1916)]; "Notizie," 94–96.

Common Ground, West Somerville, Mass., Volume 1, April, 1919: Mathematics in the elementary school and the junior high school," 7–15 [Report of the Committee on Curriculum of the Massachusetts Teachers' Federation].

Industrial Arts Management, New York, volume 8, February, 1919: "Mathematics for the apprentice," 58–62.

Journal of the Indian Mathematical Society, volume 11, no. 3, June, 1919: "Some definite integrals" by S. Ramanujan, 81–87; "Fractional differentiation" by M. T. Naraniengar, 88–95; "On a system of three similar triangles, connected with a given triangle," 96–99; "The theory of linear systems of points," 100–108; Addition formulas for $sn(u_1 + u_2 + u_3)$ etc." by F. H. V. Gulasekharam, 109–111; "On question 963" by V. R. Aiyar, 111–112; "Astronomical notes" by T. P. Sastri, 113; Solutions and questions for solution, 114–120; Periodicals received, i–iii.

Mathematical Gazette, Volume 9, July, 1919: "Mathematics in business and industry" by H. T. H. Piaggio, 361–364; "Gleanings far and near," 364, 368, 371; "Extension of the theory of inversion to conics" by J. J. Milne, 365–368; "Notes for lessons introductory to differential geometry" by E. H. Neville, 369–371; [Greeks, and the theorem coaxial triangles are copolar] by J. J. Milne, 372–373; "Geometrical construction for the trisection of an angle to any required degree of accuracy" by D. F. Ferguson, 373; "Reviews," 374–376.

Messenger of Mathematics, Volume 48, no. 4, August, 1918: "On Nielsen's functional equations" by G. N. Watson, 49–53; "On a simple summation of the series $\sum_{s=0}^{n-1} e^{2s^2\pi i/n}$" by L. J. Mordell, 54–56; "Radiation from a moving magneton" by H. Bateman, 56–64—No. 5, September: "Radiation from a moving magneton," 65–76; "On a diophantine problem" by H. Holden, 77–80—No. 6, October: "On a diophantine problem" by H. Holden, 81–87; "A simple proof of the fundamental equality in the theory of the gamma function" by S. Pollard, 87–89; "Notes on some points in the integral calculus" by G. H. Hardy, 90–96—No. 7, November: "Notes on some points in the integral calculus" (50) by G. H. Hardy, 97–100; "Method of expressing the cross-ratio of the range given by the roots of a biquadratic equation in terms of an auxiliary angle connected with the roots of the reducing cubic of the biquadratic" by A. Lodge, 100–107; "Notes on some points in the integral calculus" (51) by G. H. Hardy, 107–112—No. 8, December, 1918: "General pentaspherical coördinates" by T. C. Lewis, 113–128.

Mind, Volume 44, July, 1919: "Professor John Cook Wilson" by H. A. Prichard, 297–318 ["The death of John Cook Wilson, Wykeham Professor of Logic in Oxford since 1889, is a serious loss for philosophy. . . . The following summary of his life is condensed from a notice by Mr. H. W. B. Joseph in Vol. VII of the *Proceedings of the British Academy*, to which the reader is also referred for a sketch of his philosophy." Professor Wilson died in August, 1915. He was the author of *On the traversing of geometrical figures*, Clarendon Press, 1905]; "On occupying Space" by H. W. B. Joseph, 336–339; "A proof that any aggregate can be well ordered" by P. E. B. Jourdain, 382–384 [First sentence: "The account of my process for well-ordering any given aggregate M described in *Mind* for July, 1918 . . . has been criticised by some on grounds which show, I think, that the point of the process has not been grasped."]

Nouvelles Annales de Mathématiques, Volume 78, March, 1919: "Sur l'helicoëde gauche" by M. d'Ocagne, 81–89; "Sur une homographie particulière et son application a la perspective" by M. d'Ocagne, 89–92; "Sur une classe de courbes planes remarquables" by R. Goormaghtigh, 93–111; "Foyers rationnels des courbes" 112–113; "Applications géométriques de la théorie des infiniment petits" by M. d'Ocagne, 113–114; "Sur les courbes algébriques singulières sous le rapport des barycentres cycliques" by M. d'Ocagne, 115–116; Questions and solutions, 116–120—April: "Sur une application élémentaire d'une méthode générale donnant les équations du mouvement d'un système" by P. Appell, 121–131; "Sur les centres de courbure des lignes décrites par les points d'une figure plane mobile dans son plan" by M. d'Ocagne, 131–134; "Sur l'extraction, a une unité près, de la racine $m^{\text{ième}}$ d'un nombre quelconque a l'aide des logarithmes" by P. Deleus, 134–137; "Sur l'orthopole" by R. Goormaghtigh, 137–140; "Sur quelques intégrales trigonométriques" by M. F. Egan, 140–145; Questions and solutions, 145–160. —May: "Sur la méthode de Poincaré pour étudier la stabilité de l'équilibre" by P. Appell,

161–168; "Sur la détermination des surfaces minima" by C. Clapier, 169–176; "Sur un problème de géométrie infinitésimale" by R. Goormaghtigh, 177–184; "Sur un problème de mécanique" by E. Delassus, 184–188; problems and solutions, 188–200.

Psychological Bulletin, Princeton, N. J., Volume 15, No. 12, December, 1918: "An easy method of determining the coefficient of correlation" by H. F. Adams, 456–459.

Revue de Metaphysique et de Morale, Year 26, No. 3, May–June, 1919: "La question de la sincérité de Descartes" by G. Milhaud, 297–311; "Sur une définition possible des ordinaux transfinis" by A. Reymond, 313–334; "Sur la méthode d'enseignement des mathématiques et des sciences pour la formation du futur maître" by E. Rignano, 389–399.

Revue du Mois, Tome 20, April, 1919: "L'orientation actuelle des mathématiques" by A. Denjoy, 18–28 [Discours d' installation dans la chaire de théorie des fonctions à l'université d'Utrecht, 3 Octobre, 1917].

Revue Générale des Sciences, Volume 30, No. 2, June 30, 1919: Review, by R. d'Adhémar, of Goursat's *Cours d'analyse mathématique,* tome 2, 3e éd. (Paris, 1918), p. 385; review by L. Potin of H. Bouasse's *Géographie mathématique* (Paris, 1919), 385–386.

Science, second series, Volume 50, August 8, 1919: "The problem of the boy in the swing" by H. Crew, 139–140.

Science Progress, Volume 14, No. 53, July, 1919: "Recent advances in mathematics" by P. E. B. Jourdain, 1–10; "Recent advances in applied mathematics" by S. Brodetsky, 10–18; Essay-review, "The nature of number" by J. C. Gregory [on B. Russell's *Introduction to Mathematical Philosophy* (London, 1919)], 141–146; Reviews by P. E. B. Jourdain of A. Gray's *A Treatise on Gyrostatics and Rotational Motion: Theory and Applications* (London, 1918); of E. B. Elliott's *An Introduction to the Algebra of Quantics* (2d ed., Oxford, 1913); of H. B. Phillips's *Integral Calculus* (New York and London, 1917); of R. S. Heath's *Solid Geometry* (4th ed., London, 1919); and of W. P. Milne and G. J. B. Westcott's *A First Course in the Calculus* (London, 1918), 150–153.

Tôhoku Mathematical Journal, Volume 16, Nos. 1–2, June, 1919: "On measure of discontinuity and approximative representation of a function" by Y. Okada, 1–15; "On the approximation of a function of two variables by polynomials" by S. Takenaka, 16–25; "The popular books, Jinkôki and Kaisanki, in the Japanese mathematics" [in Japanese] by T. Hayashi, 26–40; "On some methods of construction in elementary geometry" by K. Yanagihara, 41–49 [section headings: 'Constructions with a ruler and compasses of given aperture'; 'Poncelet-Steiner's constructions'; 'constructions with a ruler of limited length and compasses of given aperture']; "Notes in elementary geometry" by T. Kubota, 50–52; "On the sum of two quadratic rests with respect to a prime ideal" by T. Matsumoto, 53–61; "On the theory of finite differences" by T. Kameda, 62–72; "A note on the convergence of the continued fraction with positive elements" by K. Shibata, 73–77; "On the mean center of the contact points of tangents to an algebraic curve and of tangent planes to an algebraic surface" by K. Shibata, 78–88; "Rational determination of the 'Wertigkeit' of algebraical correspondence on an algebraic curve" by M. Shibayama, 89–91; "Mathematical notes" by T. Kubota, 92–98; "On special systems of line equations having infinite unknowns" by K. Ogura, 99–102; "On the theory of approximating functions with applications to geometry, law of errors and conduction of heat" by K. Ogura, 103–154; "Some problems of circle and sphere divisions" by T. Hayashi, 155–159; "On the maximum modulus of norm-function" [in Japanese] by T. Hayashi, 160–171; New Books, Contents of Periodicals, Miscellaneous Notes, 172–183.

United States Naval Institute Proceedings, Volume 45, July, 1919: "Great circle sailing—a few 'wrinkles' to save time" by H. G. S. Wallace, 1197–1199.—August: "Trajectories and their corrections" by A. G. Kirk, 1375–1395.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht aller Schulgattungen, Volume 50, No. 1, published Jan. 28, 1919: "Zum fünfzigsten Jahrgang der Zeitschrift" by W. Lietzmann, 1–7; "45 Jahre Aufgaben-Repertorium" by C. Müsebeck, 7–13; "Erinnerungen aus den ersten Jahren dieser Zeitschrift" by Hubert Müller, 13–18; "Die geometrische Periode in dem mathematischen Unterricht an der Breslauer Universität" by O. Toeplitz, 18–26; "Mathematische Wanderungen und Wandelungen in der Provinz Hessen-Nassau" by Carl H. Müller, 27–34; "Erinnerungen" by A. Schülke, 34–40; "Die Entwickelung des mathematischen Unterrichts im Sinne der Reformbewegung am Gymnasium in Göttingen" by E. Götting, 40–47; "Winkelmessung durch Umlauf" by K. Schwering, 47–49; "Begabung und Studium" by W. Lorey [dedicated to F. Klein on the 50th anniversary

of his doctorate and in honor of his 70th birthday], 49–56; "Der mathematische Verein München" by H. Wieleitner, 56–58; "Emil Lampe zum Gedächtnis" [with portrait] by E. Haentzschel, 58–60; "Bücherbesprechungen," "Zeitschriftenschau," and "Neuerscheinungen," 62–64—No. 2, published Feb. 20: "Einteilung der Dreiflache" by R. Sturm, 65–66; "Ueber Winkelteilung" by R. Sturm, 66–69; "Versuch einer Bestimmung der mittleren Strafzeit" by R. Sturm, 70–71; "Zum geometrischen Anfangsunterricht" by G. Kewitsch, 71–78; "Winkel an den Parallelen" by Münch, 78–81; "Kleine Mitteilungen," 81–93; "Aufgaben-Repertorium," 93–97; "Bücherbesprechungen," 100–112—No. 3, published March 21: "Ueber das 'Linearzeichnen' an den höheren Schulen und sein Verhältnis zum 'Freihandzeichnen'" by F. Schilling, 113–131; "Die Kollineation in Prima" by A. Gerlach, 131–137; "Kleine Mitteilungen," "Bücherbesprechungen," "Zeitschriftenschau," and "Neuerscheinungen," 138–152.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

2793. Proposed by J. L. RILEY, Stephenville, Texas.

If a, b, and c are complex and α, β, and γ real constants, the point

$$x = \frac{at^2 + 2bt + c}{\alpha t^2 + 2\beta t + \gamma}$$

traces a conic or a straight line when t takes all real values. *Stolz und Gmeiner.*

2794. Proposed by B. J. BROWN, Kansas City, Mo.

Find the value of $x^{e^x} \div x^{x^x}$ when $x \doteq 0$ and when $x \doteq \infty$. *I. C. S.* 1902.

2795. Proposed by C. N. SCHMALL, New York City.

A square is described touching the ellipse $x^2/a^2 + y^2/b^2 = 1$, at the ends of its minor axis; a second ellipse is drawn circumscribing the square and tangent to the given ellipse at the ends of the major axis. The new ellipse is treated as the first and the process is continued until there are n new ellipses. Show that the last ellipse is a circle if the eccentricity of the original ellipse is $\sqrt{n/(n+1)}$.

2796. Proposed by N. P. PANDYA, Amreli, India.

Construct a triangle ABC having its centroid on a given ellipse, AB being a fixed diameter of the ellipse and C lying on one of the directrices.

2797. Proposed by E. J. OGLESBY, College of William and Mary.

Solve for x and y the simultaneous equations, $x^3 + y^3 = 35$ and $x^2 + y^2 = 13$.

2798. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that every positive integer is equal to the sum of at most four squares.

SOLUTIONS OF PROBLEMS.

406 (Algebra) [March, 1914]. Proposed by S. A. COREY, Albia, Iowa.

Solve the system of equations:

$$(1-x)(a_1 + a_2y + a_3z) = d, \qquad (1-y)(b_1 + b_2x + b_3z) = g, \qquad (1-z)(c_1 + c_2x + c_3y) = h.$$

Discussion by H. S. Uhler, Yale University.

In order to introduce cyclic order we shall interchange two of the coefficients in the second equation and rewrite it as

$$(1 - y)(b_1 + b_2 z + b_3 x) = g.$$

Obviously, this modification will have no essential influence on the final result.

Solving the third equation for z we obtain

$$z = 1 - \frac{h}{c_1 + c_2 x + c_3 y}.$$

Substituting this expression for z in the first equation and performing elementary reductions, we find

$$\alpha y^2 + \beta y + \gamma = 0 \tag{1}$$

where

$$\begin{aligned}
\alpha &\equiv a_2 c_3(1 - x),\\
\beta &\equiv [c_3(a_1 + a_3) + a_2 c_1 - c_3 d] - [c_3(a_1 + a_3) + a_2(c_1 - c_2)]x - a_2 c_2 x^2,\\
\gamma &\equiv [c_1(a_1 + a_3) - c_1 d - a_3 h] - [(c_1 - c_2)(a_1 + a_3) + c_2 d - a_3 h]x - c_2(a_1 + a_3)x^2.
\end{aligned}$$

The second equation (modified), when treated in the same manner, gives

$$Ay^2 + By + C = 0 \tag{2}$$

where

$$\begin{aligned}
A &\equiv c_3[(b_1 + b_2) + b_3 x],\\
B &\equiv [(c_1 - c_3)(b_1 + b_2) + c_3 g - b_2 h] + [c_2(b_1 + b_2) + b_3(c_1 - c_3)]x + b_3 c_2 x^2,\\
C &\equiv -\{[c_1(b_1 + b_2) - c_1 g - b_2 h] + [c_2(b_1 + b_2) + b_3 c_1 - c_2 g]x + b_3 c_2 x^2\}.
\end{aligned}$$

Eliminating y from equations (1) and (2) we get

$$(\alpha C - \gamma A)^2 - (\alpha B - \beta A)(\beta C - \gamma B) = 0. \tag{3}$$

When the expressions for α, β, γ, A, B, C are substituted in relation (3) it is found that the coefficient of x^6 vanishes identically, but that all the remaining coefficients have extremely complicated values other than zero, in general. Therefore, since the equation in x is of the fifth degree with literal coefficients, the original set of equations cannot be "solved."

In the particular case where $a_1 = 2$, $a_2 = 3$, $a_3 = 4$, $b_1 = 5$, (new) $b_2 = 6$, (new) $b_3 = 7$, $c_1 = 8$, $c_2 = 9$, $c_3 = 10$, $d = 11$, $g = 12$, and $h = 13$, I found

$$1{,}628{,}991x^5 + 3{,}399{,}760x^4 - 3{,}151{,}759x^3 - 8{,}498{,}412x^2 - 8{,}179{,}912x - 8{,}133{,}008 = 0,$$

which has at least one positive real root.

411 (Algebra) [April, 1914, June, 1919]. Proposed by V. M. SPUNAR, Chicago, Ill.

Determine $x_1, x_2, x_3, \cdots x_p$, from the equations:

$$\begin{aligned}
x_1 + x_2 + x_3 + \cdots + x_p &= a_0,\\
b_1 x_1 + b_2 x_2 + b_3 x_3 + \cdots + b_p x_p &= a_1,\\
b_1^2 x_1 + b_2^2 x_2 + b_3^2 x_3 + \cdots + b_p^2 x_p &= a_2,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
b_1^{p-1} x_1 + b_2^{p-1} x_2 + b_3^{p-1} x_3 + \cdots + b_p^{p-1} x_p &= a_{p-1}.
\end{aligned}$$

Solution by H. S. Uhler, Yale University.

From the theory of determinants, we know that

$$x_i = \begin{vmatrix} a_0 & 1 & \cdots & 1 & 1 & \cdots & 1\\ a_1 & b_1 & \cdots & b_{i-1} & b_{i+1} & \cdots & b_p\\ a_2 & b_1^2 & \cdots & b_{i-1}^2 & b_{i+1}^2 & \cdots & b_p^2\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ a_{p-1} & b_1^{p-1} & \cdots & b_{i-1}^{p-1} & b_{i+1}^{p-1} & \cdots & b_p^{p-1} \end{vmatrix} \div \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1\\ b_i & b_1 & \cdots & b_{i-1} & b_{i+1} & \cdots & b_p\\ b_i^2 & b_1^2 & \cdots & b_{i-1}^2 & b_{i+1}^2 & \cdots & b_p^2\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ b_i^{p-1} & b_1^{p-1} & \cdots & b_{i-2}^{p-1} & b_{i+1}^{p-1} & \cdots & b_p^{p-1} \end{vmatrix} \equiv \frac{D_1}{D_2}$$

Consider the determinant in the denominator, D_2. If any b were equal to any one of the $p - 1$ remaining b's, two columns would be identical and D_2 would vanish. Hence,

$$D_2 = k\Delta(b_i - b_1)(b_i - b_2) \cdots (b_i - b_{i-1})(b_i - b_{i+1}) \cdots (b_i - b_p),$$

where $\Delta \equiv (b_1 - b_2) \cdots (b_1 - b_p)(b_2 - b_3) \cdots (b_2 - b_p) \cdots (b_{p-1} - b_p)$, and k can only be a non-literal factor. To determine k, we note that the product of all the first terms in the $\frac{1}{2}p(p - 1)$ binomial factors equals $b_i^{p-1}b_1^{p-2}b_2^{p-3} \cdots b_{p-2}^2 b_{p-1}$ which differs only in sign from the algebraic value of the negative diagonal of D_2; hence, $k = -1$ or

$$D_2 = -\Delta(b_i - b_1)(b_i - b_2) \cdots (b_i - b_{i-1})(b_i - b_{i+1}) \cdots (b_i - b_p). \tag{1}$$

Next, consider D_1. If $a_0, a_1, a_2, \cdots, a_{p-2}, a_{p-1}$ were replaced, respectively, by $1, x, x^2, \cdots, x^{p-2}, x^{p-1}$ we should have (by the same argument)

$$D_x = -\Delta(x - b_1)(x - b_2) \cdots (x - b_{i-1})(x - b_{i+1}) \cdots (x - b_p)$$
$$\equiv -\Delta(x^{p-1} + c_1x^{p-2} + c_2x^{p-3} + \cdots + c_{p-2}x + c_{p-1}).$$

From the theory of equations, we know that $-c_1$ is the sum of $b_1, b_2, \cdots, b_{i-1}, b_{i+1}, \cdots, b_p$, that $+c_2$ is the sum of the products of these b's taken two at a time, etc., and that $\pm c_{p-1}$ is the product of all these b's, the upper or lower sign to be taken according as p is odd or even. Now the first minors that multiply the a's in D_2 are identical with the corresponding minors which multiply the various powers of x in D_x. Consequently,

$$D_1 = -\Delta(a_{p-1} + c_1a_{p-2} + c_2a_{p-3} + \cdots + c_{p-2}a_1 + c_{p-1}a_0). \tag{2}$$

Finally, dividing equation (1) by equation (2) we get the required formula

$$x_i = \frac{a_{p-1} - (\Sigma b)a_{p-2} + (\Sigma bb)a_{p-3} + \cdots \pm (\Pi b)a_0}{\Pi(b_i - b)},$$

the interpretation of the notation being clear from the preceding discussion.

Also solved by NORMAN ANNING.

455 (Geometry) [February, 1915; June, 1919]. Proposed by R. P. BAKER, University of Iowa.

Find the minimum triangle of assigned angles inscribed in a given triangle.

SOLUTION BY R. C. ARCHIBALD, Brown University.

This problem is closely allied to the problem: Find the maximum triangle of assigned angles circumscribed to a given triangle. Both problems were proposed for solution in 1811 on the last page (exclusive of index), 384, of the first volume of *Annales de mathématiques pures et appliquées*. Solutions by Rochat, and others, were given in volume 2, pages 88–93.

Let $D_1E_1F_1$ be the triangle of assigned angles and ABC be the given triangle. On CA (on the side opposite from B) describe an arc of a circle containing an angle equal to the angle E_1. On CB (on the side opposite from A) describe an arc of a circle containing an angle equal to D. Then if through C a line is drawn parallel to the line of centers of the circles it will meet the circles again in D and E such that DE is the greatest of all line-segments drawn through C. Then if EA, DB be produced to meet in F, the triangle DEF will clearly be the maximum triangle circumscribing the triangle ABC, and the triangle ABC will be the minimum triangle inscribed in the triangle DEF.

Hence to solve the given problem find the maximum triangle, $A_1B_1C_1$, similar to the triangle ABC and circumscribing the triangle $D_1E_1F_1$ (the point F_1 falling in A_1B, and the point E_1 in A_1C_1; also the angles A_1, B_1, C_1 correspond respectively to A, B, and C). Then divide AB at F in the ratio $A_1F_1 : F_1B$; also AC at E in the ratio $A_1E_1 : E_1C$; and CB at D as $C_1D_1 : D_1B_1$. Then D, E, and F are the vertices of the required minimum triangle.

In general, there are, as Rochat pointed out, six solutions of the problem, depending upon the arrangement of the vertices of the triangle DEF on the sides of the given triangle.

Another solution is given in *A Key to the Exercises in the First Six Books of Casey's Elements of Euclid* by Joseph B. Casey, second edition, 1887, pp. 120–121.

Note. A solution by John Casey is given in his *Sequel to . . . the Elements of Euclid*, second edition, 1882, pp. 38–39.—Editors.

Also solved by A. PELLETIER.

2737. Proposed by C. N. SCHMALL, New York City.

Employing Maclaurin's theorem, or otherwise, expand the following three functions (1) $e^{\tan^{-1} x}$ as far as x^6; (2) $e^{\sin x}$ as far as x^8; and (3) $\tan x$ as far as x^9.

The solution of this problem published in our issue for September was incorrectly ascribed to Elmer Latshaw; it was by G. A. OSBORNE, professor emeritus of the Massachusetts Institute of Technology—EDITORS.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Mr. H. E. WOLFE has been appointed assistant professor of mathematics in Indiana University.

Dr. L. S. SHIVELY, who has been professor of mathematics in Mount Morris College, is now president of the college.

T. W. JACKSON, formerly principal of the high school in Fulton, Missouri, has been made head of the department of mathematics in Jamestown College.

H. L. OLSON has been elected to an instructorship in mathematics in the University of Wisconsin.

W. H. SHERK has been appointed professor of mathematics in the University of Buffalo.

Mr. NORMAN ANNING, recently discharged from the Canadian railway troops after service in France, has been appointed instructor in mathematics at the University of Maine.

Dr. H. C. GOSSARD, formerly instructor in the U. S. Naval Academy, has been made assistant professor of mathematics at the University of Oklahoma.

DR. H. C. WOLFF, formerly assistant professor of mathematics at the University of Wisconsin, has accepted an appointment as head of the department of mathematics at the Drexel Institute.

At Washington University, Professor GEORGE O. JAMES has been made head of the department of mathematics and Dr. J. R. MUSSELMAN, formerly of the University of Illinois, has accepted a position as instructor in mathematics.

Assistant professor GLENN JAMES, of Purdue University, has resigned to accept a similar position in the mathematics department at the Carnegie Institute of Technology.

The following men have been appointed instructors in mathematics at Purdue University: Messrs. R. S. UNDERWOOD, G. D. JAMES, F. H. HODGE and T. E. RAIFORD.

Dr. T. McN. SIMPSON has returned from Y. M. C. A. service in France and has been appointed professor of mathematics in his *alma mater*, Randolph-Macon College.

Mr. J. A. CAPARO, professor of electrical engineering and physics at the University of Notre Dame, received the A.M. degree in mathematics at the University of Chicago at the September convocation.

Mr. ELBERT ALLEN, formerly of the University of Chicago High School, has been appointed instructor in mathematics at the University of Missouri.

Dr. S. CHAPMAN, chief assistant at Greenwich Observatory, has been appointed professor of mathematics at the University of Manchester.

Professor W. H. YOUNG, formerly of the University of Liverpool, has been appointed professor of mathematics in the University College of Wales.

At Princeton, Professor PIERRE BOUTROUX and Assistant professor J. H. M. WEDDERBURN have returned after five years of absence in the French and British war service respectively.

Lieutenant F. L. CARMICHAEL, associate professor of mathematics at the University of Alabama, was in charge of range table calculations in the Ordnance Department at Washington during the past summer.

At the University of Illinois, Assistant Professor ARNOLD EMCH has been promoted to an associate professorship of mathematics and Mr. E. H. VANCE has been appointed instructor in mathematics.

Corrigenda: It should have been stated (*1919*, 419) that the value of the Querini-Stampalia prize was 3,000 *lire*—Mr. H. L. SMITH, of Cornell University, is instructor in mathematics at the University of Wisconsin and not as previously noted (1919, 322)—While it is true that Professor L. L. SILVERMAN resigned to enter a banking house in Boston (*1919*, 418) he later decided to remain at Dartmouth College.

THE WORK OF THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS, NOVEMBER, 1919.

Following a previous statement regarding the work of the National Committee on Mathematical Requirements (*1919*, 439), the following items may be of interest.

A preliminary report on "The Reorganization of Introductory Courses in Mathematics in Secondary Schools" was issued towards the end of November.

This report has been prepared by a representative sub-committee. It has not as yet been considered by the National Committee but its publication as a basis for discussion by teachers, organizations, committees, and local groups has been authorized.

A report on "The Valid Aims and Purposes of the Teaching of Mathematics in the Light of Recent Criticisms" will, it is hoped, be ready for distribution by January. In it an attempt will be made to state precisely and succinctly the mathematical training that every citizen should secure. The findings of this report can then be made a basis for the determination of precisely what and how much mathematics should be required of all students.

An extended investigation of "Experimental Schools and Courses" is being undertaken for the Committee by Mr. Raleigh Schorling of the Lincoln School. Detailed plans for this investigation were approved by the National Committee at its last meeting in New York City on November 1st.

Mr. J. A. Foberg is preparing a report on "Mathematics in Junior High Schools."

Professor A. R. Crathorne has recently submitted a report giving the results of his investigation of "Change of Mind Between High School and College as to Life Work." A summary of these results will be published in *School and Society* in the very near future. Professor Crathorne is still at work on an extended investigation entitled "A Critical Study of the Correlation Method as Applied to Grades."

A statement of general principles to govern the proposed revision of college entrance requirements has been tentatively approved by the Committee. This statement has been sent out to some 50 representative colleges and universities for their criticism and comment.

The Committee has sent letters to all teachers organizations having mathematical interests, of which it has been able to learn, asking their coöperation and offering the assistance of the Committee. The response has been very enthusiastic. A considerable number have already appointed committees to receive reports from the National Committee for study and criticism. The plans of the National Committee have been, or will be, presented to 22 such organizations during the months of October to December in the following states: Indiana, Illinois, Iowa, Kentucky, Massachusetts, Missouri, New Jersey, New York, North Carolina, North Dakota, Ohio, Oklahoma, Pennsylvania, Rhode Island, Texas, West Virginia and Wisconsin. Little has yet been done with reference to the promotion of new organizations. It was felt that this Committee should wait until it had definite material on hand for distribution before taking up this phase of its work. The formation of mathematical clubs in several of the larger cities, where they did not exist before, is, however, under way and material regarding the formation of such clubs is in hand. Any individuals interested in the formation of a mathematical club may secure this material by addressing the Chairman of the Committee (J. W. Young, Hanover, New Hampshire).